Algebraic characterizations of distance-regular graphs

(Master of Science Thesis)

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Outline 1

1 Basic results from algebraic graph theory
   (a.1) Perron-Frobenius theorem
   (a.2) The number of walks
   (a.3) The total number of rooted closed walks
   (a.4) The adjacency (or Bose-Mesner) algebra $A(\Gamma)$
   (a.5) Hoffman polynomial

2 Distance-regular graphs
   Definition of distance-regular graph
   Definition of local distance-regular graph
   Characterization of DRG involving the distance matrices
   Examples of distance-regular graphs
   Characterization of DRG involving the distance polynomials
Outline II

3 Characterizations involving the spectrum

Characterization of DRG involving the principal idempotent matrices

Characterizations involving the spectrum
Perron-Frobenius theorem

- Background of problem: Assume that someone give us some matrix $A$. What we can say about maximum eigenvalue of $A$, and appropriate eigenvector for that eigenvalue?

**Theorem (Perron-Frobenius)**

Let $M$ be a nonnegative irreducible symmetric matrix. Then the largest eigenvalue $\lambda_0$ has algebraic multiplicity 1 and has an eigenvector whose entries are all positive. For all other eigenvalues we have $|\lambda_i| \leq \lambda_0$. 
Perron-Frobenius theorem

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**Theorem (Perron-Frobenius)**

Let $M$ be a nonnegative irreducible symmetric matrix. Then the largest eigenvalue $\lambda_0$ has algebraic multiplicity 1 and has an eigenvector whose entries are all positive. For all other eigenvalues we have $|\lambda_i| \leq \lambda_0$. 
Since $\Gamma$ is connected, $A$ is an irreducible nonnegative matrix. Then, by the Perron-Frobenius theorem, the maximum eigenvalue $\lambda_0$ is simple, positive (in fact, it coincides with the spectral radius of $A$), and has a positive eigenvector $v$, say, which is useful to normalize in such a way that $\min_{u \in V} v_u = 1$. Moreover, $\Gamma$ is regular if and only if $v = j$, the all-$1$ vector (then $\lambda_0 = \delta$, the degree of $\Gamma$).
Perron-Frobenius theorem - example

a) Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Characteristic polynomial of $A$ is

$$\text{char}(\lambda) = \lambda^2(\lambda - \sqrt{3})(\lambda + \sqrt{3}).$$

It follow that, maximum eigenvalue $\lambda_0 = \sqrt{3}$ is simple, positive and coincides with spectral radius of $A$. Eigenvector for eigenvalue $\lambda_0$ is $\mathbf{v} = (1, 1, 1, \sqrt{3})^\top$, so it is positive.
Perron-Frobenius theorem - example

Simple graph $\Gamma_1$ and its adjacency matrix.

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 1 \\
4 & 1 & 1 & 1 & 0
\end{bmatrix}$$
Perron-Frobenius theorem - example

\( b) \) Consider matrix \( A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \). Characteristic polynomial of \( A \) is

\[
\text{char}(\lambda) = \lambda^2 (\lambda - 1)(\lambda - 2)(\lambda + 1)(\lambda + 2).
\]

It follows that, maximum eigenvalue \( \lambda_0 = 2 \) is simple, positive and coincides with spectral radius of \( A \). Eigenvector for eigenvalue \( \lambda_0 \) is \( \mathbf{v} = (1, 1, 1, 2, 2, 1)^\top \), so it is positive.
Perron-Frobenius theorem - example

Simple graph $\Gamma_2$ and its adjacency matrix.

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 & 0 \\
5 & 1 & 0 & 0 & 1 & 0 & 1 \\
6 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$
The number of walks of a given length

- Background of problem: Assume that someone give us some graph $\Gamma$. What is easiest way to count the number of walks of given length $k \geq 0$ between vertices $u$ and $v$?

- The number of walks of length $k \geq 0$ between vertices $u$ and $v$ is $a_{uv}^k := (A^k)_{uv}$. 

