Univerzitet u Zenici
Ekonomski fakultet
Odsjek: Menadžment preduzeća, Računovodstveni i revizijski menadžment
Zenica, 16.02.2010.

Pismeni ispit iz predmeta Matematika

1. Dokazati matematičkom indukcijomtvrdnju $7|(n^7 - n)$, gdje je $n \in \mathbb{N}$.

2. Odrediti član u razvoju binoma $\left(\sqrt[3]{\left(\frac{a}{b}\right)^2} + \frac{\sqrt[4]{b}}{\sqrt[8]{a^3}}\right)^{35}$ koji sadrži b^6.

3. Izračunati: $(1 - \frac{\sqrt{3} - i}{2})^{24}(2 + \sqrt{3})^{12}$.

4. Riješiti matričnu jednačinu $(XA + B)^{-1}(XC + B) = C$, ako su $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$,
$B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i $C = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

5. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra:

$$\begin{aligned} x + y + bz &= 1 - b \\ x - by - z &= 2 \\ bx - y + z &= 2b \ . \end{aligned}$$

6. Dati su vektori u četverodimenzionalnom vektorskom prostoru: $\vec{a} = (1, 1, 2, 3)$, $\vec{b} = (1, 2 - x^2, 2, 3)$, $\vec{c} = (2, 3, 1, 5)$, $\vec{d} = (2, 3, 1, 9 - x^2)$. Odrediti x tako da ti vektori budu linearno zavisni.

7. Ispitati funkciju i nacrtati joj grafik: $y = \frac{x^2}{2} + 8x^{-2}$.

8. Ispitati funkciju i nacrtati joj grafik: $y = \frac{x^3 + 1}{2x^2 - 2}$.

9. Ispitati funkciju i nacrtati joj grafik: $y = e^{\frac{1}{x^2 - 4x + 3}}$.

10. Ispitati funkciju i nacrtati joj grafik: $y = \frac{x^2 - 1}{e^{x^2}}$.

11. Ispitati funkciju i nacrtati joj grafik: $y = (x - 1)ln^2(x - 1)$.

12. Ispitati funkciju i nacrtati joj grafik: $y = \frac{3lnx - 5}{x^2}$.

13. Izračunati integral $\int \sqrt{\frac{x-2}{x+2}}\,dx$.

14. Izračunati integral $\int \frac{1-sin\,x+cos\,x}{1+sin\,x-cos\,x}\,dx$.

15. Izračunati integral $\int_0^{\frac{\pi}{4}} sin^5 x\,cos^7 x\,dx$.

16. Izračunati integral $\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$.

17. Izračunati površinu figure koja je određena linijama $y=\sqrt{x}$, $y=1$, $y=10-2x$.

18. Izračunati površinu figure koja je određena linijama $y=\frac{3}{x-2}$, $x+y=6$.

19. Naći ekstreme funkcije $z=x^2-2x-y-ln(2-y)+4$.

20. Naći ekstreme funkcije $z=(x^2+y)\sqrt{e^y}$.

21. Naći uslovne ekstreme funkcije $z=xy$, ako je $x^2+y^2=2ax$, $a>0$.

22. Riješiti diferencijalnu jednačinu $(x^2+2x-2y)\,dx-dy=0$.

23. Riješiti diferencijalnu jednačinu $y'-\frac{xy}{1+x^2}=\frac{x\sqrt{1+x^2}}{x^2-2x+2}$.

24. Riješiti diferencijalnu jednačinu $2y-2xy'=a(\sqrt{1+(y')^2}-y')$.

Rješeni zadaci su skinuti sa stranice **pf.unze.ba\nabokov**.
Za uočene greške pisati na **infoarrt@gmail.com**.

(#) Dokazati matematičkom indukcijom tvrdnju

$7 \mid (n^7 - n)$, $n \in \mathbb{N}$.

R: BAZA INDUKCIJE

Dokažimo da je tvrdnja tačna za broj 1.

$n=1$: $n^7 - n = 1^7 - 1 = 0$, $\quad 7 \mid 0$ (7 dijeli 0)

$0 = 7 \cdot 0$ Tvrdnja je tačna za broj 1.

