Algebraic characterizations of distance-regular graphs

(Master of Science Thesis)

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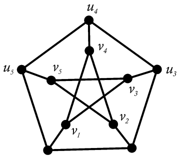

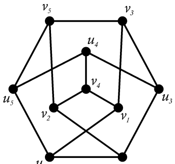

Outline I

Outline II

3. Characterizations involving the spectrum
   - Characterization of DRG involving the principal idempotent matrices
   - Characterizations involving the spectrum

Perron-Frobenius theorem

- Background of problem: Assume that someone give us some matrix A. What we can say about maximum eigenvalue of A, and appropriate eigenvector for that eigenvalue?

Theorem (Perron-Frobenius)

Let M be a nonnegative irreducible symmetric matrix. Then the largest eigenvalue λ_0 has algebraic multiplicity 1 and has an eigenvector whose entries are all positive. For all other eigenvalues we have $|\lambda_i| \leq \lambda_0$.

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Perron-Frobenius theorem

- Since Γ is connected, $\boldsymbol{A}$ is an irreducible nonnegative matrix. Then, by the Perron-Frobenius theorem, the maximum eigenvalue λ_0 is simple, positive (in fact, it coincides with the spectral radius of $\boldsymbol{A}$), and has a positive eigenvector $\boldsymbol{v}$, say, which is useful to normalize in such a way that $\min_{u\in V} \boldsymbol{v}_u = 1$. Moreover, Γ is regular if and only if $\boldsymbol{v} = \boldsymbol{j}$, the all-1 vector (then $\lambda_0 = \delta$, the degree of Γ).

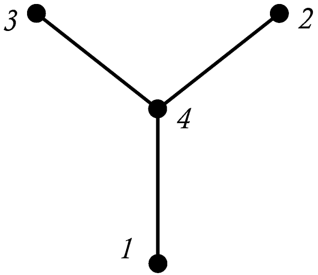
Perron-Frobenius theorem - example

- *a*) Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Characteristic polynomial of A is

$$\mathrm{char}(\lambda) = \lambda^2(\lambda - \sqrt{3})(\lambda + \sqrt{3}).$$

It follow that, maximum eigenvalue $\lambda_0 = \sqrt{3}$ is simple, positive and coincides with spectral radius of A. Eigenvector for eigenvalue λ_0 is $\boldsymbol{v} = (1, 1, 1, \sqrt{3})^\top$, so it is positive.

Perron-Frobenius theorem - example



Γ_1

$$\mathbf{A}= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Simple graph Γ_1 and its adjacency matrix.

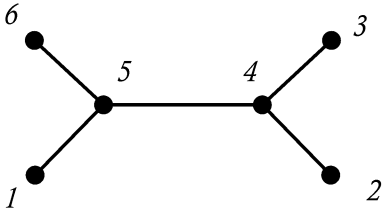
Perron-Frobenius theorem - example

- *b*) Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Characteristic polynomial of A is

$$\mathrm{char}(\lambda) = \lambda^2(\lambda - 1)(\lambda - 2)(\lambda + 1)(\lambda + 2).$$

It follows that, maximum eigenvalue $\lambda_0 = 2$ is simple, positive and coincides with spectral radius of A. Eigenvector for eigenvalue λ_0 is $\boldsymbol{v} = (1, 1, 1, 2, 2, 1)^\top$, so it is positive.

Perron-Frobenius theorem - example



Γ_2

$$\mathbf{A}=\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Simple graph Γ_2 and its adjacency matrix.

The number of walks of a given length

- Background of problem: Assume that someone give us some graph Γ. What is easiest way to count the number of walks of given length $k \geq 0$ between vertices u and v?
- The number of walks of length $k \geq 0$ between vertices u and v is $a_{uv}^{k} := (\boldsymbol{A}^{k})_{uv}$.

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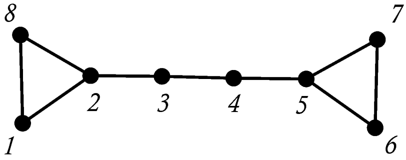
The number of walks of a given length - example

- Consider graph Γ_3.



Γ_3

$$\mathbf{A}= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{bmatrix} 0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \end{array}$$

Simple graph Γ_3 and its adjacency matrix.

The number of walks of a given length - example

- Let's say that we want to find number of walks of length 4 and 5, between vertices 3 and 7. Then first that we need to do is to find adjacency matrix for Γ_3. After that we need to find $(3,7)$-entry (or $(7,3)$-entry) of A^4 and A^5.

The number of walks of a given length - example

- We have $A^4 = \begin{bmatrix} 7 & 6 & 5 & 1 & 1 & 0 & 0 & 6 \\ 6 & 12 & 2 & 5 & 0 & 1 & 1 & 6 \\ 5 & 2 & 7 & 0 & 5 & 1 & 1 & 5 \\ 1 & 5 & 0 & 7 & 2 & 5 & 5 & 1 \\ 1 & 0 & 5 & 2 & 12 & 6 & 6 & 1 \\ 0 & 1 & 1 & 5 & 6 & 7 & 6 & 0 \\ 0 & 1 & 1 & 5 & 6 & 6 & 7 & 0 \\ 6 & 6 & 5 & 1 & 1 & 0 & 0 & 7 \end{bmatrix}$ and

$$A^5 = \begin{bmatrix} 12 & 18 & 7 & 6 & 1 & 1 & 1 & 13 \\ 18 & 14 & 17 & 2 & 7 & 1 & 1 & 18 \\ 7 & 17 & 2 & 12 & 2 & 6 & 6 & 7 \\ 6 & 2 & 12 & 2 & 17 & 7 & 7 & 6 \\ 1 & 7 & 2 & 17 & 14 & 18 & 18 & 1 \\ 1 & 1 & 6 & 7 & 18 & 12 & 13 & 1 \\ 1 & 1 & 6 & 7 & 18 & 13 & 12 & 1 \\ 13 & 18 & 7 & 6 & 1 & 1 & 1 & 12 \end{bmatrix}.$$

The total number of closed walks of a given length

- Background of problem: Assume that someone give us some simple graph Γ. How can we compute the total number of rooted closed walks of given length?
- If $\Gamma = (V, E)$ has spectrum

$$\mathrm{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$$

then the total number of (rooted) closed walks of length $l \geq 0$ is $\mathrm{tr}(\boldsymbol{A}^l) = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^l$.

