Pismeni ispit iz Matematike za ekonomiste, 08. 09. 2010.

GRUPA A

1. Dokazati matematičkom indukcijom tvrdnju

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1}=\frac{2\left(n^2+n+1\right)}{3n(n+1)}\ (n=2,3,4...).$$

2. Ispitati funkciju i nacrtati njen grafik: $y=\dfrac{x^4-5x^2+4}{x^2-5}$.
3. Izračunati površinu figure određene linijama: $y=\dfrac{x}{x-2},\ x+y+1=0.$
4. Riješiti diferencijalnu jednačinu $\left(2x+y+5\right)y'=3x+6.$

GRUPA B

1. Izračunati x ako je četvrti član u razvoju binoma $\left[\left(\sqrt{x}\right)^{\frac{1}{\log x+1}}+\sqrt[12]{x}\right]^6$ jednak 200.
2. Ispitati funkciju i nacrtati njen grafik: $y=\left(2x-4\right)e^{\frac{1}{1-2x}}.$
3. Izračunati integral $\displaystyle\int\frac{5x^2+6x+9}{\left(x^2-2x-3\right)^2}dx.$
4. Naći ekstreme funkcije $z=\ln\left(x^2+2xy+3y^2-4x-5y+6\right).$

GRUPA C

1. Izračunati $\left[\dfrac{1-\sqrt{3}+i\left(1+\sqrt{3}\right)}{1-i}\right]^4.$
2. Ispitati funkciju i nacrtati njen grafik: $y=\dfrac{x^2}{ax^2+7x+b}$, ako se zna da funkcija nije definisana u tačkama $x=-3$ i $x=-\dfrac{1}{2}.$
3. Izračunati površinu figure određene linijama: $y=\ln\left(x-1\right),\ y=1,\ y=-1,\ x=0.$
4. Riješiti diferencijalnu jednačinu $y'=\dfrac{x^2+8}{\left(x^2-5x+6\right)y^2\cos y}.$

GRUPA D

1. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra:

$$\begin{aligned}ax+y+z&=1\\x+y+az&=1\\2x+2ay+2z&=3.\end{aligned}$$

2. Ispitati funkciju i nacrtati njen grafik: $y=\ln\dfrac{x^2-2}{x}.$
3. Izračunati integral $\displaystyle\int\frac{dx}{x\left(\sqrt{x}+3\sqrt[3]{x}-4\right)}.$
4. Naći ekstreme funkcije $z=x^3+y^3-63\left(x+y\right)+12xy.$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Dokazati matematičkom indukcijom tvrdnju

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\ldots\cdot\frac{n^3-1}{n^3+1}=\frac{2(n^2+n+1)}{3n(n+1)},\quad n=2,3,4,\ldots$$

Rj.

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\ldots\cdot\frac{k^3-1}{k^3+1}=\frac{2(k^2+k+1)}{3k(k+1)},\quad k=2,3,4,\ldots$$

BAZA INDUKCIJE

$k=2$: $$\frac{2^3-1}{2^3+1}=\frac{2(2^2+2+1)}{3\cdot 2(2+1)}\Rightarrow \frac{7}{9}=\frac{14}{18}$$ Jednakost je tačna za $k=2$.

KORAK INDUKCIJE

Pretpostavimo da je jednakost tačna za sve brojeve od 1 do n ($k=1,2,\ldots,n$) i na osnovu te pretpostavke pokažimo da je jednakost tačna za $n+1$, tj. trebamo pokazati da je

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\ldots\cdot\frac{n^3-1}{n^3+1}\cdot\frac{(n+1)^3-1}{(n+1)^3+1}=\frac{2\overbrace{\left((n+1)^2+(n+1)+1\right)}^{n^2+2n+1+n+2}}{\underbrace{3(n+1)((n+1)+1)}_{3(n+1)(n+2)}}\left(=\frac{2(n^2+3n+3)}{3(n+1)(n+2)}\right)$$

Krenimo od lijeve strane

$$\underbrace{\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\ldots\cdot\frac{n^3-1}{n^3+1}}\cdot\frac{(n+1)^3-1}{(n+1)^3+1}\overset{\text{na osnovu pretpostavke}}{=}\frac{2(n^2+n+1)}{3n(n+1)}\cdot\frac{(n+1)^3-1}{\underbrace{(n+1)^3+1}_{((n+1)+1)((n+1)^2-(n+1)+1)}}=$$

$$(n+1)^3+1=[(n+1)+1][\overbrace{(n+1)^2}^{n^2+2n+1}-(n+1)+1]=(n+2)(n^2+n+1)$$

$$(n+1)^3-1=((n+1)-1)(\overbrace{(n+1)^2}^{n^2+2n+1}+n+1+1)=n(n^2+3n+3)$$

$$=\frac{2\cancel{(n^2+n+1)}\cdot\cancel{n}\cdot(n^2+3n+3)}{3\cancel{n}(n+1)(n+2)\cancel{(n^2+n+1)}}=\frac{2(n^2+3n+3)}{3(n+1)(n+2)}=\frac{2((n+1)^2+(n+1)+1)}{3(n+1)((n+1)+1)}$$

što je i trebalo dobiti.

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve $n\geq 2$.

(#) Izračunati x ako je četvrti član u razvoju binoma $\left[(\sqrt{x})^{\frac{1}{\log x+1}} + \sqrt[12]{x}\right]^6$ jednak 200.

Rj.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\left[(\sqrt{x})^{\frac{1}{\log x+1}} + \sqrt[12]{x}\right]^6 = \left(x^{\frac{1}{2\log x+2}} + x^{\frac{1}{12}}\right)^6 =$$

$$= \sum_{k=0}^{6} \binom{6}{k} \left(x^{\frac{1}{2\log x+2}}\right)^{6-k} \cdot \left(x^{\frac{1}{12}}\right)^k = \sum_{k=0}^{6} \binom{6}{k} x^{\frac{6-k}{2\log x+2} + \frac{k}{12}} =$$

$\cdot(\log x+1)$

$$= \sum_{k=0}^{6} \binom{6}{k} x^{\frac{36-6k+k(\log x+1)}{12\log x+12}}$$

$$\binom{6}{3} = \frac{\cancel{6}\cdot 5\cdot 4}{2\cdot \cancel{3}}$$

Četvrti član dobijemo za $k=3$

$$\binom{6}{3} x^{\frac{6-k}{2\log x+2} + \frac{k}{12}} = 200$$

$$20\, x^{\frac{6-k}{2\log x+2} + \frac{k}{12}} = 200 \quad /:20$$

$$x^{\frac{6-k}{2\log x+2} + \frac{k}{12}} = 10$$

$(\log x+1)$

$k=3$ pa imamo $x^{\frac{3}{2\log x+2} + \frac{\cancel{3}^{1}}{\cancel{12}_4}} = 10$

$$x^{\frac{6+\log x+1}{4\log x+4}} = 10 \Rightarrow \frac{7+\log x}{4\log x+4} \cdot \log x = 1$$

$\log x = t$

$(7+t)\cdot t = 4t+4$

$t^2+7t-4t-4=0$

$t^2+3t-4=0$

$(t-1)(t+4)=0$

$t_1=1 \quad t_2=-4$

$\log x = 1$

$\log x = 1$

$x = 10$

$\log x = -4$

$x = 10^{-4}$

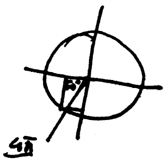
\# Izračunati $\left(\dfrac{1-\sqrt{3}+i(1+\sqrt{3})}{1-i}\right)^4$.

Rj.

$$\frac{1-\sqrt{3}+i(1+\sqrt{3})}{1-i}\cdot\frac{1+i}{1+i}=\frac{\cancel{1}-\sqrt{3}+i\cancel{+i\sqrt{3}}+i\cancel{-i\sqrt{3}}-\cancel{1}-\sqrt{3}}{1-i^2}=$$

$$=\frac{2i-2\sqrt{3}}{2}=i-\sqrt{3}$$

$\sin 60° = \frac{\sqrt{3}}{2}$



$z=-8-8i\sqrt{3}$

$a=-8 \quad b=-8\sqrt{3}$

$|z|=\sqrt{256} \quad \cos\varphi=-\frac{1}{2}$

$|z|=16 \quad \sin\varphi=-\frac{\sqrt{3}}{2}$

$$(i-\sqrt{3})^2=i^2-2i\sqrt{3}+3=2-2i\sqrt{3}$$

$$(i-\sqrt{3})^4=(2-2i\sqrt{3})^2=4-8i\sqrt{3}+\overbrace{4i^2\cdot 3}^{-12}=-8-8i\sqrt{3}=-8(1+i\sqrt{3})$$

$$\left(\frac{1-\sqrt{3}+i(1+\sqrt{3})}{1-i}\right)^4=-8-8i\sqrt{3}=16\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

traženo rješenje

Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra

$$
\begin{aligned}
ax+y+z&=1\\
x+y+az&=1\\
2x+2ay+2z&=3
\end{aligned}
$$

Rj. Sistem ću riješiti Cramerovom metodom

$$D=\begin{vmatrix} a & 1 & 1\\ 1 & 1 & a\\ 2 & 2a & 2\end{vmatrix}=2\begin{vmatrix} a & 1 & 1\\ 1 & 1 & a\\ 1 & a & 1\end{vmatrix}\xrightarrow{I_k+(II_k+III_k)}2\begin{vmatrix} a+2 & 1 & 1\\ a+2 & 1 & a\\ a+2 & a & 1\end{vmatrix}\xrightarrow{I_v-II_v}2\begin{vmatrix} 0 & 0 & 1-a\\ a+2 & 1 & a\\ a+2 & a & 1\end{vmatrix}$$

$$=2(1-a)(a+2)\begin{vmatrix} 1 & 1\\ 1 & a\end{vmatrix}=-2(a-1)(a+2)(a-1)=(-2)(a-1)^2(a+2)$$

$$D_x=\begin{vmatrix} 1 & 1 & 1\\ 1 & 1 & a\\ 3 & 2a & 2\end{vmatrix}\xrightarrow{II_v-I_v}\begin{vmatrix} 1 & 1 & 1\\ 0 & 0 & a-1\\ 3 & 2a & 2\end{vmatrix}=-(a-1)\begin{vmatrix} 1 & 1\\ 3 & 2a\end{vmatrix}=(1-a)(2a-3)=(3-2a)(a-1)$$

$$D_y=\begin{vmatrix} a & 1 & 1\\ 1 & 1 & a\\ 2 & 3 & 2\end{vmatrix}\xrightarrow[III_v-II_v\cdot 3]{I_v-II_v}\begin{vmatrix} a-1 & 0 & 1-a\\ 1 & 1 & a\\ -1 & 0 & 2-3a\end{vmatrix}=\begin{vmatrix} a-1 & \overset{-(a-1)}{1-a}\\ -1 & 2-3a\end{vmatrix}=(a-1)\overbrace{\begin{vmatrix} 1 & -1\\ -1 & 2-3a\end{vmatrix}}^{2-3a-1=1-3a}=$$

$$=(a-1)(1-3a)$$

$$D_z=\begin{vmatrix} a & 1 & 1\\ 1 & 1 & 1\\ 2 & 2a & 3\end{vmatrix}\xrightarrow{I_v-II_v}\begin{vmatrix} a-1 & 0 & 0\\ 1 & 1 & 1\\ 2 & 2a & 3\end{vmatrix}=(a-1)\begin{vmatrix} 1 & 1\\ 2a & 3\end{vmatrix}=(a-1)(3-2a)$$