KORAK INDUKCIJE

Pretpostavimo da je tvrdnja tačna za brojeve od 1 do n tj. $7 \mid (k^7 - k)$ za $k = 1, 2, 3, \dots, n-1, n$. Na osnovu ove pretpostavke dokažimo da je tvrdnja tačna za $n+1$ tj: da $7 \mid [(n+1)^7 - (n+1)]$.

$$n^7 - n = n(n^6 - 1) = n(n^3-1)(n^3+1) = \underline{\underline{n}}(n-1)\underline{(n^2+n+1)}\underline{\underline{(n+1)}}(n^2-n+1)$$

$$(n+1)^7 - (n+1) = (n+1)\left[(n+1)^6 - 1\right] = (n+1)\left[(n+1)^3-1\right]\left[(n+1)^3+1\right] =$$

$$= (n+1)\left[(n+1)-1\right]\left[(n+1)^2+n+1+1\right]\left[(n+1)+1\right]\left[(n+1)^2-(n+1)+1\right]$$

$$= \underline{\underline{(n+1)}}\,\underline{\underline{n}}\,(n^2+3n+3)(n+2)\underline{(n^2+n+1)}$$

Pronađimo vezu između $(n-1)(n^2-n+1)$ i $(n^2+3n+3)(n+2)$

$$\left.\begin{aligned} (n-1)(n^2-n+1) &= n^3 - n^2 + n - n^2 + n - 1 = n^3 - 2n^2 + 2n - 1 \\ (n+2)(n^2+3n+3) &= n^3 + \underline{3n^2} + \underline{\underline{3n}} + \underline{2n^2} + \underline{\underline{6n}} + 6 = n^3 + 5n^2 + 9n + 6 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (n+2)(n^2+3n+3) = (n-1)(n^2-n+1) - 7n^2 - 7n - 7$$

pa imamo: $(n+1)^7 - (n+1) = (n+1)n(n^2+n+1)\left[(n-1)(n^2-n+1) - 7(n^2+n+1)\right]$

$$= (n+1)n(n^2+n+1)(n-1)(n^2-n+1) - 7(n+1)n(n^2+n+1)^2$$

$$= \underbrace{(n^7-n)}_{A} - \underbrace{7n(n+1)(n^2+n+1)^2}_{B}$$

$\left.\begin{array}{l} A \text{ je prema pretpostavci djeljivo sa } 7 \\ B \text{ je očigledno djeljivo sa } 7 \end{array}\right\} \Rightarrow$ $(n+1)^7 - (n+1)$ je djeljivo sa 7 tj. $7 \mid (n+1)^7 - (n+1)$

ZAKLJUČAK

Tvrdnja $7 \mid (n^7 - n)$ je tačna za sve prirodne brojeve

(#) Odrediti član u razvoju binoma $\left(\sqrt[3]{\left(\frac{a}{b}\right)^2}+\frac{\sqrt[4]{b}}{\sqrt[8]{a^3}}\right)^{35}$ koji sadrži b^6.

Rj. $\left(\sqrt[3]{\left(\frac{a}{b}\right)^2}+\frac{\sqrt[4]{b}}{\sqrt[8]{a^3}}\right)^{35}=\left(\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}+\frac{b^{\frac{1}{4}}}{a^{\frac{3}{8}}}\right)^{35}=\left(a^{\frac{2}{3}}b^{-\frac{2}{3}}+a^{-\frac{3}{8}}b^{\frac{1}{4}}\right)^{35}$

$$=\sum_{k=0}^{35}\binom{35}{k}\left(a^{\frac{2}{3}}b^{-\frac{2}{3}}\right)^{35-k}\cdot\left(a^{-\frac{3}{8}}b^{\frac{1}{4}}\right)^{k}.$$

Napisani izraz će sadržavati b^6 ako i samo ako je $\left(b^{-\frac{2}{3}}\right)^{35-k}\cdot b^{\frac{k}{4}}=b^6$ tj. $b^{\frac{-70+2k}{3}}\cdot b^{\frac{k}{4}}=b^6$

$$\Rightarrow b^{\frac{-70+2k}{3}+\frac{k}{4}}=b^6 \Rightarrow \frac{-70+2k}{3}+\frac{k}{4}=6 \quad /\cdot 12$$

$$-280+8k+3k=72$$

$$11k=352$$

$$k=32$$

$\Rightarrow$ Tridesettreći član u razvoju binoma sadrži b^6.

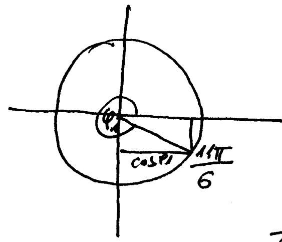
\# Izračunati $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}(2+\sqrt{3})^{12}$.