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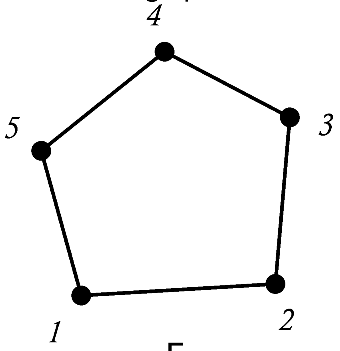
The total number of closed walks - example

- Consider graph Γ_4



Γ_4

A=

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	0	0
3	0	1	0	1	0
4	0	0	1	0	1
5	1	0	0	1	0

Simple graph Γ_4 and its adjacency matrix.

The total number of closed walks - example

- This graph have three eigenvalues $\lambda_0 = 2$, $\lambda_1 = \frac{\sqrt{5}}{2} - \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} - \frac{1}{2}$, and spectrum:

$$\mathrm{spec}(\Gamma_4) = \{2^1, \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2, \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^2\}$$

- Total number of rooted closed walks of length 3, 4 and 5 is

$$\mathrm{tr} A^3 = 1 \cdot 2^3 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^3 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^3 = 0,$$

$$\mathrm{tr} A^4 = 1 \cdot 2^4 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^4 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^4 = 30$$

and

$$\mathrm{tr} A^5 = 1 \cdot 2^5 + 2 \cdot \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^5 + 2 \cdot \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^5 = 10.$$

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The adjacency (or Bose-Mesner) algebra $\mathcal{A}(\Gamma)$

- Background of problem: Assume that someone give us some simple graph Γ. Can we form (define) some algebra that is connected with this graph? Can we say something about connection between the number of distinct eigenvalues and the diameter of graph?

Definition (adjacency algebra)

The *adjacency* (or *Bose-Mesner*) *algebra* of a graph Γ is algebra of matrices which are polynomials in $\boldsymbol{A}$ under the usual matrix operations. We shall denote this algebra by $\mathcal{A} = \mathcal{A}(\Gamma)$. Therefore

$$\mathcal{A}(\Gamma) = \{p(\boldsymbol{A}) : p \in \mathbb{R}[x]\}$$

(elements in $\mathcal{A}$ are matrices).

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The adjacency (or Bose-Mesner) algebra $\mathcal{A}(\Gamma)$

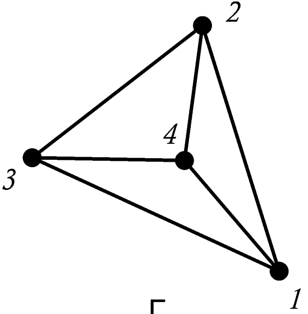
- If Γ has $d+1$ distinct eigenvalues, then $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$ is a basis of the adjacency or Bose-Mesner algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in $\boldsymbol{A}$. Moreover, if Γ has diameter D,

$$dim \mathcal{A}(\Gamma) = d + 1 \geq D + 1,$$

because $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^D\}$ is a linearly independent set of $\mathcal{A}(\Gamma)$. Hence, the diameter is always less than the number of distinct eigenvalues: $D \leq d$.

The diameter is always less than the number of distinct eigenvalues - example

- *a*) Consider graph Γ_5.



Γ_5

$$\mathbf{A}= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

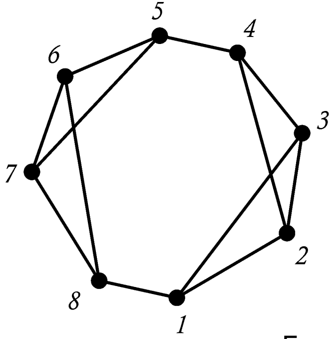
Simple graph Γ_5 and its adjacency matrix.

The diameter is always less than the number of distinct eigenvalues - example

- Eigenvalues of Γ_5 are $\lambda_0 = 3$ and $\lambda_1 = -1$, so $d + 1 = 2$. Diameter is $D = 1$. Therefore $D = d$.

The diameter is always less than the number of distinct eigenvalues - example

- *b*) Consider graph Γ_6.



Γ_6

$$\mathbf{A}= \begin{array}{c|cccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 7 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 8 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Simple graph Γ_6 and its adjacency matrix.

The diameter is always less than the number of distinct eigenvalues - example

- Eigenvalues of Γ_6 are $\lambda_0 = 3$, $\lambda_1 = \sqrt{5}$, $\lambda_2 = 1$, $\lambda_3 = -1$ and $\lambda_4 = -\sqrt{5}$, so $d + 1 = 5$. Diameter of Γ_6 is $D = 3$. Therefore $D < d$.

Hoffman polynomial

- Background of problem: Assume that someone give us some simple graph Γ. We want to know is there exists a polynomial $H(x)$ such that
$$\boldsymbol{J} = H(\boldsymbol{A})$$
where $\boldsymbol{J}$ is square matrix of order n, which every entry is one, and $\boldsymbol{A}$ is adjacency matrix of Γ?

Hoffman polynomial

- A graph $\Gamma = (V, E)$ with eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$ is a regular graph if and only if there exists a polynomial $H \in \mathbb{R}_d[x]$ such that $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix. This polynomial is unique and it is called the Hoffman polynomial. It has zeros at the eigenvalues λ_i, $i \neq 0$, and $H(\lambda_0) = n := |V|$. Thus,

$$H = \frac{n}{\pi_0} \prod_{i=1}^{d} (x - \lambda_i),$$

where $\pi_0 := \prod_{i=0}^{d} (\lambda_0 - \lambda_i)$.