1° za $D\neq 0$ tj. za $a\neq 1$ i $a\neq -2$ sistem ima jedinstveno rješenje:

$$x=\frac{D_x}{D}=\frac{-(2a-3)(a-1)}{(-2)(a-1)^2(a+2)}=\frac{2a-3}{2(a-1)(a+2)},\quad y=\frac{D_y}{D}=\frac{3a-1}{2(a-1)(a+2)},\quad z=\frac{D_z}{D}=\frac{2a-3}{2(a-1)(a+2)}$$

2° za $a=-2$ imamo da je $D_x\neq 0 \Rightarrow$ sistem nema rješenja

3° za $a=1$ imamo $D=D_x=D_y=D_z=0$ pa sistem postaje

$$
\begin{aligned}
x+y+z&=1 \;/\cdot 2\\
x+y+z&=1 \;/\cdot 2\\
2x+2y+2z&=3
\end{aligned}
\qquad
\begin{aligned}
2x+2y+2z&=1\\
-\;2x+2y+2z&=3\\
\hline
0&=-2
\end{aligned}
$$

sistem nema rješenja

(#) Ispitati f-ju i nacrtati njen grafik $y = \dfrac{x^4-5x^2+4}{x^2-5}$.

Rj. definiciono područje

$x^2-5 \neq 0$

$x^2 \neq 5$ $\qquad \sqrt{5} \approx 2{,}24$

$x \neq \pm\sqrt{5}$

nule, presjek sa y-osom, znak

$y=0$ akko $x^4-5x^2+4=0$

$x^2=t, \quad t^2-5t+4=0$

$D=25-16=9$

$t_{1,2}=\dfrac{5\pm3}{2} \qquad t_1=1, \; t_2=4$

$(x^2-1)(x^2-4)=0$

$(x-1)(x+1)(x-2)(x+2)=0$

Nule f-je su $(-1,0), (1,0), (-2,0), (2,0)$

$f(0)=-\dfrac{4}{5}$ Presjek sa y-osom je $(0,-\dfrac{4}{5})$

parnost (neparnost), periodičnost

$f(-x)=\dfrac{(-x)^4-5(-x)^2+4}{(-x)^2-5}=\dfrac{x^4-5x^2+4}{x^2-5}=f(x)$

f-ja je parna (simetrična u odnosu na y-osu)

f-ja nije periodična

0 1 2 $\sqrt{5}$

x	$(0,1)$	$(1,2)$	$(2,\sqrt{5})$	$(\sqrt{5},+\infty)$
x^2-1	$-$	$+$	$+$	$+$
x^2-4	$-$	$-$	$+$	$+$
x^2-5	$-$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

Za $x=\pm\sqrt{5}$ f-ja ima prekid

$\lim\limits_{x\to\sqrt{5}+0} f(x) = \lim\limits_{x\to\sqrt{5}+0}\dfrac{x^4-5x^2+4}{x^2-5}=\dfrac{25+0-25-0+4}{5+0-5}=+\infty$

$\lim\limits_{x\to\sqrt{5}-0} f(x)=\dfrac{25-0-25-0+4}{5-0-5}=-\infty$

$\Rightarrow x=\sqrt{5}$ je V. A.

$\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{x^4-5x^2+4}{x^2-5}\;\dfrac{/:x^2}{/:x^2}=\pm\infty \Rightarrow$ f-ja nema H. A. $\qquad$ tražimo kosu asimptotu u obliku $y=kx+n$

$\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{x^4-5x^2+4}{x^3-5x}\;\dfrac{/:x^3}{/:x^3}=+\infty \Rightarrow$ f-ja nema K. A.

Poslije ovog koraka počinjemo skicirati graf.

rast i opadanje

$y'=\left(\dfrac{x^4-5x^2+4}{x^2-5}\right)'=\dfrac{(4x^3-10x)(x^2-5)-(x^4-5x^2+4)\cdot 2x}{(x^2-5)^2}$

$y'=\dfrac{4x^5\cancel{-10x^3}-20x^3+50x-2x^5\cancel{+10x^3}-8x}{(x^2-5)^2}$

$y'=2\dfrac{x(x^4-10x^2+21)}{(x^2-5)^2}=2\dfrac{x(x^2-7)(x^2-3)}{(x^2-5)^2}$

$y'=0$ akko $x=0, \; x=\pm\sqrt{7}, \; x=\pm\sqrt{3}$

$\sqrt{3}\approx 1{,}73$

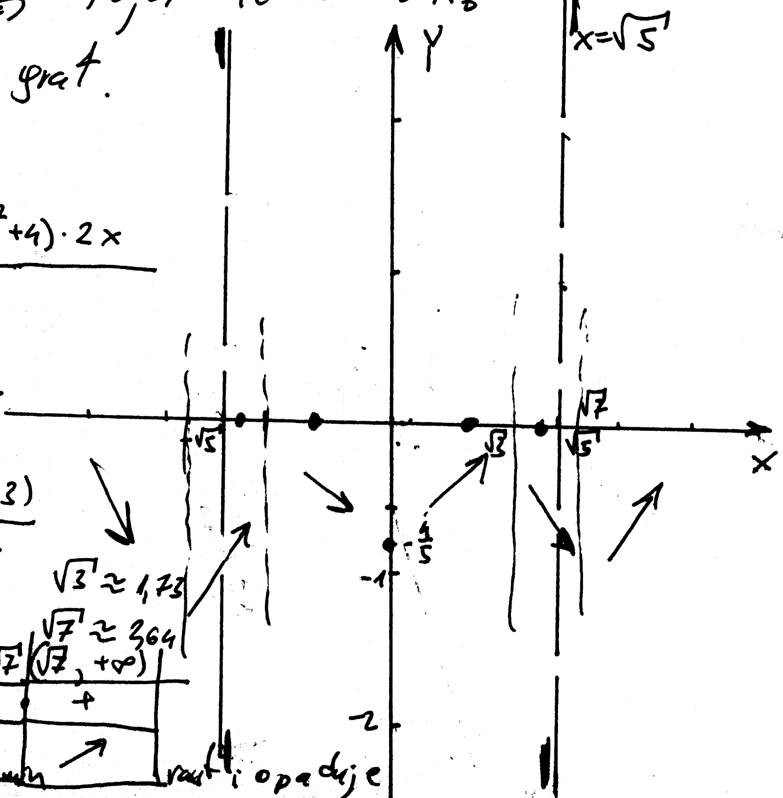
$\sqrt{7}\approx 2{,}64$

0 $\sqrt{3}$ $\sqrt{5}$ $\sqrt{7}$

↑ prekidi y + nule y'

x	$(0,\sqrt{3})$	$(\sqrt{3},\sqrt{5})$	$(\sqrt{5},\sqrt{7})$	$(\sqrt{7},+\infty)$
y'	$+$	$-$	$-$	$+$
y	↗ max	↘	↘ min	↗

rast i opadanje



$f(0) = -\frac{4}{5}$, $f(\sqrt{3}) = \frac{9-15+4}{3-5} = \frac{-2}{-2} = 1$, $f(\sqrt{7}) = \frac{49-35+4}{7-5} = \frac{18}{2} = 9$

ekstremi f-je

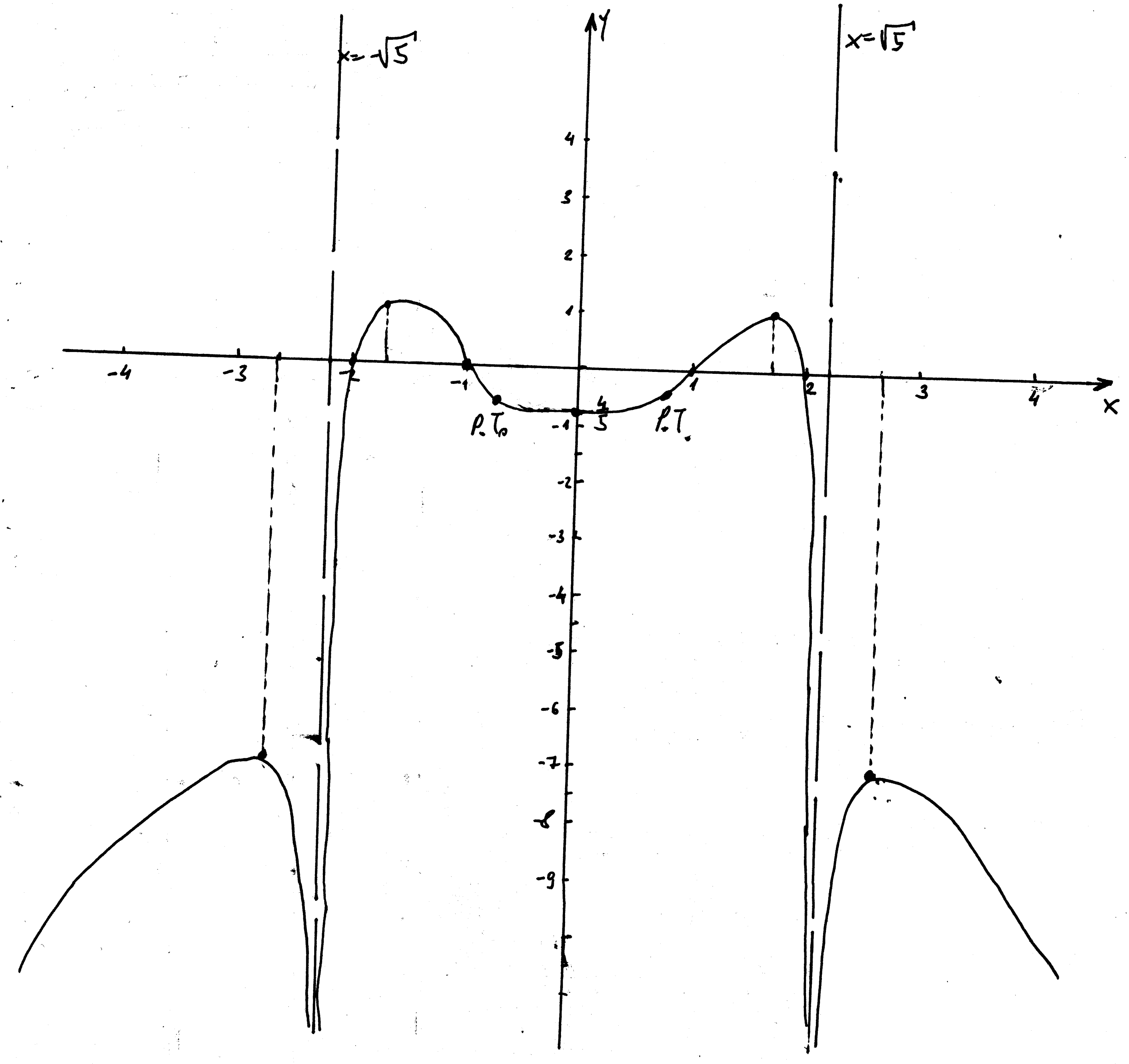
Na osnovu tabele rasta i opadanja vidimo da f-ja ima ekstreme i to maksimume u tačkama $(-\sqrt{3},1)$, $(\sqrt{3},1)$ i minimume u tačkama $(-\sqrt{7},9)$, $(\sqrt{7},9)$, $(0,-\frac{4}{5})$.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(2\,\frac{x^5-10x^3+21x}{(x^2-5)^2}\right)' = 2\,\frac{(5x^4-30x^2+21)(x^2-5)^{\cancel{2}} - (x^5-10x^3+21x)\,2\cancel{(x^2-5)}\cdot 2x}{(x^2-5)^{\cancel{4}\,3}}$$

