Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 03.02.2010.

Pismeni ispit iz predmeta **Uvod u linearnu algebru i analitičku geometriju**

1. Kroz središte S duži određene tačkama A(1,3,0) i B(-3,7,2) postaviti pravu p paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$. Prava $q: x=t+2,\ y=t+2,\ z=t+1,\ t\in\mathbf{R}$ je zadana parametarski. Ispitati odnos između pravih p i q. Ukoliko nisu mimoilazne, napisati jednačinu ravni koja ih sadrži.

2. Odrediti vektor visine $\vec{h}_a$ iz vrha A trougla ABC ako je: $\overrightarrow{BC}=\vec{m}+2\vec{n}$, $\overrightarrow{CA}=2\vec{m}-\vec{n}$, $|\vec{m}|=|\vec{n}|=\sqrt{3},\ \angle(\vec{m},\vec{n})=\frac{\pi}{2}$.

3. Riješiti sistem jednačina za razne vrijednosti parametra $a\in\mathbf{R}$:

$$\begin{aligned}2x_1-x_2+3x_3-7x_4&=15\\ 6x_1-3x_2+x_3-4x_4&=7\\ 4x_1-2x_2+14x_3-31x_4&=a\end{aligned}.$$

4. Riješiti matričnu jednačinu: $AX-2B=3X+A$, gdje je $A=\begin{pmatrix}6&1&0\\0&5&2\\0&0&4\end{pmatrix}$, $B=\begin{pmatrix}-1&2&0\\2&3&1\\4&0&3\end{pmatrix}$.

(Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti pravu p paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$.
Prava $g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases}$, $t\in\mathbb{R}$ je zadana parametarski. Ispitati odnos između pravih p i g. Ukoliko nisu mimoilazne, napisati jednačinu ravni koja ih sadrži.

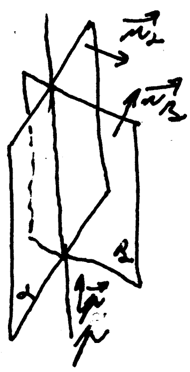
R: Nađimo središte S duži AB

$A(1,3,0)$
$B(-3,7,2)$ $\Rightarrow S(-1,5,1)$

$S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$\alpha: 6x-4y+z=16$
$\beta: \quad y+2z=-1$

Pronađimo koeficijent pravca prave koja je presjek ove dvije ravni



$\vec{n}_\alpha=(6,-4,1)$

$\vec{n}_\beta=(0,1,2)$

$\left.\begin{matrix}\vec{p}\perp\vec{n}_\alpha \\ \vec{p}\perp\vec{n}_\beta\end{matrix}\right\} \Rightarrow \vec{p}\parallel\vec{n}_\alpha\times\vec{n}_\beta$

$\Downarrow$

$\exists k: \vec{p}=k(\vec{n}_\alpha\times\vec{n}_\beta)$

$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2\end{vmatrix}=$

$=-9\vec{i}-12\vec{j}+6\vec{k}$

$\Downarrow$

$\vec{p}=(-3,-4,2)$

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$
jednačina prave kroz jednu tačku

$\frac{x+1}{-3}=\frac{y-5}{-4}=\frac{z-1}{2}$ jednačina tražene prave p

$g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases} \ldots(*)$, $t\in\mathbb{R}$, $g: \begin{cases} x-2=t \\ y-2=t \\ z-1=t \end{cases}$, $t\in\mathbb{R}$ $\Rightarrow g: \frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{1}$

Koeficijent pravca prave g je $\vec{p}_g=(1,1,1)$.
Prave p i g nisu paralelne (nije $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$)

Pokušajmo naći presječnu tačku pravih p i g.

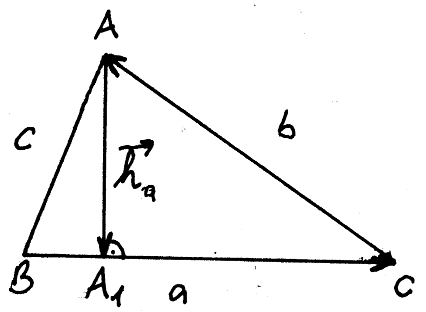
$p: \begin{cases} x=-3s-1 \\ y=-4s+5 \\ z=2s+1 \end{cases}$, $s\in\mathbb{R}$ $\ldots(**)$ $\quad (*)$ i $(**) \Rightarrow$

$-3s-1=t+2$ (1)
$-4s+5=t+2$ (2)
$2s+1=t+1$ (3)

(1)-(2):
$s-6=0$
$s=6 \Rightarrow$
$\Rightarrow t+2=-19$
$t=-\frac{19}{2}$
Kako ovaj t ne zadovoljava (3) sistem nema rješenja.
Prave p i g su mimoilazne.