Some notation before definition of DRG

- Let $\Gamma = (V, E)$ denote a simple connected graph with vertex set V, edge set E and diameter D. Let ∂ denotes the path-length distance function for Γ.
- $\Gamma_i(x) := \{y \in V : \partial(x, y) = i\}$
- $|\Gamma_i(x) \cap \Gamma_j(y)|$ denote the number of elements of the set $\Gamma_i(x) \cap \Gamma_j(y)$
- The *eccentricity* of a vertex u is $\mathrm{ecc}(u) := \max_{v \in V} \partial(u, v)$ and the *diameter* of the graph is $D := \max_{u \in V} \mathrm{ecc}(u)$.

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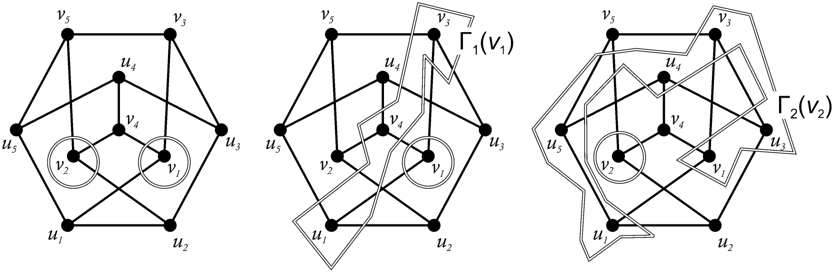
Some notation before definition of DRG - example



Petersen graph. We have $\partial(v_1, v_2) = 2$, $\Gamma_1(v_1) = \{u_1, v_3, v_4\}$, $\Gamma_2(v_2) = \{u_1, u_3, u_4, u_5, v_1, v_3\}$, $|\Gamma_1(v_1) \cap \Gamma_2(v_2)| = |\{u_1, v_3\}| = 2$.

Distance-regular graphs

Definition (distance-regular graphs)

A simple connected graph $\Gamma = (V, E)$ with diameter D is called *distance-regular* whenever there exist numbers p^h_{ij} $(0 \leq h, i, j \leq D)$ such that for any $x, y \in V$ with $\partial(x, y) = h$ we have

$$|\Gamma_i(x) \cap \Gamma_j(y)| = p^h_{ij}.$$

Local distance-regular graph

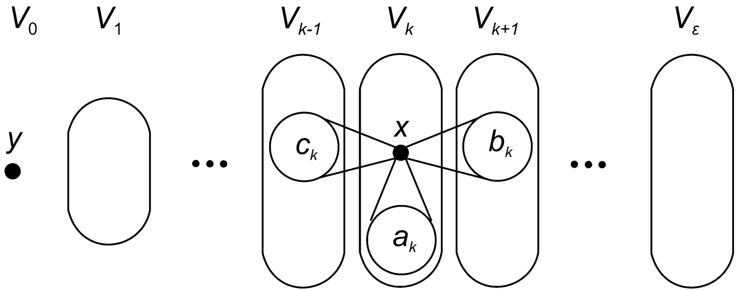
Definition (local distance-regular graph)

Let $y \in V$ be a vertex with eccentricity $\mathrm{ecc}(y) = \varepsilon$ of a regular graph Γ. Let $V_k := \Gamma_k(y)$ and consider the numbers

$$c_k(x) := |\Gamma_1(x) \cap V_{k-1}|,$$
$$a_k(x) := |\Gamma_1(x) \cap V_k|,$$
$$b_k(x) := |\Gamma_1(x) \cap V_{k+1}|,$$

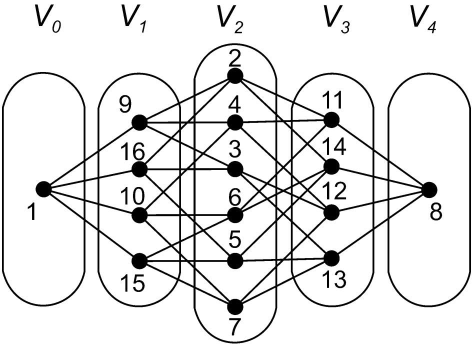
defined for any $x \in V_k$ and $0 \leq k \leq \varepsilon$ (where, by convention, $c_0(x) = 0$ and $b_\varepsilon(x) = 0$ for any $x \in V_\varepsilon$). We say that Γ is *distance-regular around y* whenever $c_k(x)$, $a_k(x)$, $b_k(x)$ do not depend on the considered vertex $x \in V_k$ but only on the value of k.

Local distance-regular graph



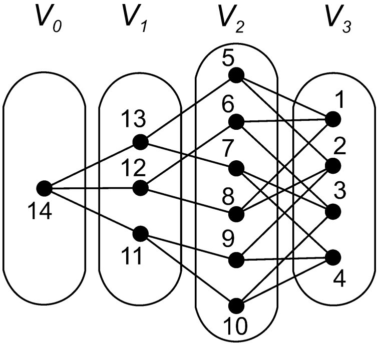
Intersection numbers around y.

Local distance-regular graph - example



Simple connected regular graph that is distance-regular around vertices 1 and 8. This graph is known as Hoffman graph.

Local distance-regular graph - example



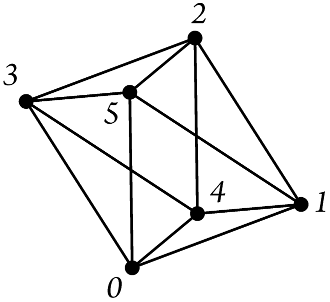
Simple connected graph that is distance-regular around vertex 14.

Distance-i matrix

Definition (distance-i matrix)

Let $\Gamma = (V, E)$ denote a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and let $\mathrm{Mat}_\Gamma(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by the vertices of Γ. For $0 \leq i \leq D$ we define *distance-i matrix* $\boldsymbol{A}_i \in \mathrm{Mat}_\Gamma(\mathbb{R})$ with entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\partial(u, v) = i$ and $(\boldsymbol{A}_i)_{uv} = 0$ otherwise. Note that $\boldsymbol{A}_0$ is the identity matrix and $\boldsymbol{A}_1 = \boldsymbol{A}$ is the usual adjacency matrix of Γ.