$$y'' = 2\,\frac{x^6-15x^4+87x^2-105}{(x^2-5)^3}$$

Kako je brojnik polinom 6 stepena, nećemo tražiti nule y''. Prevojne tačke ćemo odrediti približno tačno.



(#) Ispitati f-ju i nacrtati njen grafik $y=\dfrac{x^2}{ax^2+7x+b}$ ako se zna da f-ja nije definisana u tačkama $x=-3$ i $x=-\frac{1}{2}$.

R: F-ja $f(x)=\dfrac{x^2}{ax^2+7x+b}$ nije definisana kada je

$ax^2+7x+b=0$.

$x=-3$: $\quad 9a-21+b=0$

$x=-\frac{1}{2}$: $\quad \frac{1}{4}a-\frac{7}{2}+b=0 \quad /\cdot 4$

$$\begin{aligned} 9a+b&=21 \quad /\cdot 4\\ a+4b&=14\\ \hline 36a+4b&=84\\ -\ a+4b&=14\\ \hline 35a&=70\\ a&=2 \end{aligned}$$

$a+4b=14$
$4b=12$
$b=3$

Naša f-ja je oblika $f(x)=\dfrac{x^2}{\underbrace{2x^2+7x+3}_{2(x+\frac{1}{2})(x+3)}}$.

definiciono područje

$D:\ x\in(-\infty,-3)\cup(-3,-\tfrac{1}{2})\cup(-\tfrac{1}{2},+\infty)$

parnost (neparnost), periodičnost

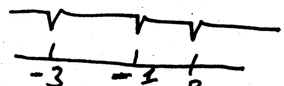
D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

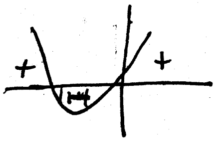
nule, presjek sa Y-osom, znak f-je

$f(0)=0$

$(0,0)$ je nula f-je i presjek sa Y-osom



prekidi + nule f-je



X	$(-\infty,-3)$	$(-3,-\frac{1}{2})$	$(-\frac{1}{2},0)$	$(0,+\infty)$
Y	+	−	+	+

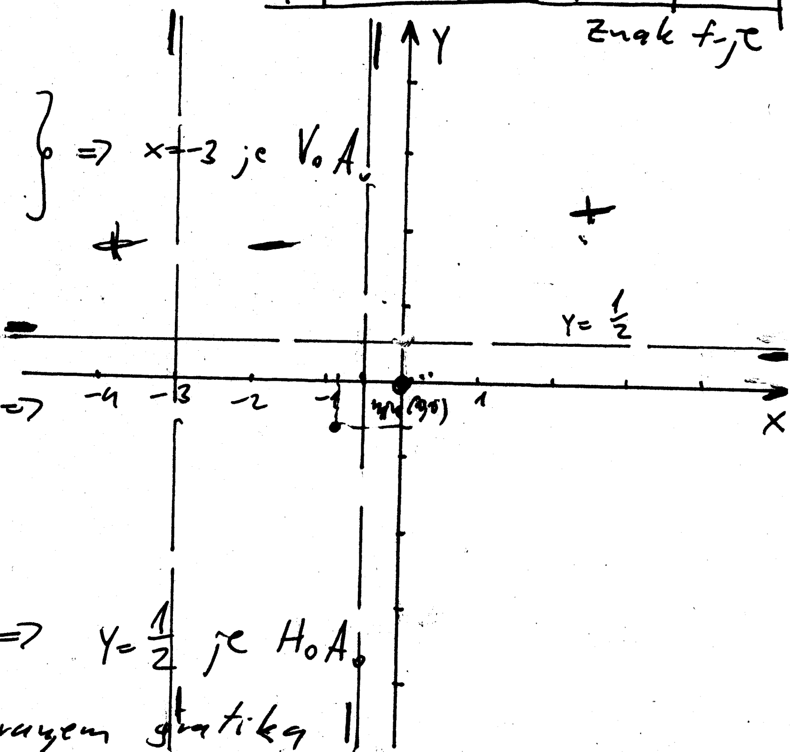
Znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-3$ i $x=-\frac{1}{2}$ f-ja ima prekid

$$\lim_{x\to-3-0} f(x)=\lim_{x\to-3-0}\frac{x^2}{(2x+1)(x+3)}=\frac{(-3-0)^2}{+0}=+\infty$$
$$\lim_{x\to-3+0} f(x)=\lim_{x\to-3+0}\frac{x^2}{(2x+1)(x+3)}=\frac{(-3-0)^2}{-0}=-\infty$$
$\Rightarrow x=-3$ je V. A.

$$\lim_{x\to-\frac{1}{2}-0} f(x)=\lim_{x\to-\frac{1}{2}-0}\frac{x^2}{(2x+1)(x+3)}=\frac{(-\frac{1}{2}-0)^2}{-0}=-\infty$$
$$\lim_{x\to-\frac{1}{2}+0} f(x)=\lim_{x\to-\frac{1}{2}+0}\frac{x^2}{(2x+1)(x+3)}=\frac{(-\frac{1}{2}-0)^2}{+0}=+\infty$$
$\Rightarrow x=-\frac{1}{2}$ je V. A.



$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^2}{2x^2+7x+3}\ \frac{/:x^2}{/:x^2}=\frac{1}{2}\ \Rightarrow\ y=\frac{1}{2} \text{ je H. A.}$$

poslije ovog koraka počinjemo sa skiciranjem grafika rast i opadanje

$$y' = \left(\frac{x^2}{2x^2+7x+3}\right)' = \frac{2x(2x^2+7x+3) - x^2\cdot(4x+7)}{(2x^2+7x+3)^2} = \frac{\cancel{4x^3}+14x^2+6x-\cancel{4x^3}-7x^2}{(2x^2+7x+3)^2}$$

$$y' = \frac{7x^2+6x}{(2x^2+7x+3)^2} \qquad y'=0 \text{ akko } 7x^2+6x=0.$$

$$x(7x+6)=0 \qquad x_1=0,\ x_2=-\frac{6}{7}$$

$-3 \quad -\frac{6}{7} \quad -\frac{1}{2} \quad 0$

prelazi nule y

x	$(-\infty,-3)$	$(-3,-\frac{6}{7})$	$(-\frac{6}{7},-\frac{1}{2})$	$(-\frac{1}{2},0)$	$(0,+\infty)$
y'	+	+	−	−	+
y	↗	↗	↘	↘	↗

max   min   rast i opadanje

$$f\left(-\frac{6}{7}\right) = \frac{\left(-\frac{6}{7}\right)^2}{2\left(-\frac{6}{7}\right)^2+7\left(-\frac{6}{7}\right)+3} = -\frac{12}{25} \approx -0{,}48$$

$$f(0)=0$$

ekstremi f-je

Na osnovu tabele rasta i opadanja f-ja ima ekstreme i to

max u $M\left(-\frac{6}{7}, -\frac{12}{25}\right)$ i min u $N(0,0)$.

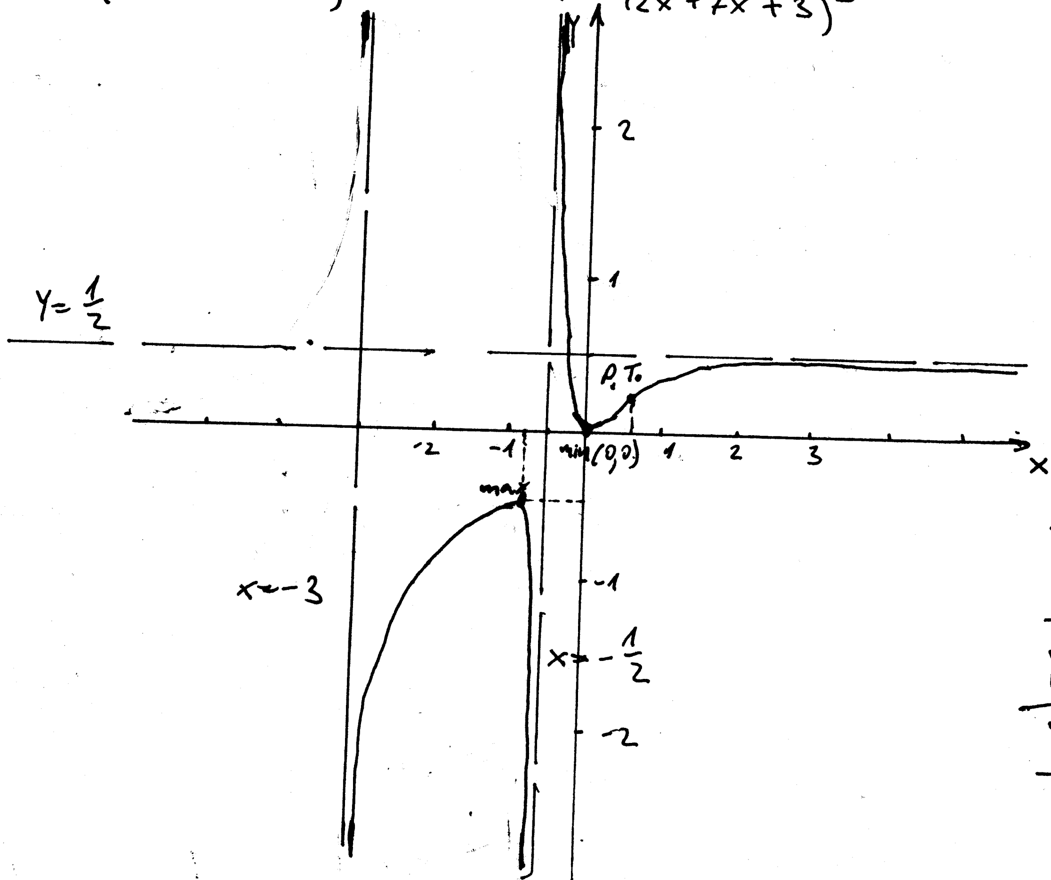
prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{7x^2+6x}{(2x^2+7x+3)^2}\right)' = \frac{(14x+6)(2x^2+7x+3)^{\cancel{2}} - (7x^2+6x)\cdot 2\cancel{(2x^2+7x+3)}\cdot\overbrace{(4x+7)}^{(8x+14)}}{(2x^2+7x+3)^{\cancel{4}\,3}}$$