#) Odrediti vektor visine $\vec{h}_a$ iz vrha A trougla ΔABC ako je $\overrightarrow{BC} = \vec{m} + 2\vec{n}$, $\overrightarrow{CA} = 2\vec{m} - \vec{n}$, $|\vec{m}| = |\vec{n}| = \sqrt{3}$, $\measuredangle(\vec{m}, \vec{n}) = \frac{\pi}{2}$.

Rj.



$$\overrightarrow{AB} = \overrightarrow{BC} + \overrightarrow{CA} = \vec{m} + 2\vec{n} + 2\vec{m} - \vec{n} = 3\vec{m} + \vec{n}$$

$$\vec{h}_a = ?$$

$$\vec{h}_a = x\vec{m} + y\vec{n}$$

$$\vec{h}_a \perp \overrightarrow{BC} \Rightarrow \vec{h}_a \cdot \overrightarrow{BC} = 0 \quad \text{tj.}$$

$$(x\vec{m} + y\vec{n}) \cdot (\vec{m} + 2\vec{n}) = x\vec{m}^2 + 2x\vec{m}\cdot\vec{n} + y\vec{m}\vec{n} + y\vec{n}^2 \overset{(*)}{=}$$

$$\vec{m}^2 = |\vec{m}|^2 = 3$$

$$\vec{n}^2 = |\vec{n}|^2 = 3 \qquad \ldots (*)$$

$$\vec{m}\cdot\vec{n} = |\vec{m}||\vec{n}| \cos\measuredangle(\vec{m}, \vec{n}) = 0$$

$$\overset{(*)}{=} 3x + 3y = 0$$

tj. $x + y = 0$

$x = -y$

$$P_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$P_{\Delta ABC} = \frac{|\vec{h}_a| \cdot |\overrightarrow{BC}|}{2}$$

$$a^2 = |\overrightarrow{BC}|^2 = \overrightarrow{BC}^2 = (\vec{m} + 2\vec{n})^2 =$$

$$= \vec{m}^2 + 4\vec{m}\vec{n} + 4\vec{n}^2 = 3 + 4 \cdot 3 = 15$$

$$b^2 = |\overrightarrow{CA}|^2 = \overrightarrow{CA}^2 = (2\vec{m} - \vec{n})^2 = 4\vec{m}^2 - 4\vec{m}\vec{n} + \vec{n}^2$$

$$= 15$$

$$c^2 = |\overrightarrow{AB}|^2 = \overrightarrow{AB}^2 = (3\vec{m} + \vec{n})^2 = 9\vec{m}^2 + 6\vec{m}\vec{n} + \vec{n}^2 = 30$$

$$a = \sqrt{15}, \quad b = \sqrt{15}, \quad c = \sqrt{30}$$

$$s = \frac{a+b+c}{2} = \frac{2\sqrt{15} + \sqrt{30}}{2} = \sqrt{15} + \frac{\sqrt{2}}{2}\sqrt{15} = \sqrt{15} + \frac{\sqrt{30}}{2}$$

$$P_{\Delta ABC} = \sqrt{\left(\sqrt{15} + \frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\frac{\sqrt{2}}{2}\sqrt{15}\right)\left(\sqrt{15} - \frac{\sqrt{30}}{2}\right)} =$$

$$= \sqrt{\left(15 - \frac{30}{4}\right) \cdot \frac{1}{4} \cdot 15} = \sqrt{\frac{30}{4} \cdot \frac{1}{4} \cdot 15} = \frac{15}{4}\sqrt{2} \qquad \ldots (\Delta)$$

$$P_{\Delta ABC} = \frac{|\vec{h}_a| \cdot \sqrt{15}}{2} \overset{(\Delta)}{\Rightarrow} |\vec{h}_a| \cdot \sqrt{15} = \frac{15}{2}\sqrt{2} \Rightarrow |\vec{h}_a| = \frac{15}{2}\sqrt{\frac{2}{15}}$$

$$|\vec{h}_a|^2 = \frac{15^2}{2^2} \cdot \frac{2}{15} = \frac{15}{2} = \vec{h}_a^2 = (x\vec{m} + y\vec{n})^2 = x^2\vec{m}^2 + 2xy\vec{m}\vec{n} + y^2\vec{n}^2$$

$$= 3x^2 + 3y^2 \quad \text{tj.} \quad 3x^2 + 3y^2 = \frac{15}{2} \Rightarrow x^2 + y^2 = \frac{5}{2} \quad \text{kako } x = -y$$

$$2y^2 = \frac{5}{2} \qquad y_{1,2} = \pm\frac{\sqrt{5}}{2} \qquad y_1 = \frac{\sqrt{5}}{2} \Rightarrow x_1 = -\frac{\sqrt{5}}{2} \qquad \vec{h}_a = \pm\left(\frac{\sqrt{5}}{2}\vec{m} - \frac{\sqrt{5}}{2}\vec{n}\right)$$

$$y^2 = \frac{5}{4} \qquad y_2 = -\frac{\sqrt{5}}{2} \Rightarrow x_2 = \frac{\sqrt{5}}{2}$$