Distance-i matrix - example



Distance-i matrices for octahedron are

$$A_0 = I, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Distance $\circ$-algebra

Lemma

Let $\boldsymbol{A}_i \in Mat_\Gamma(\mathbb{R})$ denote a distance-i matrices. Vector space $\mathcal{D}$ defined by

$$\mathcal{D} = span\{\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$$

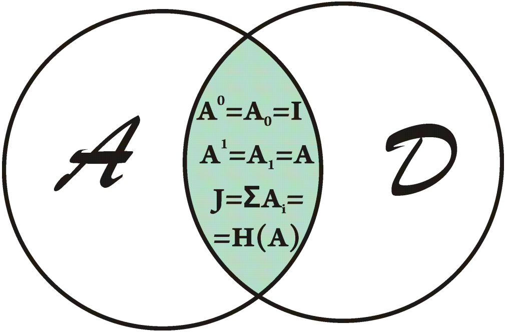
forms an algebra with the entrywise or Hadamard product of matrices, defined by $(X \circ Y)_{uv} = (X)_{uv}(Y)_{uv}$.

Definition (distance $\circ$-algebra)

Vector space $\mathcal{D}$ from Lemma above will be called the *distance $\circ$-algebra* of Γ.

An algebraic approach to DRG

- Γ regular



- Adjacency algebra "·", $\mathcal{A} = \text{span}\{\boldsymbol{A}^0, \boldsymbol{A}^1, ..., \boldsymbol{A}^d\}$
- Distance algebra "∘", $\mathcal{D} = \text{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$
- Γ is not distance-regular

Characterization A

Theorem (characterization A)

Let $\Gamma = (V, E)$ denote a graph with diameter D and let the set $\Gamma_h(u)$ represents the set of vertices at distance h from vertex u. Γ is distance-regular if and only if is distance-regular around each of its vertices and with the same intersection array (with another words if and only if for any two vertices $u, v \in V$ at distance $\partial(u, v) = h$, $0 \leq h \leq D$, the numbers

$$c_h(u,v) := |\Gamma_{h-1}(u) \cap \Gamma(v)|, \quad a_h(u,v) := |\Gamma_h(u) \cap \Gamma(v)|,$$

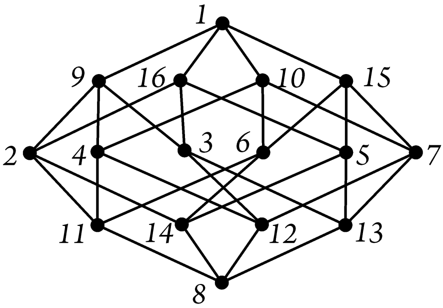
$$b_h(u,v) := |\Gamma_{h+1}(u) \cap \Gamma(v)|,$$

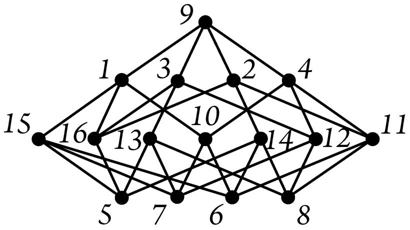
do not depend on the chosen vertices u and v, but only on their distance h; in which case they are denoted by c_h, a_h, and b_h, respectively).

What is different between Definition of DRG and Characterization A

- In definition of DRG, for fix h, we must consider numbers p_{ij}^h for all $0 \le i \le D$ and $0 \le j \le D$.
- In previous Corollary, for fix h, we consider just $p_{1,h-1}^h$, p_{1h}^h and $p_{1,h+1}^h$, that is for i we pick 1 and for j we pick $h-1$, h and $h+1$.

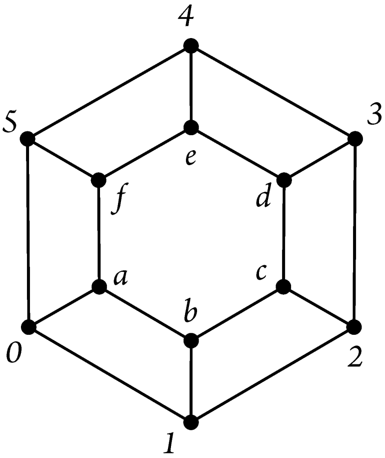
Hoffman graph





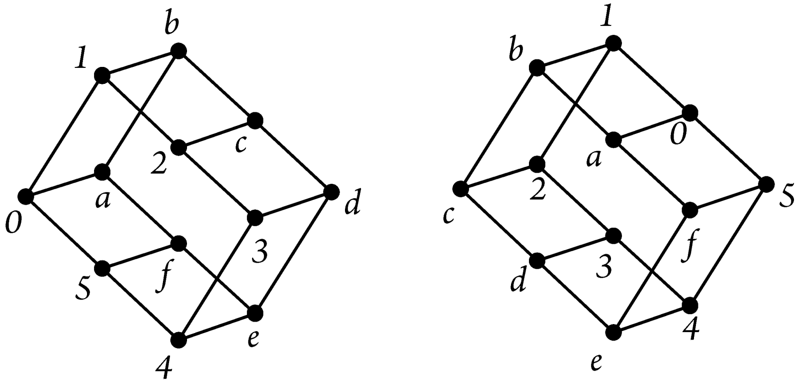
- Hoffman graph with distance partitions.

Cyclic 6-ladder



- Cyclic 6-ladder is not distance-regular.

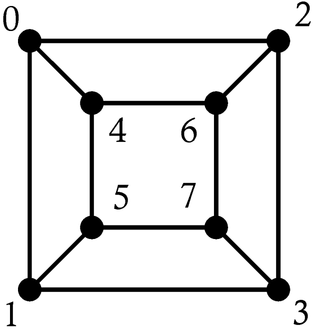
Cyclic 6-ladder is not distance-regular



- Distance partition with respect to vertices 0 and c.