$$y'' = \frac{\cancel{28x^3}+\cancel{98x^2}+\cancel{42x}+12x^2+\cancel{42x}+18-56x^3-\cancel{98x^2}-48x^2-\cancel{84x}}{(2x^2+7x+3)^3}$$

$$y'' = \frac{-28x^3-36x^2+18}{(2x^2+7x+3)^3} = (-2)\cdot\frac{14x^3+18x^2-9}{(2x^2+7x+3)^3}$$

$y''=0$ akko
$14x^3+18x^2-9=0$
(nulu nije lagano pronaći
$x \approx 0{,}58$ izračunato uz pomoć kalkulatora)



$-3 \quad -\frac{1}{2} \quad 0{,}58$

prelazi y i nule y''

x	$(-\infty,-2)$	$(-3,-\frac{1}{2})$	$(-\frac{1}{2},0{,}58)$	$(0{,}58,+\infty)$
y''	+	−	+	−
y	∪	∩	∪	∩

konveksnost i konkavnost

#) Ispitati f-ju i nacrtati njen grafik $y=\ln\frac{x^2-2}{x}$.

R: definiciono područje

$x\neq 0$

$\frac{x^2-2}{x}>0$ $\quad x^2=2$, $x=\pm\sqrt{2}\approx\pm 1{,}41$

$\mathcal{D}: x\in(-\sqrt{2},0)\cup(\sqrt{2},+\infty)$

x	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},+\infty)$
x^2-2	+	−	−	+
x	−	−	+	+
$\frac{x^2-2}{x}$	−	+	−	+

znak f-je $\frac{x^2-2}{x}$

parnost (neparnost), periodičnost

Iz $\mathcal{D}$ vidimo da f-ja nije ni parna ni neparna

F-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

$\ln\frac{x^2-2}{x}=0$

$\frac{x^2-2}{x}=1 \quad /\cdot x\ (x\neq 0)$

$x^2-2=x$

$x^2-x-2=0$

$(x-2)(x+1)=0$

Nule f-je su $(2,0)$ i $(-1,0)$

$f(0)=\dots$ $f(0)$ nije definisano

f-ja ne siječe y-osu

$\ln\frac{x^2-2}{x}>0$

$\frac{x^2-2}{x}>1$

$\frac{x^2-2}{x}-1>0$

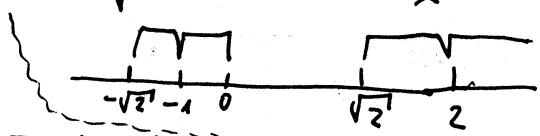
$\frac{x^2-2-x}{x}>0$

$\frac{(x^2-x-2)}{x}>0$

$\frac{(x-2)(x+1)}{x}>0$

	$(-\sqrt{2},-1)$	$(-1,0)$	$(\sqrt{2},2)$	$(2,+\infty)$
$x-2$	−	−	−	+
$x+1$	−	+	+	+
x	−	−	+	+
$\frac{(x-2)(x+1)}{x}$	−	+	−	+
$y=\ln\frac{x^2-2}{x}$	−	+	−	+

prekidi f-je y + nule f-je y



znak f-je y

$\sqrt{2}\approx 1{,}41$

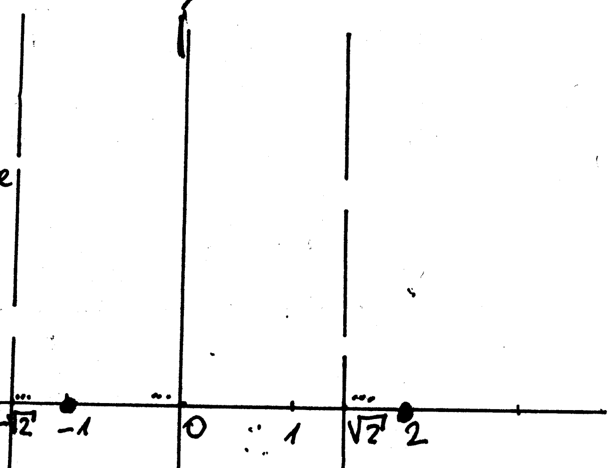
ponašanje na krajevima intervala definisanosti i asimptote

za $x=-\sqrt{2}$, $x=0$, $x=\sqrt{2}$ f-ja ima prekid

$\lim_{x\to-\sqrt{2}+0} f(x)=\ln\frac{2+0-2}{-\sqrt{2}-0}=\ln(+0)=-\infty \Rightarrow$ $x=-\sqrt{2}$ je V.A sa desne strane

$\lim_{x\to 0-0} f(x)=\ln\frac{-2}{0-}=+\infty \Rightarrow x=0$ je V.A. sa lijeve str

$\lim_{x\to\sqrt{2}+0} f(x)=\ln\frac{2+0-2}{\sqrt{2}+0}=\ln(+0)=-\infty \Rightarrow x=\sqrt{2}$ je V.A. sa desne strane



$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\ln\frac{x^2-2}{x}=\ln\lim_{x\to+\infty}\frac{x^2-2}{x}=$

$=\ln\infty=\infty \Rightarrow$ f-ja nema H.A.

Tražimo kosu asimptotu u obliku $y=kx+n$

$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\ln\frac{x^2-2}{x}}{x}\ \left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{x\to\infty}\frac{\frac{x}{x^2-2}\left(\frac{x^2-2}{x}\right)'}{1}=$

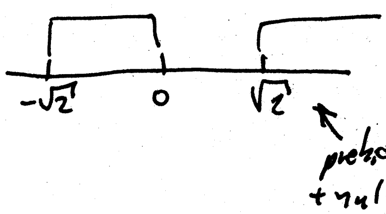
$=\lim_{x\to\infty}\frac{\cancel{x}}{x^2-2}\cdot\frac{2x\cdot x-x^2+2}{x^{\cancel{2}}}=\lim_{x\to\infty}\frac{x^2+2}{x(x^2-2)}=0 \Rightarrow$ f-ja nema K.A.

poslije četvrtog koraka počinjemo skicirati grafik f-je.

rast i opadanje

$$y'=\left(\ln\frac{x^2-2}{x}\right)'=\frac{1}{\frac{x^2-2}{x}}\left(\frac{x^2-2}{x}\right)'=\frac{\cancel{x}}{x^2-2}\cdot\frac{2x\cdot x-(x^2-2)\cdot 1}{x^{\cancel{2}}}=\frac{x^2+2}{x(x^2-2)}$$

y' nema nulu



x	$(-\sqrt{2},0)$	$(\sqrt{2},+\infty)$
y'	+	+
y	↗	↗

ekstremi f-je

$y'>0 \ \forall x\in D$

f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{x^2+2}{x^3-2x}\right)'=\frac{2x\cdot \overbrace{x(x^2-2)}^{x^3-2x}-(x^2+2)(3x^2-2)}{x^2(x^2-2)^2}=\frac{2x^4-4x^2-(3x^4+4x^2-4)}{x^2(x^2-2)^2}=$$

$$=\frac{-x^4-8x^2+4}{x^2(x^2-2)^2}=-\frac{x^4+8x^2-4}{x^2(x^2-2)^2}$$

$y''=0$ akko $x^4+8x^2-4=0$

$x^2=t \quad t^2+8t-4=0$

$D=64+16=80=4\cdot 20=16\cdot 5$

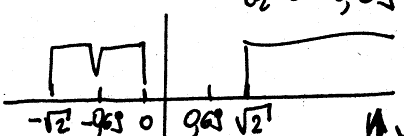
$x_{1,2}=\frac{-8\pm 4\sqrt{5}}{2}=-4\pm 2\sqrt{5}$