(a.1) Perron-Frobenius theorem
(a.2) The number of walks
(a.3) The total number of rooted closed walks
(a.4) The adjacency (or Bose-Mesner) algebra $A(\Gamma)$
(a.5) Hoffman polynomial
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The number of walks of length $k \geq 0$ between vertices $u$ and $v$ is $a_{uv}^k := (A^k)_{uv}$.
Consider graph \( \Gamma_3 \).

Simple graph \( \Gamma_3 \) and its adjacency matrix.
Let’s say that we want to find number of walks of length 4 and 5, between vertices 3 and 7. Then first that we need to do is to find adjacency matrix for $\Gamma_3$. After that we need to find $(3, 7)$-entry (or $(7, 3)$-entry) of $A^4$ and $A^5$. 
The number of walks of a given length - example

We have $A^4 =$

$$
\begin{bmatrix}
7 & 6 & 5 & 1 & 1 & 0 & 0 & 6 \\
6 & 12 & 2 & 5 & 0 & 1 & 1 & 6 \\
5 & 2 & 7 & 0 & 5 & 1 & 1 & 5 \\
1 & 5 & 0 & 7 & 2 & 5 & 5 & 1 \\
1 & 0 & 5 & 2 & 12 & 6 & 6 & 1 \\
0 & 1 & 1 & 5 & 6 & 7 & 6 & 0 \\
0 & 1 & 1 & 5 & 6 & 6 & 7 & 0 \\
6 & 6 & 5 & 1 & 1 & 0 & 0 & 7
\end{bmatrix}
$$

and

$A^5 =$

$$
\begin{bmatrix}
12 & 18 & 7 & 6 & 1 & 1 & 1 & 13 \\
18 & 14 & 17 & 2 & 7 & 1 & 1 & 18 \\
7 & 17 & 2 & 12 & 2 & 6 & 6 & 7 \\
6 & 2 & 12 & 2 & 17 & 7 & 7 & 6 \\
1 & 7 & 2 & 17 & 14 & 18 & 18 & 1 \\
1 & 1 & 6 & 7 & 18 & 12 & 13 & 1 \\
1 & 1 & 6 & 7 & 18 & 13 & 12 & 1 \\
13 & 18 & 7 & 6 & 1 & 1 & 1 & 12
\end{bmatrix}
$$
The total number of closed walks of a given length

- Background of problem: Assume that someone gives us some simple graph $\Gamma$. How can we compute the total number of rooted closed walks of given length?

- If $\Gamma = (V, E)$ has spectrum

$$\text{spec}(\Gamma) = \{\lambda_0^m(\lambda_0), \lambda_1^m(\lambda_1), ..., \lambda_d^m(\lambda_d)\}$$

then the total number of (rooted) closed walks of length $l \geq 0$ is

$$\text{tr}(A^l) = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^l.$$
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The total number of closed walks of a given length
The total number of closed walks - example

Consider graph $\Gamma_4$

Simple graph $\Gamma_4$ and its adjacency matrix.

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 0 & 1 \\
5 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}$$
Basic results from algebraic graph theory
Distance-regular graphs
Characterizations involving the spectrum

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The total number of closed walks - example

This graph have three eigenvalues $\lambda_0 = 2$, $\lambda_1 = \frac{\sqrt{5}}{2} - \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} - \frac{1}{2}$, and spectrum:

$$\text{spec}(\Gamma_4) = \{2^1, \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2, \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^2\}$$

Total number of rooted closed walks of length 3, 4 and 5 is

$$\text{tr}A^3 = 1 \cdot 2^3 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^3 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^3 = 0,$$

$$\text{tr}A^4 = 1 \cdot 2^4 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^4 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^4 = 30$$

and

$$\text{tr}A^5 = 1 \cdot 2^5 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^5 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^5 = 10.$$
This graph have three eigenvalues $\lambda_0 = 2$, $\lambda_1 = \frac{\sqrt{5}}{2} - \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} - \frac{1}{2}$, and spectrum:

$$\text{spec}(\Gamma_4) = \{2^1, \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2, \left(-\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2\}$$

Total number of rooted closed walks of length 3, 4 and 5 is

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The adjacency (or Bose-Mesner) algebra $\mathcal{A}(\Gamma)$

- Background of problem: Assume that someone give us some simple graph $\Gamma$. Can we form (define) some algebra that is connected with this graph? Can we say something about connection between the number of distinct eigenvalues and the diameter of graph?