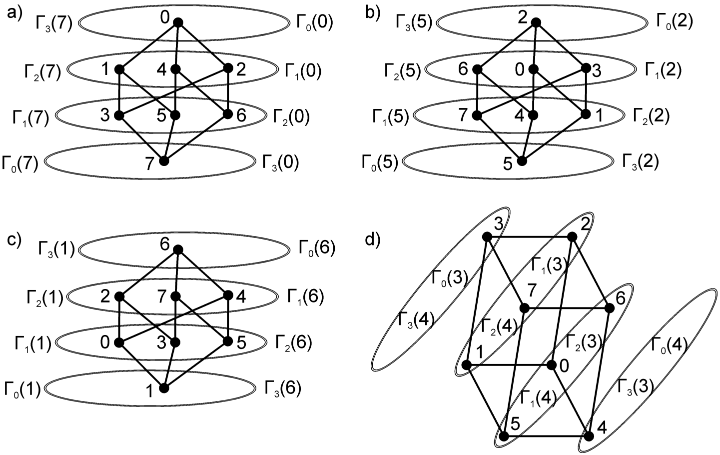
The cube

- The graph that is pictured on Figure below is called cube. The cube is distance-regular graph.



The cube

The cube - sketch of proof



- The cube drawn on four different way, and subsets of vertices at given distances from the root.

Characterization B

Theorem (characterization B)

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integers $0 \le i, j \le D$, its distance matrices satisfy

$$\boldsymbol{A}_i \boldsymbol{A}_j = \sum_{k=0}^{D} p_{ij}^k \boldsymbol{A}_k \quad (0 \le i, j \le D)$$

for some constants p_{ij}^k.

Characterization B'

Theorem (characterization B')

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for some constants a_h, b_h, c_h $(0 \leq h \leq D)$, $c_0 = b_D = 0$, its distance matrices satisfy the three-term recurrence

$$\boldsymbol{A}_h\boldsymbol{A} = b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1} \quad (0 \leq h \leq D),$$

where, by convention, $b_{-1} = c_{D+1} = 0$.

Characterization C

Theorem (characterization C)

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if $\{I, \boldsymbol{A}, \dots, \boldsymbol{A}_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$.

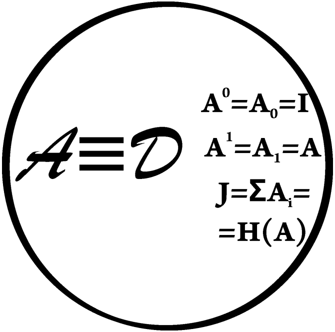
Characterization C'

Theorem (characterization C')

Let Γ be a graph of diameter D and let $\boldsymbol{A}_i$ be the distance-i matrix of Γ. Then Γ is distance-regular if and only if $\boldsymbol{A}$ acts by right (or left) multiplication as a linear operator on the vector space $\mathrm{span}\{\boldsymbol{I}, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$.

An algebraic approach to DRG

- Γ is distance-regular



- Adjacency algebra "·", $\mathcal{A} = \text{span}\{\boldsymbol{A}^0, \boldsymbol{A}^1, \ldots, \boldsymbol{A}^d\}$
- Distance algebra "∘", $\mathcal{D} = \text{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_D\}$

Hamming graphs

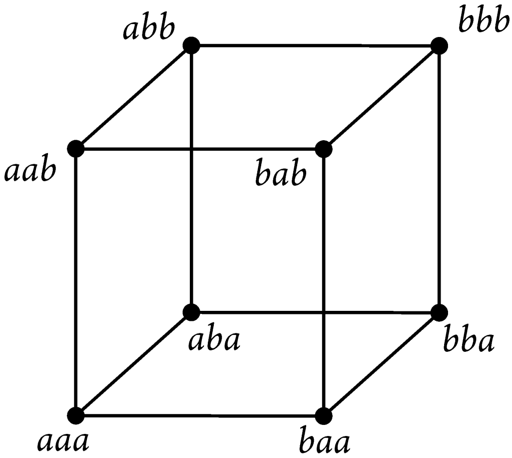
Definition (Hamming graph)

The Hamming graph $H(n,q)$ is the graph whose vertices are words (sequences or n-tuples) of length n from an alphabet of size $q \geq 2$. Two vertices are considered adjacent if the words (or n-tuples) differ in exactly one term. We observe that $|V(H(n,q))| = q^n$.

Lemma

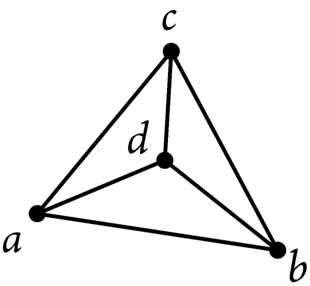
The Hamming graph $H(n,q)$ is distance-regular (with $a_i = i(q-2)$ $(0 \leq i \leq n)$, $b_i = (n-i)(q-1)$ $(0 \leq i \leq n-1)$ and $c_i = i$ $(1 \leq i \leq n)$).

Hamming graphs $H(3,2)$



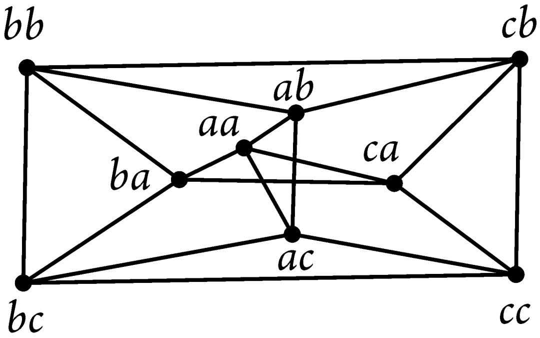
- Hamming graph $H(3,2)$.