$\pm$ zavisi od $\overrightarrow{AA_1}$ ili $\overrightarrow{A_1A}$

(#) Riješiti sistem jednačina za razne vrijednosti parametra $\lambda \in R$:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ 6x_1 - 3x_2 + x_3 - 4x_4 &= 7 \\ 4x_1 - 2x_2 + 14x_3 - 31x_4 &= \lambda \end{aligned}$$

Rj. Riješimo sistem Kroneker-Kapelijevom metodom:

$$\overline{C} = [C \mid b] = \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 6 & -3 & 1 & -4 & 7 \\ 4 & -2 & 14 & -31 & \lambda \end{array}\right] \underset{III_v - I_v \cdot 2}{\overset{II_v - I_v \cdot 3}{\sim}} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 8 & -17 & \lambda-30 \end{array}\right]$$

$$\overset{III_v + II_v}{\sim} \left[\begin{array}{cccc|c} 2 & -1 & 3 & -7 & 15 \\ 0 & 0 & -8 & 17 & -38 \\ 0 & 0 & 0 & 0 & \lambda-68 \end{array}\right]$$

1° $\lambda - 68 \neq 0$, $\lambda \neq 68$

$\text{rang}\, C = 2$, $\text{rang}\, \overline{C} = 3$ $\quad \text{rang}\, C < \text{rang}\, \overline{C}$ Prema Kroneker-Kapelijevoj teoremi sistem nema rješenje

2° $\lambda - 68 = 0$, $\lambda = 68$

$\text{rang}\, C = \text{rang}\, \overline{C} = 2 < 4$ (broj nepoznatih)

Prema Kroneker-Kapelijevoj teoremi dvije promjenjive uzimamo proizvoljno, npr. $x_4 = t$, $x_1 = s$

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 7x_4 &= 15 \\ -8x_3 + 17x_4 &= -38 \end{aligned}$$

$x_4 = t$

$-8x_3 + 17t = -38$

$-8x_3 = -17t - 38$

$x_3 = \frac{17}{8}t + \frac{38}{8} = \frac{17}{8}t + \frac{19}{4}$

$x_1 = s$

$2s - x_2 + 3\left(\frac{17}{8}t + \frac{38}{8}\right) - 7t = 15$

$x_2 = \frac{51t}{8} + \frac{114}{8} + 2s - 7t - 15$

$x_2 = -\frac{5}{8}t - \frac{6}{8} + 2s$

$x_2 = 2s - \frac{5}{8}t - \frac{3}{4}$

Za $\lambda = 68$ rješenje sistema je

$$\left(s,\ 2s - \frac{5}{8}t - \frac{3}{4},\ \frac{17}{8}t + \frac{19}{4},\ t\right),\quad t, s \in R$$

Riješiti matričnu jednačinu: $AX - 2B = 3X + A$ gdje je

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}.$$

Rj.

$$AX - 2B = 3X + A$$

$$AX - 3X = 2B + A$$

$$\underbrace{(A - 3I)}_{M} X = \underbrace{2B + A}_{N}$$

$$M = A - 3I = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$MX = N \quad / \cdot M^{-1}$ sa lijeve str.

$$M^{-1} M X = M^{-1} \cdot N$$

$$X = M^{-1} \cdot N$$

$$N = 2B + A = \begin{bmatrix} -2 & 4 & 0 \\ 4 & 6 & 2 \\ 8 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det M} M_{kof}^{T}$$

$$\det M = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 = 6$$

$$M_{11} = (-1)^2 \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 2 \qquad M_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$M_{12} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0 \qquad M_{22} = (-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$$M_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \qquad M_{23} = (1)^5 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2$$

$$M_{32} = (-1)^5 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6$$

$$M_{33} = (-1)^6 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6$$

$$M_{kof} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -6 & 6 \end{bmatrix}, \quad M_{kof}^{T} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix},$$

$$X = M^{-1} \cdot N = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 20 & -1 & 16 \\ -36 & 33 & 48 \\ 48 & 0 & 60 \end{bmatrix}$$

$8 - 4 + 16$

$10 - 11 + 0$

$0 - 4 + 20$

$0 + 12 - 48$

$0 + 33 + 0$

$12 - 60$

$$X = \begin{bmatrix} \frac{10}{3} & -\frac{1}{6} & \frac{8}{3} \\ -6 & \frac{11}{2} & 8 \\ 8 & 0 & 10 \end{bmatrix}$$

traženo rješenje