$x_1\approx -4+2\sqrt{5}\approx 0{,}47$

$x_2=-4-2\sqrt{5}\approx -8{,}47$

$t^2=0{,}47 \quad t_1\approx 0{,}69$

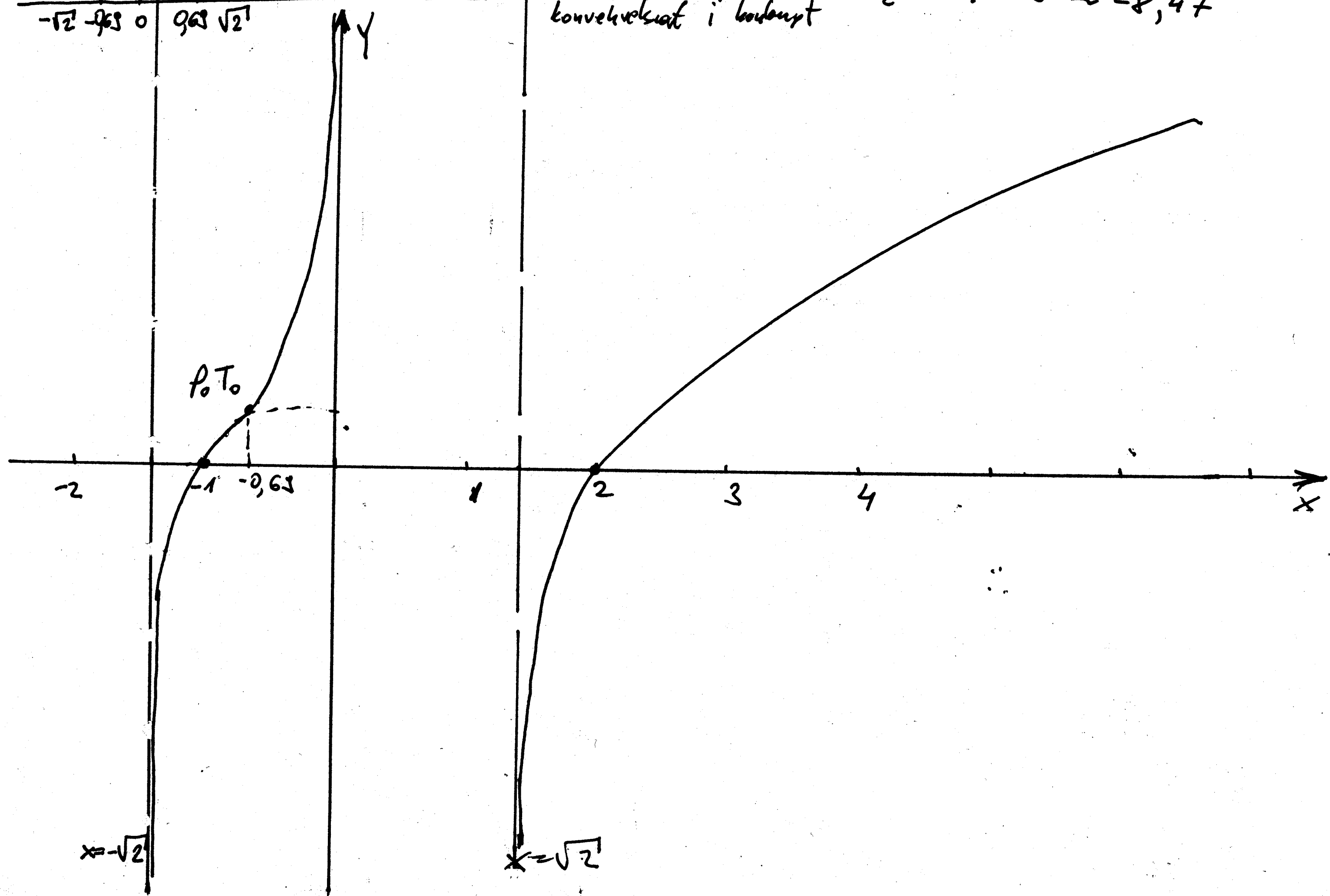
$t_2\approx -0{,}69$



x	$(-\sqrt{2},-0{,}69)$	$(-0{,}69,\ 0)$	$(\sqrt{2},+\infty)$
y''	−	+	−
y	∩	∪	∩

P_0T_0

konveksnost i konkavnost



(#) Ispitati i grafički predstaviti f-ju $y=(2x-4)e^{\frac{1}{1-2x}}$.

Rj. definiciono područje

$1-2x\neq 0$

$2x\neq 1$

$x\neq \frac{1}{2}$

$D: x\in(-\infty,\frac{1}{2})\cup(\frac{1}{2},+\infty)$

nule, presjek sa y-osom, znak f-je

$y=0$ akko $2x-4=0$

$x=2$

$(2,0)$ je nula f-je

$f(0)=(0-4)e^{1}=-4e\approx-10,87$

$(0,-4e)$ je presjek sa y-osom

parnost (neparnost), periodičnost

· D nije simetrično $(x\in D \Rightarrow -x\in D)$ pa f-ja nije ni parna ni neparna

f-ja nije periodična

$e^{\frac{1}{1-2x}}>0 \quad \forall x\in D$

prekidi y → + nule od y: $\frac{1}{2}$, 2

x	$(-\infty,\frac{1}{2})$	$(\frac{1}{2},2)$	$(2,+\infty)$
$2x-4$	−	−	+
y	−	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=\frac{1}{2}$ f-ja ima prekid $\quad 1-2(\frac{1}{2}-0)=1-1+0$

$\lim\limits_{x\to\frac{1}{2}-0}f(x)=\lim\limits_{x\to\frac{1}{2}-0}(2x-4)e^{\frac{1}{1-2x}}=(-3-0)e^{\frac{1}{+0}}=(-3-0)\cdot e^{\infty}=-\infty \Rightarrow x=\frac{1}{2}$ je V.A. (sa lijeve str.)

$\lim\limits_{x\to\frac{1}{2}+0}f(x)=\lim\limits_{x\to\frac{1}{2}+0}(2x-4)e^{\frac{1}{1-2x}}=(-3+0)e^{\frac{1}{-0}}=(-3+0)\cdot 0=0$

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}(2x-4)e^{\frac{1}{1-2x}}=\infty\cdot e^{\frac{1}{-\infty}}=\infty\cdot 1=\infty \Rightarrow$ f-ja nema H.A.

tražimo kosu asimptotu u obliku $y=kx+n$

$\left(\frac{1}{1-2x}\right)'=(-1)(1-2x)^{-2}\cdot(-2)=\frac{2}{(1-2x)^2}$

$k=\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}(2-\frac{4}{x})e^{\frac{1}{1-2x}}=2$

$n=\lim\limits_{x\to\infty}(f(x)-kx)=\lim\limits_{x\to\infty}\left((2x-4)e^{\frac{1}{1-2x}}-2x\right)=$

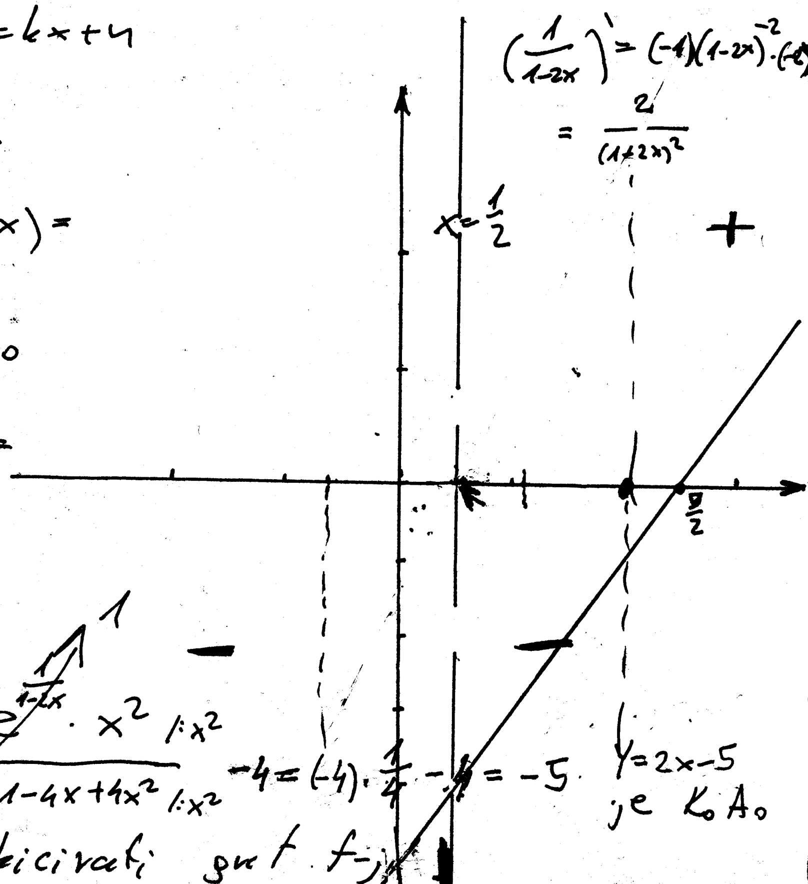
$=\lim\limits_{x\to\infty}\left(2xe^{\frac{1}{1-2x}}-4e^{\frac{1}{1-2x}}-2x\right)=$

$=\lim\limits_{x\to\infty}2x\left(e^{\frac{1}{1-2x}}-1\right)-4\lim\limits_{x\to\infty}e^{\frac{1}{1-2x}\to 0}=$

$=2\lim\limits_{x\to\infty}\frac{e^{\frac{1}{1-2x}}-1}{\frac{1}{x}}-4\left(=\frac{0}{0}-4\right)$

$\stackrel{L.P.}{=}2\lim\limits_{x\to\infty}\frac{e^{\frac{1}{1-2x}}\cdot\frac{2}{(1-2x)^2}}{-\frac{1}{x^2}}-4=-4\lim\limits_{x\to\infty}\frac{e^{\frac{1}{1-2x}\to 1}\cdot x^2 \;/:x^2}{1-4x+4x^2\;/:x^2}-4=(-4)\cdot\frac{1}{4}-4=-5$

$y=2x-5$ je K.A.



Poslije ovog koraka počinjemo skicirati graf f-je rast i opadanje

$$\left((1-2x)^{-1}\right)' = (-1)(1-2x)^{-2}(-2) = \frac{2}{(1-2x)^2}$$

$$y' = \left((2x-4)\, e^{\frac{1}{1-2x}}\right)' = 2e^{\frac{1}{1-2x}} + (2x-4)\, e^{\frac{1}{1-2x}} \cdot \left(\frac{1}{1-2x}\right)' =$$

$$= e^{\frac{1}{1-2x}}\left(2 + (2x-4)(-1)(1-2x)^{-2}\cdot(-2)\right) = e^{\frac{1}{1-2x}}\left(2 + \frac{4x-8}{(1-2x)^2}\right)$$

$$y' = 2e^{\frac{1}{1-2x}}\left(1 + \frac{2x-4}{(1-2x)^2}\right) = 2e^{\frac{1}{1-2x}}\, \frac{1-4x+4x^2+2x-4}{(1-2x)^2} = 2e^{\frac{1}{1-2x}}\, \frac{-3-2x+4x^2}{(1-2x)^2}$$

$$y' = 2e^{\frac{1}{1-2x}}\, \frac{4x^2-2x-3}{(1-2x)^2}$$

$-0{,}65 \quad \frac{1}{2} \quad 1{,}15$ ← prekid y + nule y'

$y'=0$ akko
$4x^2-2x-3=0$
$D = 4+16\cdot 3 = 52 = 4\cdot 13$

$$x_{1,2} = \frac{2 \pm 2\sqrt{13}}{8}$$

$$x_1 = \frac{1}{4} - \frac{\sqrt{13}}{4} \approx -0{,}65$$

$$x_2 = \frac{1}{4} + \frac{\sqrt{13}}{4} \approx 1{,}15$$

x	$(-\infty,-0{,}65)$	$(-0{,}65, \frac{1}{2})$	$(\frac{1}{2}, 1{,}15)$	$(1{,}15, +\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗

max — min

rast i opadanje

$$f\left(\frac{1}{4} - \frac{\sqrt{13}}{4}\right) \approx -8{,}19$$

$$f\left(\frac{1}{4} + \frac{\sqrt{13}}{4}\right) \approx -0{,}79$$

ekstremi f-je
Na osnovu tabele rasta i opadanja ekstremi f-je su $(-0{,}65; -8{,}19)$ i $(1{,}15; -0{,}79)$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(2e^{\frac{1}{1-2x}}\, \frac{4x^2-2x-3}{(1-2x)^2}\right)' =$$