Hamming graphs $H(1,4)$



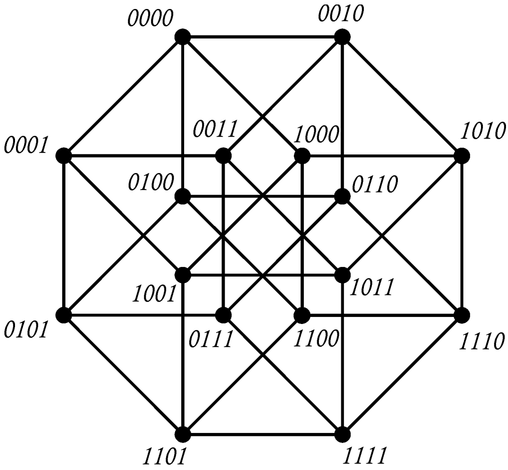
- Hamming graph $H(1,4)$.

Hamming graphs $H(2,3)$



- Hamming graph $H(2,3)$.

Hamming graphs $H(4,2)$



- Hamming graph $H(4,2)$.

Johnson graph

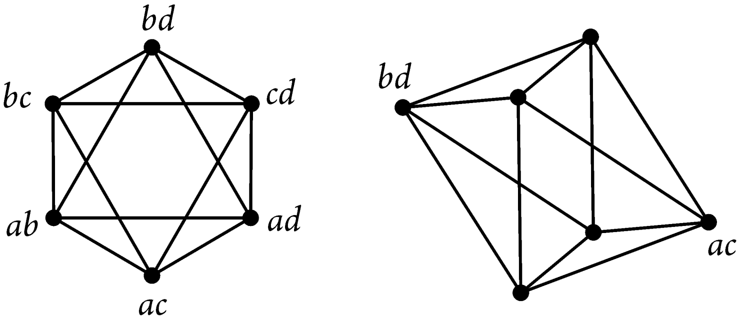
Definition (Johnson graph $J(n, r)$)

The Johnson graph $J(n, r)$, is the graph whose vertices are the r-element subsets of a n-element set S. Two vertices are adjacent if the size of their intersection is exactly $r - 1$. To put it on another way, vertices are adjacent if they differ in only one term. We observe that $|V(J(n, r))| = \binom{n}{r}$.

Lemma

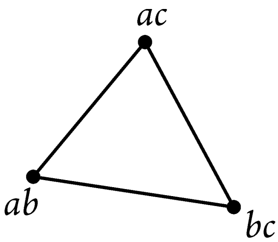
Johnson graph $J(n, r)$ is distance-regular with intersection numbers $a_i = (r - i)i + i(n - r - i)$, $b_i = (r - i)(n - r - i)$, $c_i = i^2$.

Johnson graph $J(4,2)$



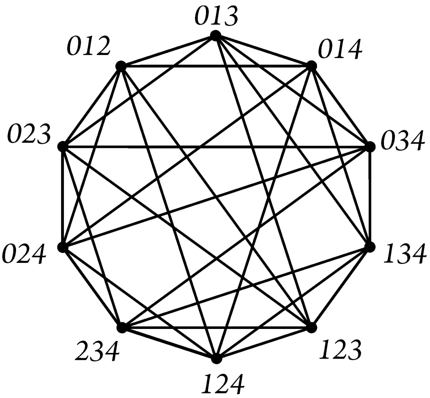
- Johnson graph $J(4,2)$, drawn in two different ways (this graph is also known as octahedron).

Johnson graph $J(3, 2)$



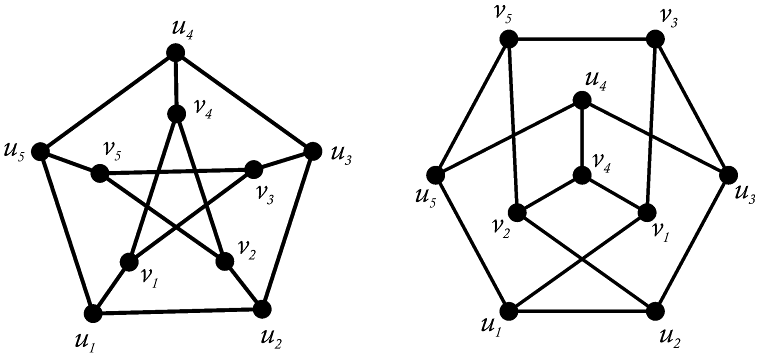
- Johnson graph $J(3, 2)$.

Johnson graph $J(5,3)$



- Johnson graph $J(5,3)$.

Petersen graph



GPG(5,2)

- Petersen graph $GPG(5, 2)$, drawn in two ways.

Distance-polynomial graphs, distance polynomials

Definition (distance-polynomial graphs, distance polynomials)

Graph Γ is called a *distance-polynomial* graph if and only if its distance matrix $\boldsymbol{A}_i$ is a polynomial in $\boldsymbol{A}$ for each $i = 0, 1, ..., D$, where D is the diameter of Γ. Polynomials $\{p_k\}_{0 \le k \le D}$ in $\boldsymbol{A}$, such that

$$\boldsymbol{A}_k = p_k(\boldsymbol{A}) \;\; (0 \le k \le D),$$

are called the *distance polynomials* (of course, $p_0 = 1$ and $p_1 = x$).

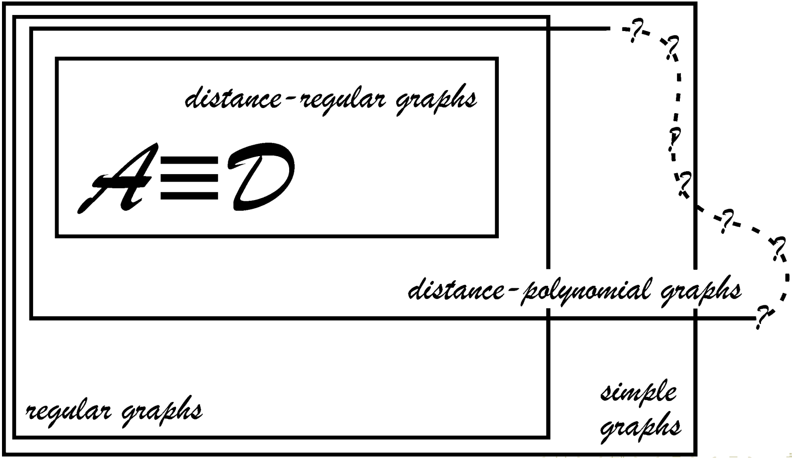
Characterization D

Theorem (characterization D)

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integer h, $0 \le h \le D$, the distance-h matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$; that is:

$$\boldsymbol{A}_h = p_h(\boldsymbol{A}) \quad (0 \le h \le D).$$

Illustration of classes for distance-regular and distance-polynomial graphs.