**Definition (adjacency algebra)**

The *adjacency* (or *Bose-Mesner*) *algebra* of a graph $\Gamma$ is algebra of matrices which are polynomials in $\mathcal{A}$ under the usual matrix operations. We shall denote this algebra by $\mathcal{A} = \mathcal{A}(\Gamma)$. Therefore

$$\mathcal{A}(\Gamma) = \{ p(\mathcal{A}) : p \in \mathbb{R}[x] \}$$

(elements in $\mathcal{A}$ are matrices).
Basic results from algebraic graph theory
Distance-regular graphs
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The adjacency (or Bose-Mesner) algebra $\mathcal{A}(\Gamma)$

If $\Gamma$ has $d + 1$ distinct eigenvalues, then \{1, $A$, $A^2$, ..., $A^d$\} is a basis of the adjacency or Bose-Mesner algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in $A$. Moreover, if $\Gamma$ has diameter $D$,

$$\text{dim} \mathcal{A}(\Gamma) = d + 1 \geq D + 1,$$

because \{1, $A$, $A^2$, ..., $A^D$\} is a linearly independent set of $\mathcal{A}(\Gamma)$. Hence, the diameter is always less than the number of distinct eigenvalues: $D \leq d$. 
The diameter is always less than the number of distinct eigenvalues - example

a) Consider graph $\Gamma_5$.

Simple graph $\Gamma_5$ and its adjacency matrix.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 \\
3 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 \\
\end{pmatrix}
\]
The diameter is always less than the number of distinct eigenvalues - example

Eigenvalues of $\Gamma_5$ are $\lambda_0 = 3$ and $\lambda_1 = -1$, so $d + 1 = 2$. Diameter is $D = 1$. Therefore $D = d$. 
The diameter is always less than the number of distinct eigenvalues - example

- Consider graph $\Gamma_6$.

Simple graph $\Gamma_6$ and its adjacency matrix.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
6 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
7 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
8 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
The diameter is always less than the number of distinct eigenvalues - example

Eigenvalues of $\Gamma_6$ are $\lambda_0 = 3$, $\lambda_1 = \sqrt{5}$, $\lambda_2 = 1$, $\lambda_3 = -1$ and $\lambda_4 = -\sqrt{5}$, so $d + 1 = 5$. Diameter of $\Gamma_6$ is $D = 3$. Therefore $D < d$. 
Background of problem: Assume that someone give us some simple graph $\Gamma$. We want to know is there exists a polynomial $H(x)$ such that

$$J = H(A)$$

where $J$ is square matrix of order $n$, which every entry is one, and $A$ is adjacency matrix of $\Gamma$?
A graph $\Gamma = (V, E)$ with eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ is a regular graph if and only if there exists a polynomial $H \in \mathbb{R}_d[x]$ such that $H(A) = J$, the all-1 matrix. This polynomial is unique and it is called the Hoffman polynomial. It has zeros at the eigenvalues $\lambda_i$, $i \neq 0$, and $H(\lambda_0) = n := |V|$. Thus,

$$H = \frac{n}{\pi_0} \prod_{i=1}^{d} (x - \lambda_i),$$

where $\pi_0 := \prod_{i=0}^{d} (\lambda_0 - \lambda_i)$. 
Some notation before definition of DRG

- Let $\Gamma = (V, E)$ denote a simple connected graph with vertex set $V$, edge set $E$ and diameter $D$. Let $\partial$ denotes the path-length distance function for $\Gamma$.
- $\Gamma_i(x) := \{y \in V : \partial(x, y) = i\}$
- $|\Gamma_i(x) \cap \Gamma_j(y)|$ denote the number of elements of the set $\Gamma_i(x) \cap \Gamma_j(y)$
- The *eccentricity* of a vertex $u$ is $\text{ecc}(u) := \max_{v \in V} \partial(u, v)$ and the *diameter* of the graph is $D := \max_{u \in V} \text{ecc}(u)$. 
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Some notation before definition of DRG - example