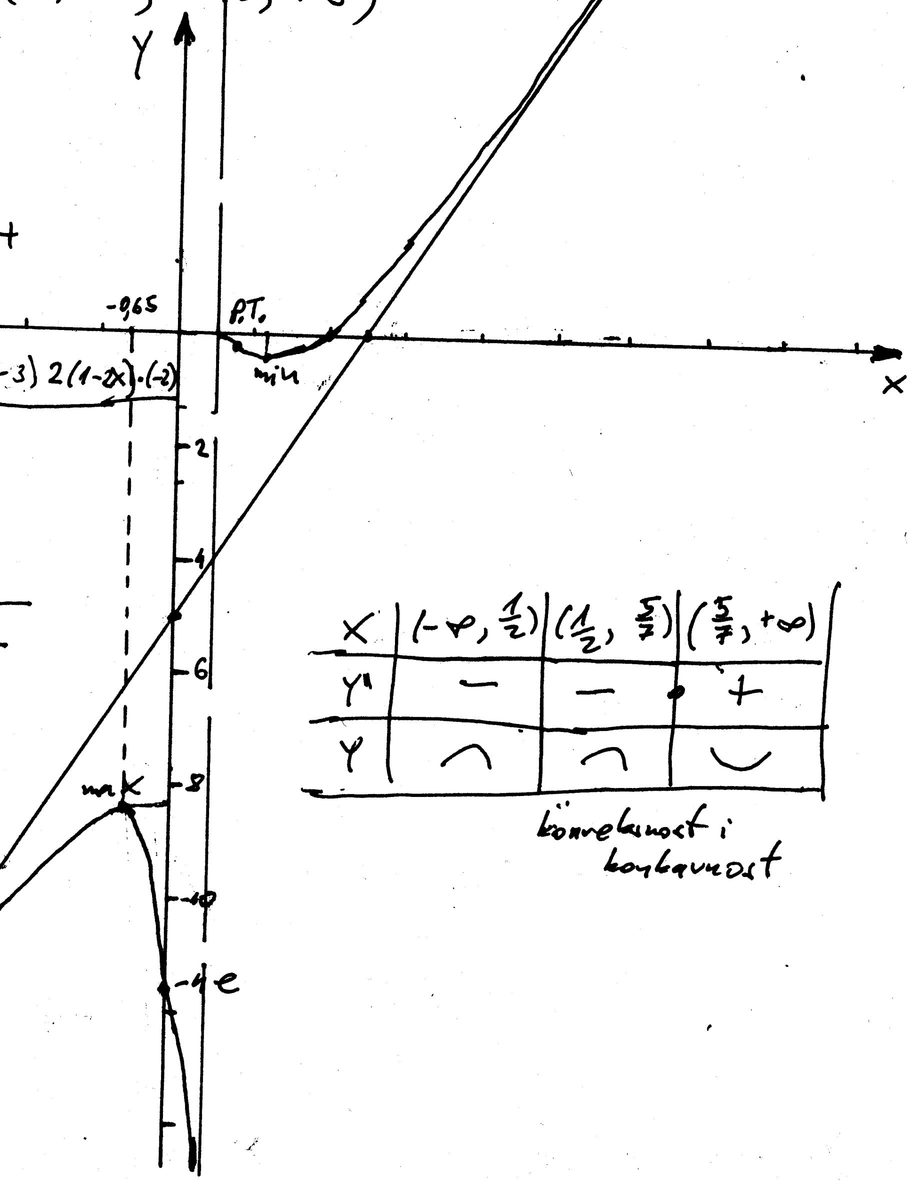
$$= 2e^{\frac{1}{1-2x}} \cdot \frac{2}{(1-2x)^2} \cdot \frac{4x^2-2x-3}{(1-2x)^2} +$$

$$+ 2e^{\frac{1}{1-2x}}\, \frac{(8x-2)(1-2x)^2 - (4x^2-2x-3)\,2(1-2x)\cdot(-2)}{(1-2x)^4}$$

$$= 8e^{\frac{1}{1-2x}}\, \frac{7x-5}{(2x-1)^4}$$

$\frac{1}{2} \quad \frac{5}{7}$

$y''=0$ akko $7x-5=0$
$x = \frac{5}{7}$



x	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, \frac{5}{7})$	$(\frac{5}{7}, +\infty)$
y''	−	−	+
y	⌒	⌒	⌣

konveksnost i konkavnost

(#) Izračunati integral $\displaystyle\int \frac{5x^2+6x+9}{(x^2-2x-3)^2}\,dx$.

Rj.

$$x^2-2x-3=(x-3)(x+1)$$

$$\frac{5x^2+6x+9}{(x^2-2x-3)^2}=\frac{5x^2+6x+9}{(x-3)^2(x+1)^2}=\frac{A}{x-3}+\frac{B}{x+1}+\frac{C}{(x-3)^2}+\frac{D}{(x+1)^2}\quad /\cdot(x-3)^2(x+1)^2$$

$$5x^2+6x+9=A(x-3)(x^2+2x+1)+B(x+1)(x^2-6x+9)+C(x+1)^2+D(x-3)^2$$

$$5x^2+6x+9=A(x^3+2x^2+x)-3A(x^2+2x+1)+B(x^3-6x^2+9x)+B(x^2-6x+9)+$$
$$+C(x^2+2x+1)+D(x^2-6x+9)$$

$$\begin{aligned}
A+B&=0\\
2A-3A-6B+B+C+D&=5\\
A-6A+9B-6B+2C-6D&=6\\
-3A+9B+C+9D&=9
\end{aligned}$$

$$\begin{aligned}
A+B&=0 \Rightarrow A=-B\\
-A-5B+C+D&=5\\
-5A+3B+2C-6D&=6\\
-3A+9B+C+9D&=9
\end{aligned}$$

$$\begin{aligned}
-4B+C+D&=5\\
8B+2C-6D&=6 \quad /:2\\
12B+C+9D&=9
\end{aligned}$$

$$\begin{aligned}
-4B+C+D&=5 \quad (1)\\
4B+C-3D&=3 \quad (2)\\
12B+C+9D&=9 \quad (3)
\end{aligned}$$

$(1)-(3)$: $-16B-8D=-4$

$(2)-(3)$: $-8B-12D=-6$

$$\begin{aligned}
16B+8D&=4\\
8B+12D&=6 \quad /\cdot 2
\end{aligned}$$

$$\begin{aligned}
16B+8D&=4\\
-\ 16B+24D&=12
\end{aligned}$$

$-16D=-8$

$D=\frac{1}{2}$

$16B+8\cdot\frac{1}{2}=4$

$16B=0$

$B=0$

$C+D=5$

$C+\frac{1}{2}=5$

$C=\frac{10}{2}-\frac{1}{2}=\frac{9}{2}$

$A=-B=0$

$$\int\frac{5x^2+6x+9}{(x^2-2x-3)^2}\,dx=\frac{9}{2}\int\frac{dx}{(x-3)^2}+\frac{1}{2}\int\frac{dx}{(x+1)^2}=-\frac{9}{2}\cdot\frac{1}{x-3}-\frac{1}{2}\cdot\frac{1}{x+1}+C$$

$$=-\frac{9}{2(x-3)}-\frac{1}{2(x+1)}+C$$ traženo rješenje

(#) Izračunati integral $\displaystyle\int \frac{dx}{x(\sqrt{x}+3\sqrt[3]{x}-4)}$.

Rj.

$$I=\int \frac{dx}{x(\sqrt{x}+3\sqrt[3]{x}-4)} = \left| \begin{matrix} x=t^6 \\ dx=6t^5\,dt\end{matrix}\right| = \int \frac{6\cancel{t^5}\,dt}{t^{\cancel{6}\,1}(t^3+3t^2-4)} = 6\int\frac{dt}{t(t^3+3t^2-4)}$$

t^3+3t^2-4

$t=1:\ 1+3-4=0$

$$\begin{array}{l} (t^3+3t^2-4):(t-1)=t^2+4t+4 \\ \underline{-\,t^3-t^2} \\ \quad 4t^2-4 \\ \quad \underline{-\,4t^2-4t} \\ \qquad 4t-4 \\ \qquad \underline{-\,4t-4} \\ \qquad // \quad // \end{array}$$

$$(t^3+3t^2-4)=(t-1)(t^2+4t+4)$$
$$=(t-1)(t+2)^2$$

$$I=6\int\frac{dt}{t(t-1)(t+2)^2}$$

$$\frac{1}{t(t-1)(t+2)^2}=\frac{A}{t}+\frac{B}{t-1}+\frac{C}{(t+2)}+\frac{D}{(t+2)^2} \quad \Big/\, t(t-1)(t+2)^2$$

$$1=A(t-1)\overset{t^2+4t+4}{(t+2)^2}+B(t)\overset{t^2+4t+4}{(t+2)^2}+C\cdot t\cdot \overset{t^2+t-2}{(t-1)(t+2)}+D(t)(t-1)$$

$$1=A(t^3+4t^2+4t)+A(-t^2-4t-4)+B(t^3+4t^2+4t)+C(t^3+t^2-2t)+D(t^2-t)$$

t^3: $A+B+C=0$

t^2: $3A+4B+C+D=0$

t: $4B-2C-D=0$

t^0: $-4A=1 \Rightarrow A=-\frac{1}{4}$

$$\begin{array}{ll} B+C=\frac{1}{4} & /\cdot 2 \\ 4B+C+D=\frac{3}{4} & /\cdot 2 \\ \underline{4B-2C-D=0} & \end{array} \qquad \begin{array}{ll} 2B+2C=\frac{1}{2} & (a) \\ 8B+2C+2D=\frac{3}{2} & (b) \\ \underline{4B-2C-D=0} & (c) \end{array}$$

$(a)+(c)$: $6B-D=\frac{1}{2}$

$(b)+(c)$: $\underline{12B+D=\frac{3}{2}}$

$18B=2$

$B=\frac{1}{9}$

$D=\frac{6}{9}-\frac{1}{2}=\frac{2}{3}-\frac{1}{2}=\frac{4-3}{6}=\frac{1}{6}$

$C=\frac{1}{4}-B=\frac{1}{4}-\frac{1}{9}=\frac{5}{36}$

$$I=6\int\frac{dt}{t(t-1)(t+2)^2}=6\cdot\left(-\frac{1}{4}\right)\int\frac{dt}{t}+6\cdot\frac{1}{9}\int\frac{dt}{t-1}+6\cdot\frac{5}{36}\int\frac{dt}{t+2}+6\cdot\frac{1}{6}\int\frac{dt}{(t+2)^2}$$