Rj: Označimo sa $z_1=\sqrt{3}-i$. Tada $|z_1|=\sqrt{3+1}=2$

$\cos\varphi_1=\frac{\sqrt{3}}{2}$ $\left(=\frac{a}{|z|}\right)$

$\sin\varphi_1=-\frac{1}{2}$ $\left(=\frac{b}{|z|}\right)$

$tg\varphi_1=\frac{b}{a}=-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$



$tg\frac{\pi}{6}=30°$

$\Rightarrow \varphi_1=\frac{11\pi}{6}=-\frac{\pi}{6}$

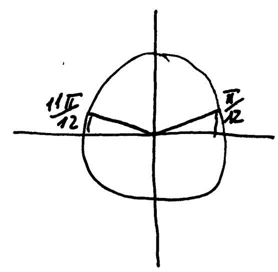
$z_1=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

$\left(1-\frac{z_1}{2}\right)=\left(1-\cos\frac{11\pi}{6}-i\sin\frac{11\pi}{6}\right)$ Znamo da je $\cos 2x=\cos^2x-\sin^2x$

$\sin 2x=2\sin x\cos x$

$$\left.\begin{array}{l}1=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\\ \cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\end{array}\right\}\Rightarrow 1-\cos x=2\sin^2\frac{x}{2}$$

$1-\cos\frac{11\pi}{6}=2\sin^2\frac{11\pi}{12}$



$\sin\frac{11\pi}{6}=2\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}$

$\left(1-\frac{1}{2}z_1\right)=\left(1-\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\left(2\sin^2\frac{11\pi}{12}-2i\sin\frac{11\pi}{12}\cos\frac{11\pi}{12}\right)=$

$=2\sin\frac{11\pi}{12}\left(\sin\frac{11\pi}{12}-i\cos\frac{11\pi}{12}\right)=2i\sin\frac{11\pi}{12}\left(-\cos\frac{11\pi}{12}-i\sin\frac{11\pi}{12}\right)=$

$=-2i\sin\frac{11\pi}{12}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$

$\sin\frac{11\pi}{12}=\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=$

$=\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{2}(\sqrt{3}-1)}{4}$, $(\sqrt{3}-1)^2=3-2\sqrt{3}+1=4-2\sqrt{3}=2(2-\sqrt{3})$

$\sin^2\frac{11\pi}{12}=\sin^2\frac{\pi}{12}=\frac{2(\sqrt{3}-1)^2}{16}=\frac{2(2-\sqrt{3})}{8}=\frac{2-\sqrt{3}}{4}$, $i^{24}=(i^2)^{12}=(-1)^{12}=1$

$\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}(2+\sqrt{3})^{12}=(-2i)^{24}\left(\sin\frac{11\pi}{12}\right)^{24}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)^{24}\cdot(2+\sqrt{3})^{12}$

$=(-2)^{24}\left(\sin^2\frac{11\pi}{12}\right)^{12}\left(\cos 24\cdot\frac{11\pi}{12}+i\sin 24\cdot\frac{11\pi}{12}\right)\cdot(2+\sqrt{3})^{12}=\cancel{2^{24}}\cdot\frac{(2-\sqrt{3})^{12}}{\cancel{2^{24}}}\cdot$

$\cdot(\cos 22\pi+i\sin 22\pi)\cdot(2+\sqrt{3})^{12}=(4-3)^{12}\cdot 1=1$ traženo rješenje

\# Riješiti matričnu jednačinu $(XA+B)^{-1}(XC+B)=C$, ako je $A=\begin{bmatrix}-1&2&3\\0&2&-1\\0&0&3\end{bmatrix}$, $B=\begin{bmatrix}2&-1&1\\0&-1&1\\0&0&1\end{bmatrix}$ i $C=\begin{bmatrix}-1&1&2\\0&-1&2\\0&0&2\end{bmatrix}$

Rj. $(XA+B)^{-1}(XC+B)=C \quad /(XA+B)$ sa lijeve strane

$$\underbrace{(XA+B)(XA+B)^{-1}}_{I}(XC+B)=(XA+B)\cdot C$$

$$XC+B=XAC+BC$$

$$X=B(C-I)(C-AC)^{-1}$$

$$XC-XAC=BC-B$$

$$C-I=\begin{bmatrix}-2&1&2\\0&-2&2\\0&0&1\end{bmatrix}$$

$$X(C-AC)=BC-B \quad /(C-AC)^{-1} \text{ sa desne strane}$$

$$B(C-I)=\begin{bmatrix}2&-1&1\\0&-1&1\\0&0&1\end{bmatrix}\begin{bmatrix}-2&1&2\\0&-2&2\\0&0&1\end{bmatrix}=\begin{bmatrix}-4&4&3\\0&2&-1\\0&0&1\end{bmatrix}$$

$$AC=\begin{bmatrix}-1&2&3\\0&2&-1\\0&0&3\end{bmatrix}\begin{bmatrix}-1&1&2\\0&-1&2\\0&0&2\end{bmatrix}=\begin{bmatrix}1&-3&8\\0&-2&2\\0&0&6\end{bmatrix}$$