Petersen graph. We have $\partial(v_1, v_2) = 2$, $\Gamma_1(v_1) = \{u_1, v_3, v_4\}$, $\Gamma_2(v_2) = \{u_1, u_3, u_4, u_5, v_1, v_3\}$, $|\Gamma_1(v_1) \cap \Gamma_2(v_2)| = |\{u_1, v_3\}| = 2$. 
Definition (distance-regular graphs)

A simple connected graph $\Gamma = (V, E)$ with diameter $D$ is called distance-regular whenever there exist numbers $p_{ij}^h$ $(0 \leq h, i, j \leq D)$ such that for any $x, y \in V$ with $\partial(x, y) = h$ we have

$$|\Gamma_i(x) \cap \Gamma_j(y)| = p_{ij}^h.$$
### Definition (local distance-regular graph)

Let $y \in V$ be a vertex with eccentricity $\text{ecc}(y) = \varepsilon$ of a regular graph $\Gamma$. Let $V_k := \Gamma_k(y)$ and consider the numbers

$$
c_k(x) := |\Gamma_1(x) \cap V_{k-1}|,
$$

$$
a_k(x) := |\Gamma_1(x) \cap V_k|,
$$

$$
b_k(x) := |\Gamma_1(x) \cap V_{k+1}|,
$$

defined for any $x \in V_k$ and $0 \leq k \leq \varepsilon$ (where, by convention, $c_0(x) = 0$ and $b_\varepsilon(x) = 0$ for any $x \in V_\varepsilon$). We say that $\Gamma$ is **distance-regular around** $y$ whenever $c_k(x)$, $a_k(x)$, $b_k(x)$ do not depend on the considered vertex $x \in V_k$ but only on the value of $k$. 

Local distance-regular graph

Intersection numbers around $y$. 

$V_0 \quad V_1 \quad V_{k-1} \quad V_k \quad V_{k+1} \quad V_\infty$ 

$y \quad \bullet \quad \cdots \quad c_k \quad b_k \quad \cdots$
Local distance-regular graph - example

Simple connected regular graph that is distance-regular around vertices 1 and 8. This graph is known as Hoffman graph.
Local distance-regular graph - example

Simple connected graph that is distance-regular around vertex 14.
Distance-i matrix

**Definition (distance-i matrix)**

Let $\Gamma = (V, E)$ denote a graph with diameter $D$, adjacency matrix $A$ and let $\text{Mat}_\Gamma(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by the vertices of $\Gamma$. For $0 \leq i \leq D$ we define distance-$i$ matrix $A_i \in \text{Mat}_\Gamma(\mathbb{R})$ with entries $(A_i)_{uv} = 1$ if $\partial(u, v) = i$ and $(A_i)_{uv} = 0$ otherwise. Note that $A_0$ is the identity matrix and $A_1 = A$ is the usual adjacency matrix of $\Gamma$. 
Distance-i matrix - example

Distance-i matrices for octahedron are

\[ A_0 = I, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \]
Distance ◦-algebra

**Lemma**

Let $A_i \in \text{Mat}_\Gamma(\mathbb{R})$ denote a distance-$i$ matrices. Vector space $D$ defined by

$$D = \text{span}\{A_0, A_1, A_2, \ldots, A_D\}$$

forms an algebra with the entrywise or Hadamard product of matrices, defined by $(X \circ Y)_{uv} = (X)_{uv}(Y)_{uv}$.

**Definition (distance ◦-algebra)**

Vector space $D$ from Lemma above will be called the *distance ◦-algebra* of $\Gamma$. 