Characterization E

Theorem (characterization E)

A graph $\Gamma = (V, E)$ is distance-regular if and only if, for each non-negative integer ℓ, the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$.

Characterization E'

Theorem (characterization E')

A regular graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if there are constants a_h^h and a_h^{h+1} such that, for any two vertices $u, v \in V$ at distance h, we have $a_{uv}^h = a_h^h$ (a_{uv}^h - number of walks of length h) and $a_{uv}^{h+1} = a_h^{h+1}$ for any $0 \leq h \leq D-1$, and $a_{uv}^D = a_D^D$ for $h = D$.

Difference between Characterization E and E

- $h = \partial(u, v)$
- In characterization E we consider the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$, where $\ell = 1, 2, 3, \ldots, h, h+1, \ldots$
- In characterization E' we consider the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$, where $\ell = h, h+1$.

Principal idempotents

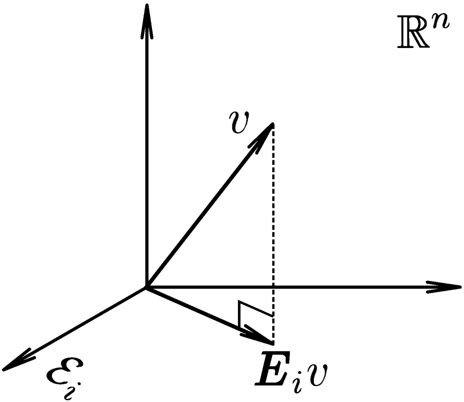
Definition (principal idempotents)

Let $\Gamma = (V, E)$ denote simple graph with adjacency matrix $\boldsymbol{A}$, and let $\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_d$ be distinct eigenvalues. For each eigenvalue λ_i, $0 \leq i \leq d$, let U_i be the matrix whose columns form an orthonormal basis of its eigenspace $\mathcal{E}_i := ker(\boldsymbol{A} - \lambda_i I)$. The *principal idempotents* of $\boldsymbol{A}$ are matrices $\boldsymbol{E}_i := U_i U_i^\top$.

Theorem

Principal idempotents of Γ *represents the orthogonal projectors onto* $\mathcal{E}_i$.

Principal idempotents of Γ represents the orthogonal projectors onto $\mathcal{E}_i$.



Some easy results

Proposition

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$ and with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_d$. Principal idempotents $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_d$ satisfy the following properties:

(i) $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i = \begin{cases} \boldsymbol{E}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$;

(ii) $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i \boldsymbol{E}_i$, where $\lambda_i \in \sigma(\boldsymbol{A})$;

(iii) $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, $\forall p \in \mathbb{R}[x]$, where $\lambda_i \in \sigma(\boldsymbol{A})$;

(iv) $\boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d = \sum_{i=0}^{d} \boldsymbol{E}_i = I$;

(v) $\sum_{i=0}^{d} \lambda_i \boldsymbol{E}_i = \boldsymbol{A}$, where $\lambda_i \in \sigma(\boldsymbol{A})$.

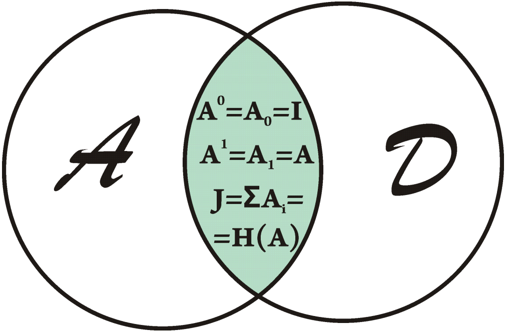
Some easy results

Proposition

Set $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$ *is an orthogonal basis of* $\mathcal{A}(\Gamma)$.

An algebraic approach to DRG

- Γ regular



- Adjacency algebra "·",
  $\mathcal{A} = \mathrm{span}\{\boldsymbol{A}^0, \boldsymbol{A}^1, ..., \boldsymbol{A}^d\} = \mathrm{span}\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$
- Distance algebra "∘", $\mathcal{D} = \mathrm{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$
- Γ is not distance-regular

Characterization D'

Theorem (characterization D')

A graph $\Gamma = (V, E)$ with diameter D and $d + 1$ distinct eigenvalues is distance-regular if and only if Γ is regular, has spectrally maximum diameter ($D = d$) and the matrix $\boldsymbol{A}_D$ is polynomial in $\boldsymbol{A}$.

Characterization F

Theorem (characterization F)

$$\Gamma \text{ distance-regular} \iff \boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j, \quad i,j = 0,1,\dots,d(=D),$$

$$\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j, \quad i = 0,1,\dots,d(=D),$$

$$\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j, \quad i = 0,1,\dots,d(=D),$$

$$\iff \boldsymbol{A}_i \in \mathcal{A}, \quad i = 0,1,\dots,d(=D).$$

Characterization G

Theorem (characterization G)

A graph Γ with diameter D and $d+1$ distinct eigenvalues is a distance-regular graph if and only if for every $0 \leq i \leq d$ and for every pair of vertices u, v of Γ, the (u, v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v.