Označimo sa

$$D=C-AC=\begin{bmatrix}-2&4&-6\\0&1&0\\0&0&-4\end{bmatrix}$$

Izračunajmo D^{-1}!

$$D^{-1}=\frac{1}{\det D}D_{kof}^{T}$$

$D_{11}=(-1)^2\begin{vmatrix}1&0\\0&-4\end{vmatrix}=-4$ $\quad D_{21}=(-1)^3\begin{vmatrix}4&-6\\0&-4\end{vmatrix}=16$ $\quad D_{31}=(-1)^4\begin{vmatrix}4&-6\\1&0\end{vmatrix}=6$

$D_{12}=(-1)^3\begin{vmatrix}0&0\\0&-4\end{vmatrix}=0$ $\quad D_{22}=(-1)^4\begin{vmatrix}-2&-6\\0&-4\end{vmatrix}=8$ $\quad D_{32}=(-1)^5\begin{vmatrix}-2&-6\\0&0\end{vmatrix}=0$

$D_{13}=(-1)^4\begin{vmatrix}0&1\\0&0\end{vmatrix}=0$ $\quad D_{23}=(-1)^5\begin{vmatrix}-2&4\\0&0\end{vmatrix}=0$ $\quad D_{33}=(-1)^6\begin{vmatrix}-2&4\\0&1\end{vmatrix}=-2$

$$\det D=\begin{vmatrix}-2&4&-6\\0&1&0\\0&0&-4\end{vmatrix}=(-4)\begin{vmatrix}-2&4\\0&1\end{vmatrix}=8 \qquad D_{kof}=\begin{bmatrix}-4&0&0\\16&8&0\\6&0&-2\end{bmatrix} \qquad D_{kof}^{T}=\begin{bmatrix}-4&16&6\\0&8&0\\0&0&-2\end{bmatrix}$$

$$D^{-1}=\begin{bmatrix}-\frac{1}{2}&2&\frac{3}{4}\\0&1&0\\0&0&-\frac{1}{4}\end{bmatrix}, \quad X=B(C-I)(C-AC)^{-1}=\frac{1}{8}\begin{bmatrix}-4&4&3\\0&2&-1\\0&0&1\end{bmatrix}\begin{bmatrix}-4&16&6\\0&8&0\\0&0&-2\end{bmatrix}$$

$$X=\frac{1}{8}\begin{bmatrix}16&-32&-30\\0&16&2\\0&0&-2\end{bmatrix}=\begin{bmatrix}2&-4&-\frac{15}{4}\\0&2&\frac{1}{4}\\0&0&-\frac{1}{4}\end{bmatrix} \text{ traženo rješenje}$$

Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra

$$\begin{aligned} x+y+bz&=1-b\\ x-by-z&=2\\ bx-y+z&=2b \end{aligned}$$

Rj. Rješavamo sistem Cramerovom metodom

$$D=\begin{vmatrix}1&1&b\\1&-b&-1\\b&-1&1\end{vmatrix}\overset{I_k+III_k}{=}\begin{vmatrix}b+1&1&b\\0&-b&-1\\b+1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}1&1&b\\0&-b&-1\\1&-1&1\end{vmatrix}\overset{I_v-III_v}{=}$$

$$=(b+1)\begin{vmatrix}0&2&b-1\\0&-b&-1\\1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}2&b-1\\-b&-1\end{vmatrix}=(b+1)\Big[\overbrace{-2+(b^2-b)}^{b^2-b-2}\Big]=$$

$$=(b+1)(b+1)(b-2)$$

$$D_x=\begin{vmatrix}1-b&1&b\\2&-b&-1\\2b&-1&1\end{vmatrix}\overset{I_v+III_v}{=}\begin{vmatrix}b+1&0&b+1\\2&-b&-1\\2b&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}1&0&1\\2&-b&-1\\2b&-1&1\end{vmatrix}=$$

$$\overset{I_k-III_k}{=}(b+1)\begin{vmatrix}0&0&1\\3&-b&-1\\2b-1&-1&1\end{vmatrix}=(b+1)\begin{vmatrix}3&-b\\2b-1&-1\end{vmatrix}=(b+1)\big(\overbrace{-3+2b^2-b}^{2b^2-b-3}\big)=$$