An algebraic approach to DRG

- $\Gamma$ regular

- Adjacency algebra $\cdot$, $A = \text{span}\{A^0, A^1, \ldots, A^d\}$
- Distance algebra $\circ$, $D = \text{span}\{A_0, A_1, \ldots, A_D\}$
- $\Gamma$ is not distance-regular
Characterization A

**Theorem (characterization A)**

Let $\Gamma = (V, E)$ denote a graph with diameter $D$ and let the set $\Gamma_h(u)$ represents the set of vertices at distance $h$ from vertex $u$. $\Gamma$ is distance-regular if and only if is distance-regular around each of its vertices and with the same intersection array (with another words if and only if for any two vertices $u, v \in V$ at distance $\partial(u, v) = h$, $0 \leq h \leq D$, the numbers

$$c_h(u, v) := |\Gamma_{h-1}(u) \cap \Gamma(v)|, \quad a_h(u, v) := |\Gamma_h(u) \cap \Gamma(v)|,$$

$$b_h(u, v) := |\Gamma_{h+1}(u) \cap \Gamma(v)|,$$

do not depend on the chosen vertices $u$ and $v$, but only on their distance $h$; in which case they are denoted by $c_h$, $a_h$, and $b_h$, respectively).
What is different between Definition of DRG and Characterization A

- In definition of DRG, for fix $h$, we must consider numbers $p_{ij}^h$ for all $0 \leq i \leq D$ and $0 \leq j \leq D$.
- In previous Corollary, for fix $h$, we consider just $p_{1,h-1}^h$, $p_{1h}^h$ and $p_{1,h+1}^h$, that is for $i$ we pick 1 and for $j$ we pick $h - 1$, $h$ and $h + 1$. 
Hoffman graph with distance partitions.
Cyclic 6-ladder

- Cyclic 6-ladder is not distance-regular.
Cyclic 6-ladder is not distance-regular

Distance partition with respect to vertices 0 and c.
The graph that is pictured on Figure below is called cube. The cube is distance-regular graph.
Basic results from algebraic graph theory
Distance-regular graphs
Characterizations involving the spectrum

The cube - sketch of proof

- The cube drawn on four different ways, and subsets of vertices at given distances from the root.
Characterization B

Theorem (characterization B)

A graph $\Gamma = (V, E)$ with diameter $D$ is distance-regular if and only if, for any integers $0 \leq i, j \leq D$, its distance matrices satisfy

$$A_i A_j = \sum_{k=0}^{D} p_{ij}^k A_k \quad (0 \leq i, j \leq D)$$

for some constants $p_{ij}^k$. 
Theorem (characterization B’)

A graph $\Gamma = (V, E)$ with diameter $D$ is distance-regular if and only if, for some constants $a_h, b_h, c_h \ (0 \leq h \leq D)$, $c_0 = b_D = 0$, its distance matrices satisfy the three-term recurrence

$$A_h A = b_{h-1} A_{h-1} + a_h A_h + c_{h+1} A_{h+1} \quad (0 \leq h \leq D),$$

where, by convention, $b_{-1} = c_{D+1} = 0$. 

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Characterization B’

Definition of distance-regular graph
Definition of local distance-regular graph
Characterization of DRG involving the distance matrices
Examples of distance-regular graphs
Characterization of DRG involving the distance polynomials
A graph $\Gamma = (V, E)$ with diameter $D$ is distance-regular if and only if $\{I, A, \ldots, A_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$. 
Characterization C’

**Theorem (characterization C’)***

Let $\Gamma$ be a graph of diameter $D$ and let $A_i$ be the distance-$i$ matrix of $\Gamma$. Then $\Gamma$ is distance-regular if and only if $A$ acts by right (or left) multiplication as a linear operator on the vector space $\text{span}\{I, A_1, A_2, ..., A_D\}$.
An algebraic approach to DRG

- Γ is distance-regular

\[ \mathcal{A} = \mathcal{D} \]

- Adjacency algebra "·", \[ \mathcal{A} = \text{span}\{A^0, A^1, \ldots, A^d\} \]

- Distance algebra "◦", \[ \mathcal{D} = \text{span}\{A_0, A_1, \ldots, A_D\} \]
Hamming graphs

Definition (Hamming graph)

The Hamming graph $H(n, q)$ is the graph whose vertices are words (sequences or $n$-tuples) of length $n$ from an alphabet of size $q \geq 2$. Two vertices are considered adjacent if the words (or $n$-tuples) differ in exactly one term. We observe that $|V(H(n, q))| = q^n$.