Characterization H

Theorem (characterization H)

$$\begin{aligned}
\Gamma \text{ distance-regular} \quad &\Longleftrightarrow \quad \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i, \qquad i,j = 0,1,\dots,d(=D),\\
&\Longleftrightarrow \quad \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i, \qquad j = 0,1,\dots,d(=D),\\
&\Longleftrightarrow \quad \boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_i(\lambda_j)\boldsymbol{A}_i, \qquad j = 0,1,\dots,d(=D),\\
&\Longleftrightarrow \quad \boldsymbol{E}_j \in \mathcal{D}, \qquad j = 0,1,\dots,d(=D).
\end{aligned}$$

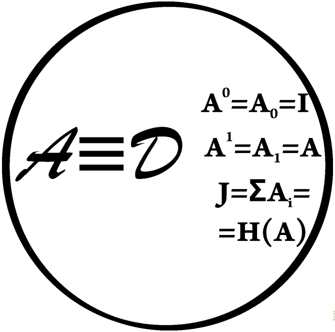
Characterization I

Theorem (characterization I)

$$\begin{aligned}
\Gamma \text{ distance-regular} \quad &\Longleftrightarrow \quad \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i, \qquad i,j = 0,1,\dots,d(=D), \\
&\Longleftrightarrow \quad \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)} \boldsymbol{A}_i, \qquad i,j = 0,1,\dots,d(=D), \\
&\Longleftrightarrow \quad \boldsymbol{A}^j = \sum_{i=0}^{d} \sum_{\ell=0}^{d} q_{i\ell} \lambda_\ell^j \boldsymbol{A}_i, \qquad j = 0,1,\dots,d(=D), \\
&\Longleftrightarrow \quad \boldsymbol{A}^j \in \mathcal{D}, \qquad j = 0,1,\dots,d.
\end{aligned}$$

An algebraic approach to DRG

- The following statements are equivalent:
  (i) Γ is distance-regular,
  (ii) $\mathcal{D}$ is an algebra with the ordinary product,
  (iii) $\mathcal{A}$ is an algebra with the Hadamard product,
  (iv) $\mathcal{A} = \mathcal{D}$.



Spectrum

- The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $\boldsymbol{A}(\Gamma)$, together with their multiplicities as eigenvalues of $\boldsymbol{A}(\Gamma)$. If the distinct eigenvalues of $\boldsymbol{A}(\Gamma)$ are $\lambda_0 > \lambda_1 > ... > \lambda_{s-1}$ and their multiplicities are $m(\lambda_0)$, $m(\lambda_1)$,...,$m(\lambda_{s-1})$, then we shall write

$$\mathrm{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_{s-1}^{m(\lambda_{s-1})}\}.$$

Predistance polynomials

Definition (predistance polynomials)

Let $\Gamma = (V, E)$ be a simple connected graph with $|V| = n$ (number of vertices is n). The *predistance polynomials* p_0, p_1, ..., p_d, $\deg p_i = i$, associated with a given graph Γ with spectrum $\mathrm{spec}(\Gamma) = \mathrm{spec}(\boldsymbol{A}) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, are orthogonal polynomials with respect to the scalar product

$$\langle p, q \rangle = \frac{1}{n}\mathrm{trace}(p(\boldsymbol{A})q(\boldsymbol{A})) = \frac{1}{n}\sum_{k=0}^{d} m_k \, p(\lambda_k)q(\lambda_k)$$

on the space of all polynomials with degree at most d, normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$.

Characterization J

Theorem (characterization J)

A regular graph Γ with n vertices and predistance polynomials $\{p_k\}_{0\leq k\leq d}$ is distance-regular if and only if

$$q_k(\lambda_0) = \frac{n}{\sum_{u\in V} \frac{1}{s_k(u)}} \quad (0 \leq k \leq d),$$

where $q_k = p_0 + \ldots + p_k$, $s_k(u) = |N_k(u)| = |\Gamma_0(u)| + |\Gamma_1(u)| + \ldots + |\Gamma_k(u)|$.

Characterization K

Theorem (characterization K)

A graph $\Gamma = (V, E)$ with predistance polynomials $\{p_k\}_{0 \le k \le d}$ is distance-regular if and only if the number of vertices at distance k from every vertex $u \in V$ is

$$p_k(\lambda_0) = |\Gamma_k(u)| \quad (0 \le k \le d).$$

Characterization J'

Theorem (characterization J')

A regular graph Γ with n vertices and spectrum $\mathrm{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\}$ is distance-regular if and only if

$$\frac{\sum_{u \in V} n/(n - k_d(u))}{\sum_{u \in V} k_d(u)/(n - k_d(u))} = \sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}.$$

where $\pi_h = \prod_{\substack{i=0 \\ i \neq h}}^{d} (\lambda_h - \lambda_i)$ and $k_d(u) = |\Gamma_d(u)|$.

Part of references I

- [4] N. Biggs: *"Algebraic Graph Theory"*, Cambridge tracts in Mathematics, 1974.
- [13] E. R. van Dam: *"The spectral excess theorem for distance-regular graphs: a global (over)view"*, The electronic journal of combinatorics 15 (#R129), 2008
- [19] M.A. Fiol, E. Garriga, and J.L.A. Yebra: *"Locally pseudo-distance-regular graphs"*, J.Combin. Theory Ser. B68 (1996), 179-205.

Part of references II

- [9] M. Cámara, J. Fàbrega, M. A. Fiol, E. Garriga: *"Some Families of Orthogonal Polynomials of a Discrete Variable and their Applications to Graphs and Codes"*, The electronic journal of combinatorics 16 (#R83), 2009
- [23] M.A. Fiol: *"Algebraic characterizations of distance-regular graphs"*, Discrete Mathematics 246(1-3), page 111-129, 2002.
- [24] M.A. Fiol: *"Algebraic characterizations of bipartite distance-regular graphs"*, 3rd International Workshop on Optimal Networks Topologies, IWONT 2010
- [38] Š. Miklavič: Part of Lectures from *"PhD Course Algebraic Combinatorics, Computability and Complexity"* in TEMPUS project SEE Doctoral Studies in Mathematical Sciences, 2011