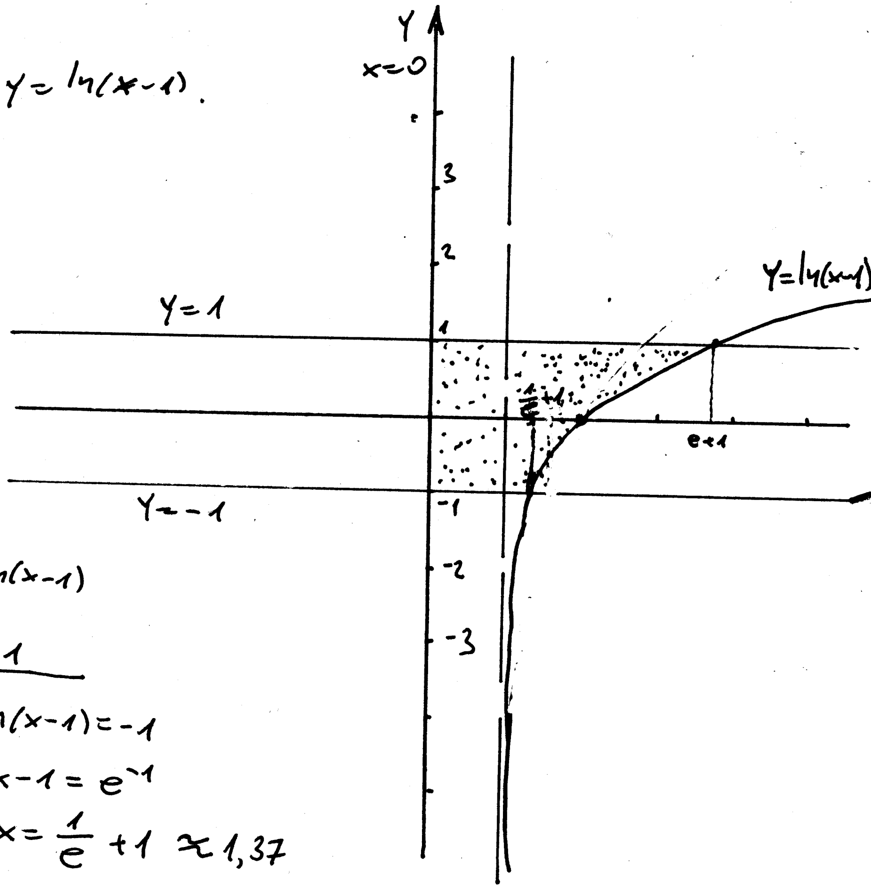
$$=-\frac{3}{2}\ln|t|+\frac{2}{3}\ln|t-1|+\frac{5}{6}\ln|t+2|-\frac{1}{t+2}$$ traženo rješenje

(#) Izračunati površinu figure određene linijama

$y = \ln(x-1)$, $y = 1$, $y = -1$, $x = 0$.

Rj. Kako grafički izgleda f-ja $y = \ln(x-1)$.

def. podr. $x - 1 > 0$
$x > 1$



$\ln(x-1) > 0$
$\ln(x-1) > \ln 1$
$x - 1 > 1$
$x > 2$

x	(1, 2)	(2, +∞)
Y	−	+

znak f-je

(2, 0) nula f-je.
ne siječe Y-osu

Pronađimo presječne tačke pravih $y = \ln(x-1)$, $y = 1$, $y = -1$

$\ln(x-1) = 1$
$x - 1 = e$
$x = e + 1$

$e + 1 \approx 3{,}72$

$y = \ln(x-1)$
$x - 1 = e^y$

$\ln(x-1) = -1$
$x - 1 = e^{-1}$
$x = \frac{1}{e} + 1 \approx 1{,}37$

$$P = \int_{-1}^{1} (e^y + 1)\,dy = e^y\Big|_{-1}^{1} + y\Big|_{-1}^{1} = e - e^{-1} + (1+1) = 2 + e + \frac{1}{e}$$ tražena površina

Izračunati površinu figure određene linijama

$$y=\frac{x}{x-2}, \quad x+y+1=0.$$

Rj. ispitajmo ukratko f-ju $y=\frac{x}{x-2}$

$D: x\in\mathbb{R}\setminus\{2\}$

(0,0) je presjek sa y-osom i nula

x	$(-\infty,0)$	$(0,2)$	$(2,+\infty)$
$x-2$	$-$	$-$	$+$
y	$+$	$-$	$+$

znak f-je

$$\lim_{x\to 2-0} f(x)=\frac{2-0}{2-0-2}=-\infty$$

$$\lim_{x\to 2+0} f(x)=\frac{2+0}{2+0-2}=+\infty$$

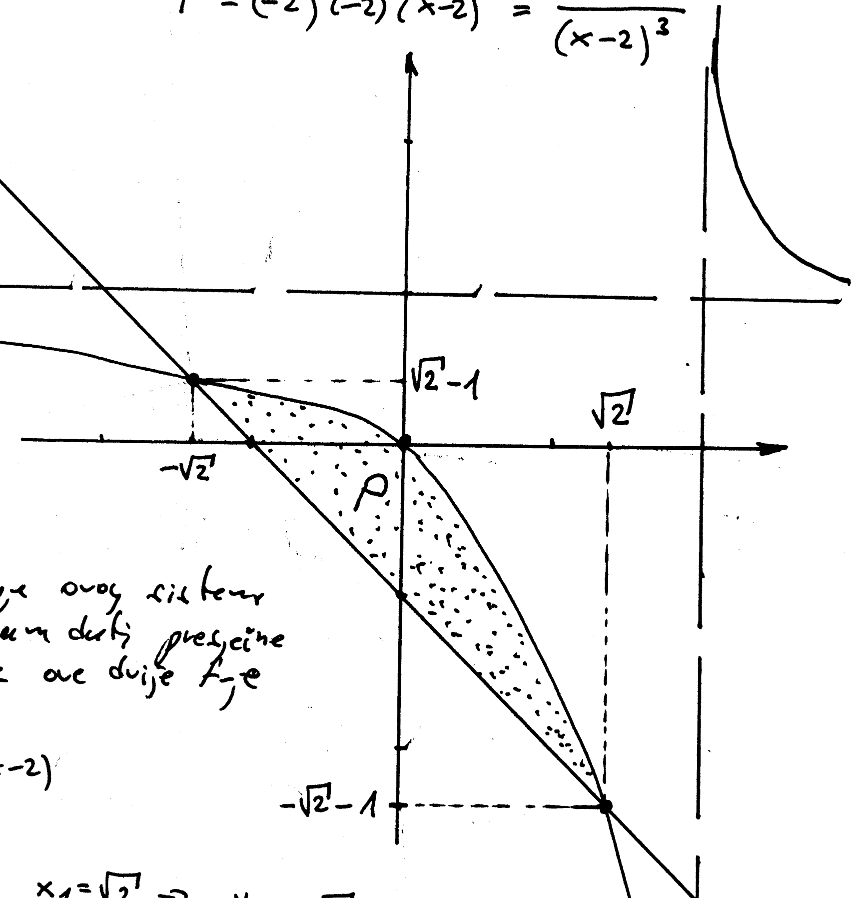
$x=2$ je V.A.

$$\lim_{x\to\pm\infty} f(x)=1 \Rightarrow y=1 \text{ je H.A.}$$

$$y'=\left(\frac{x}{x-2}\right)'=\frac{x-2-x}{(x-2)^2}=\frac{-2}{(x-2)^2}$$

$y'<0 \overset{\forall x\in D}{\Rightarrow}$ f-ja uvijek opada i nema ekstrema

$$y''=(-2)(-2)(x-2)^{-3}=\frac{4}{(x-2)^3}$$



$x+y+1=0 \Rightarrow x=-y-1$

$y=-x-1$

$x=0 \Rightarrow y=-1$

$y=0 \Rightarrow x=-1$

$$\left.\begin{aligned} y&=\frac{x}{x-2}\\ y&=-x-1\end{aligned}\right\}$$ rješenje ovog sistema će nam dati presječne tačke ove dvije f-je

$$-x-1=\frac{x}{x-2} \quad /\cdot(x-2)$$

$$(-1)(x+1)(x-2)=x$$

$$(-1)(x^2-x-2)=x$$

$$-x^2+x+2=x$$

$$-x^2+2=0$$

$$x_{1,2}=\pm\sqrt{2}\approx\pm 1{,}41$$

$x_1=\sqrt{2} \Rightarrow y_1=-\sqrt{2}-1\approx -2{,}41$

$x_2=\sqrt{2} \Rightarrow y_2=\sqrt{2}-1\approx 0{,}41$

$$\frac{x^2}{2}\Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$P=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{x}{x-2}-(-x-1)\right)dx=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{x}{x-2}+x+1\right)dx=2\sqrt{2}+2\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}+0+2\sqrt{2}$$

$$\int_{-\sqrt{2}}^{\sqrt{2}}\frac{\overset{-2+2}{x}}{x-2}\,dx=\int_{-\sqrt{2}}^{\sqrt{2}}\left(1+\frac{2}{x-2}\right)dx=x\Big|_{-\sqrt{2}}^{\sqrt{2}}+2\ln|x-2|\Big|_{-\sqrt{2}}^{\sqrt{2}}=\sqrt{2}+\sqrt{2}+2\left(\ln|\sqrt{2}-2|-\ln|-\sqrt{2}-2|\right)$$

$$P=4\sqrt{2}+2\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}$$ tražena površina

(#) Nađi ekstreme f-je $z=\ln(x^2+2xy+3y^2-4x-5y+6)$

R:

$$\frac{\partial z}{\partial x}=\frac{2x+2y-4}{x^2+2xy+3y^2-4x-5y+6}$$

$$\frac{\partial z}{\partial y}=\frac{2x+6y-5}{x^2+2xy+3y^2-4x-5y+6}$$

$$\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial y}=0$$

$$2x+2y-4=0$$

$$-\quad 2x+6y-5=0$$

$$-4y+1=0$$

$$y=\frac{1}{4}$$

$$2x+2y-4=0$$

$$2x=4-2y$$

$$x=2-y$$

$$x=2-\frac{1}{4}$$

$$x=\frac{7}{4}$$

Stacionarna tačka je $M(\frac{7}{4},\frac{1}{4})$.

$$\frac{\partial^2 z}{\partial x^2}=\frac{2(x^2+2xy+3y^2-4x-5y+6)-(2x+2y-4)(2x+2y-4)}{(x^2+2xy+3y^2-4x-5y+6)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{2(x^2+2xy+3y^2-4x-5y+6)-(2x+2y-4)\cdot(2x+6y-5)}{(x^2+2xy+3y^2-4x-5y+6)^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{6(x^2+2xy+3y^2-4x-5y+6)-(2x+6y-5)(2x+6y-5)}{(x^2+2xy+3y^2-4x-5y+6)^2}$$

Za $x=\frac{7}{4}$, $y=\frac{1}{4}$ imamo

$$x^2+2xy+3y^2-4x-5y+6=\frac{49}{16}+\frac{14}{16}+\frac{3}{16}-\frac{28}{4}-\frac{5}{4}+\frac{24}{4}=\frac{15}{8}$$

$$(x^2+2xy+3y^2-4x-5y+6)^2=\frac{255}{64}=\left(\frac{15}{8}\right)^2$$

$$2x+2y-4=\frac{7}{2}+\frac{1}{2}-\frac{8}{2}=0,\qquad 2x+6y-5=0$$

Za $M(\frac{7}{4},\frac{1}{4})$, $A=\frac{2\cdot\cancel{\frac{15}{8}}}{(\frac{15}{8})^{\cancel{2}}}=\frac{16}{15}$, $B=\frac{2\cdot\cancel{\frac{15}{8}}}{(\frac{15}{8})^{\cancel{2}}}=\frac{16}{15}$, $C=\frac{6\cdot\cancel{\frac{15}{8}}}{(\frac{15}{8})^{\cancel{2}}}=\frac{48}{15}$

$$D=AC-B^2=\frac{16}{15}\cdot\frac{48}{15}-\left(\frac{16}{15}\right)^2=\frac{768-256}{255}=\frac{512}{255}>0$$

f-ja ima ekstrem u tački $M(\frac{7}{4},\frac{1}{4})$, $A>0$ f-ja ima minimum

$$z_{min}\left(\frac{7}{4},\frac{1}{4}\right)=\ln\frac{15}{8}\approx 0,63$$

(#) Naći ekstreme f-je $z = x^3 + y^3 - 63(x+y) + 12xy$.