Lemma

The Hamming graph $H(n, q)$ is distance-regular (with $a_i = i(q - 2)$ $(0 \leq i \leq n)$, $b_i = (n - i)(q - 1)$ $(0 \leq i \leq n - 1)$ and $c_i = i$ $(1 \leq i \leq n)$).
Hamming graphs $H(3, 2)$.
Hamming graphs $H(1, 4)$.
Hamming graphs \(H(2, 3)\)

- Hamming graph \(H(2, 3)\).
Hamming graphs $H(4,2)$.

- Hamming graph $H(4,2)$. 
**Johnson graph**

**Definition (Johnson graph $J(n, r)$)**

The Johnson graph $J(n, r)$, is the graph whose vertices are the $r$-element subsets of a $n$-element set $S$. Two vertices are adjacent if the size of their intersection is exactly $r - 1$. To put it on another way, vertices are adjacent if they differ in only one term. We observe that $|V(J(n, r))| = \binom{n}{r}$.

**Lemma**

*Johnson graph $J(n, r)$ is distance-regular with intersection numbers $a_i = (r - i)i + i(n - r - i)$, $b_i = (r - i)(n - r - i)$, $c_i = i^2$.***
Johnson graph $J(4, 2)$

- Johnson graph $J(4, 2)$, drawn in two different ways (this graph is also known as octahedron).
Johnson graph $J(3, 2)$.

- Johnson graph $J(3, 2)$. 

$\hat{ }$
Johnson graph $J(5, 3)$.
Petersen graph

GPG(5,2)

• Petersen graph $GPG(5, 2)$, drawn in two ways.
Graph $\Gamma$ is called a **distance-polynomial** graph if and only if its distance matrix $A_i$ is a polynomial in $A$ for each $i = 0, 1, \ldots, D$, where $D$ is the diameter of $\Gamma$. Polynomials $\{p_k\}_{0 \leq k \leq D}$ in $A$, such that

$$A_k = p_k(A) \ (0 \leq k \leq D),$$

are called the **distance polynomials** (of course, $p_0 = 1$ and $p_1 = x$).
Theorem (characterization D)

A graph $\Gamma = (V, E)$ with diameter $D$ is distance-regular if and only if, for any integer $h$, $0 \leq h \leq D$, the distance-$h$ matrix $A_h$ is a polynomial of degree $h$ in $A$; that is:

$$A_h = p_h(A) \quad (0 \leq h \leq D).$$
Illustration of classes for distance-regular and distance-polynomial graphs.
Theorem (characterization E)

A graph $\Gamma = (V, E)$ is distance-regular if and only if, for each non-negative integer $\ell$, the number $a^\ell_{uv}$ of walks of length $\ell$ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$. 
Characterization E’

**Theorem (characterization E’)**

A regular graph $\Gamma = (V, E)$ with diameter $D$ is distance-regular if and only if there are constants $a^h_h$ and $a^{h+1}_{h}$ such that, for any two vertices $u, v \in V$ at distance $h$, we have $a^h_{uv} = a^h_h \left( a^h_{uv} - \text{number of walks of length } h \right)$ and $a^{h+1}_{uv} = a^{h+1}_h$ for any $0 \leq h \leq D - 1$, and $a^D_{uv} = a^D_D$ for $h = D$. 
Difference between Characterization E and E

- $h = \partial(u, v)$

- In characterization E we consider the number $a_{uv}^\ell$ of walks of length $\ell$ between two vertices $u, v \in V$, where $\ell = 1, 2, 3, ..., h, h + 1, ...$

- In characterization E′ we consider the number $a_{uv}^\ell$ of walks of length $\ell$ between two vertices $u, v \in V$, where $\ell = h, h + 1$. 
Principal idempotents

**Definition (principal idempotents)**

Let $\Gamma = (V, E)$ denote simple graph with adjacency matrix $A$, and let $\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_d$ be distinct eigenvalues. For each eigenvalue $\lambda_i$, $0 \leq i \leq d$, let $U_i$ be the matrix whose columns form an orthonormal basis of its eigenspace $E_i := \ker(A - \lambda_i I)$. The principal idempotents of $A$ are matrices $E_i := U_i U_i^T$.

**Theorem**

Principal idempotents of $\Gamma$ represents the orthogonal projectors onto $E_i$. 
Principal idempotents of $\Gamma$ represents the orthogonal projectors onto $\mathcal{E}_i$. 

![Diagram showing principal idempotents and projectors](image-url)
Some easy results

**Proposition**

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $A$ and with $d + 1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$. Principal idempotents $E_0, E_1, \ldots, E_d$ satisfy the following properties:

(i) $E_i E_j = \delta_{ij} E_i = \begin{cases} E_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

(ii) $A E_i = \lambda_i E_i$, where $\lambda_i \in \sigma(A)$;

(iii) $p(A) = \sum_{i=0}^{d} p(\lambda_i) E_i$, $\forall p \in \mathbb{R}[x]$, where $\lambda_i \in \sigma(A)$;

(iv) $E_0 + E_1 + \ldots + E_d = \sum_{i=0}^{d} E_i = I$;

(v) $\sum_{i=0}^{d} \lambda_i E_i = A$, where $\lambda_i \in \sigma(A)$. 
Some easy results

**Proposition**

Set \( \{E_0, E_1, \ldots, E_d\} \) is an orthogonal basis of \( A(\Gamma) \).
An algebraic approach to DRG

- $\Gamma$ regular

- Adjacency algebra "·", $A = \text{span}\{A^0, A^1, \ldots, A^d\} = \text{span}\{E_0, E_1, \ldots, E_d\}$

- Distance algebra "○", $D = \text{span}\{A_0, A_1, \ldots, A_D\}$

- $\Gamma$ is not distance-regular
Characterization D’

**Theorem (characterization D’)**

A graph \( \Gamma = (V, E) \) with diameter \( D \) and \( d + 1 \) distinct eigenvalues is distance-regular if and only if \( \Gamma \) is regular, has spectrally maximum diameter \( (D = d) \) and the matrix \( A_D \) is polynomial in \( A \).
Characterization F

Theorem (characterization F)

Γ distance-regular ⇐⇒ $A_iE_j = p_{ji}E_j$, $i, j = 0, 1, \ldots, d (= D)$,

$\iff A_i = \sum_{j=0}^{d} p_{ji}E_j$, $i = 0, 1, \ldots, d (= D)$,

$\iff A_i = \sum_{j=0}^{d} p_i(\lambda_j)E_j$, $i = 0, 1, \ldots, d (= D)$,

$\iff A_i \in \mathcal{A}$, $i = 0, 1, \ldots, d (= D)$. 
Characterization G

Theorem (characterization G)

A graph $\Gamma$ with diameter $D$ and $d + 1$ distinct eigenvalues is a distance-regular graph if and only if for every $0 \leq i \leq d$ and for every pair of vertices $u, v$ of $\Gamma$, the $(u, v)$-entry of $E_i$ depends only on the distance between $u$ and $v$. 
Characterization H

Theorem (characterization H)

\[ \Gamma \text{ distance-regular} \iff E_j \circ A_i = q_{ij} A_i, \quad i, j = 0, 1, \ldots, d (= D), \]

\[ \iff E_j = \sum_{i=0}^{D} q_{ij} A_i, \quad j = 0, 1, \ldots, d (= D), \]

\[ \iff E_j = \frac{1}{n} \sum_{i=0}^{d} q_i(\lambda_j) A_i, \quad j = 0, 1, \ldots, d (= D), \]

\[ \iff E_j \in \mathcal{D}, \quad j = 0, 1, \ldots, d (= D). \]
Characterization I