Rj.

$$\frac{\partial z}{\partial x} = 3x^2 - 63 + 12y$$

$$\frac{\partial z}{\partial y} = 3y^2 - 63 + 12x$$

$$\begin{aligned} 3x^2 + 12y &= 63 \quad /:3 \\ 3y^2 + 12x &= 63 \quad /:3 \end{aligned}$$

$$\begin{aligned} x^2 + 4y &= 21 \\ - \quad y^2 + 4x &= 21 \end{aligned}$$

$$x^2 - y^2 + 4(y - x) = 0$$

$$(x-y)(x+y) - 4(x-y) = 0$$

$$(x-y)(x+y-4) = 0$$

$x = y$ ili $x = 4 - y$

$x = y$:
$$x^2 + 4x - 21 = 0$$
$$(x-3)(x+7) = 0$$
$$x_1 = y_1 = 3$$
$$x_2 = y_2 = -7$$

$x = 4 - y$
$$y^2 + 4(4-y) - 21 = 0$$
$$y^2 - 4y - 5 = 0$$
$$(y+1)(y-5) = 0$$

$y_3 = -1 \Rightarrow x_3 = 5$

$y_4 = 5 \Rightarrow x_4 = -1$

Stacionarne tačke su $M_1(3,3)$, $M_2(-7,-7)$, $M_3(5,-1)$ i $M_4(-1,5)$.

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = 12$$

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

Za $M_1(3,3)$, $A = 18$, $B = 12$, $C = 18$

$D = AC - B^2 = 18^2 - 12^2 > 0$ f-ja u M_1 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(3,3) = 27 + 27 - 63 \cdot 6 + 12 \cdot 9 = -216$

Za $M_2(-7,-7)$, $A = -42$, $B = 12$, $C = -42$, $D = AC - B^2 = 42^2 - 12^2 > 0$

$\Rightarrow$ f-ja u M_2 ima ekstrem, a kako je $A < 0$ f-ja ima maksimum

$z_{max}(-7,-7) = -343 - 343 - 63 \cdot (-14) + 12 \cdot 49 = 784$

Za $M_3(5,-1)$, $A = 30$, $B = 12$, $C = -6$, $D = AC - B^2 = -180 + 144 < 0$

$\Rightarrow$ f-ja u tački M_3 nema ekstrem

Za $M_4(-1,5)$, $A = -6$, $B = 12$, $C = 30$, $D = AC - B^2 = -180 + 144 < 0$

$\Rightarrow$ f-ja u tački M_4 nema ekstrem

(#) Riješiti diferencijalnu jednačinu $y' = \dfrac{x^2+8}{(x^2-5x+6)y^2\cos y}$.

Rj. $y' = \dfrac{x^2+8}{(x^2-5x+6)\,y^2\cos y} \quad / \cdot y^2\cos y$

$$y^2\cos y \frac{dy}{dx} = \frac{x^2+8}{x^2-5x+6} \quad / \cdot dx$$

$$y^2\cos y\, dy = \frac{x^2+8}{x^2-5x+6}dx$$ diferencijalna jednačina sa razdvojenim promjenjiv.

$$\int y^2\cos y\, dy = \int \frac{x^2+8}{x^2-5x+6}dx$$

$$\int y^2\cos y\, dy = \begin{vmatrix} u=y^2 & dv=\cos y\, dy \\ du=2y\,dy & v=\sin y \end{vmatrix} = y^2\sin y - 2\int y\sin y\, dy = \begin{vmatrix} u=y & dv=\sin y\, dy \\ du=dy & v=-\cos y \end{vmatrix}$$

$$= y^2\sin y - 2\left(y\cos y + \int \cos y\, dy\right) = y^2\sin y - 2y\cos y + 2\sin y + C$$

$$\int \frac{x^2+8}{x^2-5x+6}dx = \int\left(1+\frac{5x+2}{x^2-5x+6}\right)dx \overset{(*)}{=} \int\left(1+2\cdot\frac{2x-5}{x^2-5x+6}+\frac{15}{x-3}-\frac{14}{x-2}\right)dx$$

$(x^2-5x+6)' = 2x-5$

$x^2-5x+6 = (x-3)(x-2)$

$$(x^2+8):(x^2-5x+6) = 1$$
$$\underline{-\ x^2-5x+6}$$
$$= 5x+2$$

$5x+2 = 2x-5+3x+7$

$2(2x-5)+x+12$

$$\frac{5x+2}{x^2-5x+6} = 2\cdot\frac{2x-5}{x^2-5x+6} + \frac{x+12}{x^2-5x+6}$$

$$\frac{x+12}{x^2-5x+6} = \frac{A}{x-3}+\frac{B}{x-2} \quad /(x-3)(x-2)$$

$$x+12 = A(x-2)+B(x-3)$$

$A+B=1 \quad /\cdot 2$

$-2A-3B=12$

$$\begin{aligned} &2A+2B=2 \\ +\ &\underline{-2A-3B=12} \\ &-B=14 \\ &B=-14 \end{aligned}$$

$A = 1-B$

$A = 15 \quad \ldots(*)$

$$\int\frac{x^2+8}{x^2-5x+6}dx = x + 2\ln|x^2-5x+6| + 15\ln|x-3| - 14\ln|x-2| + C$$

Mogao sam odmah rastaviti $\dfrac{5x+2}{x^2-5x+6} = \dfrac{17}{x-3} + \dfrac{(-12)}{x-2} \Rightarrow$

$$\Rightarrow \int\frac{x^2+8}{x^2-5x+6}dx = x + 17\ln|x-3| - 12\ln|x-2|$$

$$y^2\sin y - 2y\cos y + 2\sin y = x + 17\ln|x-3| - 12\ln|x-2| + C$$ rješenje diferenc. jednačine

(#) Riješiti diferencijalnu jednačinu $(2x+y+5)y' = 3x+6$.

Rj.

$$(2x+y+5)\,y' = 3x+6$$

$$y' = \frac{3x+6}{2x+y+5}$$ ovo je diferencijalna jednačina koja se svodi na homogenu $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

Kako je $a_1b_2 - b_1a_2 = 3-0 = 3 \neq 0$ uvodimo smjenu $x = u+\alpha$, $y = v+\beta$

gdje ćemo α i β izračunati iz sistema

$$\begin{aligned} 3\alpha + 6 &= 0 \\ 2\alpha + \beta + 5 &= 0 \\ \hline \alpha = -\frac{6}{3} &= -2 \end{aligned}$$

$$2\cdot(-2) + \beta + 5 = 0$$
$$\beta + 1 = 0$$
$$\beta = -1$$

Uvodimo smjenu $x = u-2$, $y = v-1$

$$y' = \frac{3(u-2)+6}{2(u-2)+v-1+5}$$

$$y' = \frac{3u}{2u+v}, \quad y' = v'$$

$$v' = \frac{3u}{2u+v} \quad \begin{matrix} /:u \\ /:u \end{matrix}$$

$$v' = \frac{3}{2+\frac{v}{u}}$$ ovo je homogena diferencijalna jednačina $y' = f\left(\frac{y}{x}\right)$

uvodimo smjenu $\frac{v}{u} = z$

$$v = uz, \quad v' = z'u + z$$

$$z'u + z = \frac{3}{2+z}$$

$$z'u = \frac{3}{2+z} - z$$

$$z'u = \frac{3 - z(2+z)}{2+z}$$

$$z'u = \frac{-z^2-2z+3}{z+2}$$

$$\frac{dz}{du}u = \frac{-z^2-2z+3}{z+2}$$

$$\frac{z+2}{-z^2-2z+3}\,dz = \frac{du}{u}$$

$$(-1)\cdot\frac{z+2}{z^2+2z-3}\,dz = \frac{du}{u} \quad \ldots(*)$$

$$(-1)\cdot\frac{z+2}{(z-1)(z+3)}\,dz = \frac{du}{u}$$

$$\frac{z+2}{(z-1)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3} \quad /(z-1)(z+3)$$

$$z+2 = A(z+3) + B(z-1)$$

$$\begin{aligned} A + B &= 1 \\ + \quad 3A - B &= 2 \\ \hline 4A &= 3 \\ A &= \frac{3}{4} \end{aligned}$$

$$B = \frac{1}{4}$$

$$(*) \Rightarrow (-1)\int \frac{z+2}{z^2+2z-3}\,dz = \int \frac{du}{u} \quad /\cdot(-1)$$

$$\frac{3}{4}\ln|z-1| + \frac{1}{4}\ln|z+3| = -\ln u$$

$$\ln|z-1|^{\frac{3}{4}} + \ln|z+3|^{\frac{1}{4}} = \ln u^{-1}$$

$$\ln \sqrt[4]{(z-1)^3 \cdot (z+3)} = \ln \frac{1}{u}$$

$$\sqrt[4]{\left(\frac{v}{u}-1\right)^3 \cdot \left(\frac{v}{u}+3\right)} = \frac{1}{u}$$

$$x = u-2 \Rightarrow u = x+2$$

$$y = v-1 \Rightarrow v = y+1$$

$$\left(\frac{y+1}{x+2}-1\right)^3 \left(\frac{y+1}{x+2}\right) = \frac{1}{(x+2)^4}$$

rješenje diferencijalne jednačine