Theorem (characterization I)

\[ \Gamma \text{ distance-regular} \iff A^j \circ A_i = a_i^{(j)} A_i, \quad i, j = 0, 1, \ldots, d(= D), \]

\[ \iff A^j = \sum_{i=0}^{d} a_i^{(j)} A_i, \quad i, j = 0, 1, \ldots, d(= D), \]

\[ \iff A^j = \sum_{i=0}^{d} \sum_{\ell=0}^{d} q_{i\ell} \lambda_{\ell}^{j} A_i, \quad j = 0, 1, \ldots, d(= D), \]

\[ \iff A^j \in D, \quad j = 0, 1, \ldots, d. \]
An algebraic approach to DRG

- The following statements are equivalent:
  1. $\Gamma$ is distance-regular,
  2. $\mathcal{D}$ is an algebra with the ordinary product,
  3. $\mathcal{A}$ is an algebra with the Hadamard product,
  4. $\mathcal{A} = \mathcal{D}$.

\[
\begin{align*}
\mathcal{A} &\equiv \mathcal{D} \\
A^0 &= A_0 = I \\
A^1 &= A_1 = A \\
J &= \Sigma A_i = H(A)
\end{align*}
\]
The *spectrum* of a graph $\Gamma$ is the set of numbers which are eigenvalues of $A(\Gamma)$, together with their multiplicities as eigenvalues of $A(\Gamma)$. If the distinct eigenvalues of $A(\Gamma)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$ and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\text{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_{s-1}^{m(\lambda_{s-1})}\}.$$
Predistance polynomials

Definition (predistance polynomials)

Let $\Gamma = (V, E)$ be a simple connected graph with $|V| = n$ (number of vertices is $n$). The predistance polynomials $p_0, p_1, \ldots, p_d$, $\deg p_i = i$, associated with a given graph $\Gamma$ with spectrum $\text{spec}(\Gamma) = \text{spec}(A) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, are orthogonal polynomials with respect to the scalar product

$$\langle p, q \rangle = \frac{1}{n} \text{trace}(p(A)q(A)) = \frac{1}{n} \sum_{k=0}^{d} m_k p(\lambda_k)q(\lambda_k)$$

on the space of all polynomials with degree at most $d$, normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$. 

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Basic results from algebraic graph theory
Distance-regular graphs
Characterizations involving the spectrum

Characterization of DRG involving the principal idempotent matrix
Characterizations involving the spectrum
Characterization J

**Theorem (characterization J)**

A regular graph $\Gamma$ with $n$ vertices and predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ is distance-regular if and only if

$$q_k(\lambda_0) = \frac{n}{\sum_{u \in V} \frac{1}{s_k(u)}} \quad (0 \leq k \leq d),$$

where $q_k = p_0 + \ldots + p_k$, $s_k(u) = |N_k(u)| = |\Gamma_0(u)| + |\Gamma_1(u)| + \ldots + |\Gamma_k(u)|$. 
Characterization K

**Theorem (characterization K)**

A graph $\Gamma = (V, E)$ with predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ is distance-regular if and only if the number of vertices at distance $k$ from every vertex $u \in V$ is

$$p_k(\lambda_0) = |\Gamma_k(u)| \quad (0 \leq k \leq d).$$
Theorem (characterization J’)

A regular graph $\Gamma$ with $n$ vertices and spectrum $\text{spec}(\Gamma) = \{\lambda_0^m(\lambda_0), \lambda_1^m(\lambda_1), \ldots, \lambda_d^m(\lambda_d)\}$ is distance-regular if and only if

$$\frac{\sum_{u \in V} n/(n - k_d(u))}{\sum_{u \in V} k_d(u)/(n - k_d(u))} = \sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}.$$ 

where $\pi_h = \prod_{i=0}^{d} (\lambda_h - \lambda_i)$ and $k_d(u) = |\Gamma_d(u)|$. 
Part of references I


Part of references II

- [38] Š. Miklavič: Part of Lectures from "PhD Course Algebraic Combinatorics, Computability and Complexity" in TEMPUS project SEE Doctoral Studies in Mathematical Sciences, 2011