$D=1+24=25$, $b_{1,2}=\frac{1\pm5}{4}$

$$=(b+1)\cdot2\left(b-\frac{3}{2}\right)(b+1)$$

$$=(b+1)(2b-3)(b+1)$$

$$D_y=\begin{vmatrix}1&1-b&b\\1&2&-1\\b&2b&1\end{vmatrix}\overset{I_k+III_k}{=}\begin{vmatrix}b+1&1-b&b\\0&2&-1\\b+1&2b&1\end{vmatrix}=(b+1)\begin{vmatrix}1&1-b&b\\0&2&-1\\1&2b&1\end{vmatrix}\overset{III_v-I_v}{=}$$

$$=(b+1)\begin{vmatrix}1&1-b&b\\0&2&-1\\0&3b-1&1-b\end{vmatrix}=(b+1)\begin{vmatrix}2&-1\\3b-1&1-b\end{vmatrix}=(b+1)(2-2b+3b-1)=$$

$$=(b+1)(b+1)$$

$$D_z=\begin{vmatrix}1&1&1-b\\1&-b&2\\b&-1&2b\end{vmatrix}\overset{I_v+III_v}{=}\begin{vmatrix}b+1&0&b+1\\1&-b&2\\b&-1&2b\end{vmatrix}=(b+1)\begin{vmatrix}1&0&1\\1&-b&2\\b&-1&2b\end{vmatrix}\overset{I_k-III_k}{=}$$

$$=(b+1)\cdot\begin{vmatrix}0&0&1\\-1&-b&2\\-b&-1&2b\end{vmatrix}=(b+1)\begin{vmatrix}-1&-b\\-b&-1\end{vmatrix}=(b+1)(1-b^2)=-(b+1)(b^2-1)$$

$$=-(b+1)(b-1)(b+1)$$

Diskusija: a) $D\neq0$ tj. $b\neq-1$; $b\neq2$

sistem ima jedinstveno rješenje $x=\frac{D_x}{D}=\frac{(2b-3)(b+1)^2}{(b+1)^2(b-2)}=\frac{2b-3}{b-2}$

$$y=\frac{D_y}{D}=\frac{(b+1)^2}{(b+1)^2(b-2)}=\frac{1}{b-2}\quad;\quad z=\frac{D_z}{D}=\frac{-(b-1)(b+1)^2}{(b-2)(b+1)^2}=-\frac{b-1}{b-2}$$

b) $b=-1 \Rightarrow D=D_x=D_y=D_z=0$ sistem trebamo riješiti na drugi način

Za $b=-1$ sistem postaje

$$\begin{aligned} x+y-z&=2 \\ x+y-z&=2 \\ -x-y+z&=-2 \quad / \cdot(-1) \end{aligned}$$

Sve tri jednačine su iste $\Rightarrow$ Sistem ima ∞ mnogo rješenja. Ako uzmemo $x=t,\ y=s$ Rješenja sistema su

$(t,\ s,\ t+s-2)$

dvije promjenjive uzimamo proizvoljno

c) $b=2 \Rightarrow D=0,\ D_x=9\neq 0 \Rightarrow$

Sistem za $b=2$ nema rješenja

(#) Dati su vektori u četverodimenzionalnom vektorskom prostoru: $\vec{a}=(1,1,2,3)$, $\vec{b}=(1,2-x^2,2,3)$, $\vec{c}=(2,3,1,5)$, $\vec{d}=(2,3,1,9-x^2)$. Odrediti x tako da bi vektori budu linearno zavisni.

Rj. Vektori $\vec{a}, \vec{b}, \vec{c}$ i $\vec{d}$ su linearno nezavisni ako postoji bar jedan skalar α, β, γ ili δ različit od nule takav da važi $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}+\delta\vec{d}=\vec{0}$

$$\alpha(1,1,2,3)+\beta(1,2-x^2,2,3)+\gamma(2,3,1,5)+\delta(2,3,1,9-x^2)=(0,0,0,0)$$

$$\begin{aligned}
\alpha + \beta + 2\gamma + 2\delta &= 0 \quad (a)\\
\alpha + (2-x^2)\beta + 3\gamma + 3\delta &= 0 \quad (b)\\
2\alpha + 2\beta + \gamma + \delta &= 0 \quad (c)\\
3\alpha + 3\beta + 5\gamma + (9-x^2)\delta &= 0 \quad (d)
\end{aligned}$$

$$\begin{aligned}
(b)-(a):&\quad (1-x^2)\beta + \gamma + \delta = 0 \quad (I)\\
(c)-2(a):&\quad -3\gamma - 3\delta = 0 \quad (II)\\
(d)-3(a):&\quad -\gamma + (3-x^2)\delta = 0 \quad (III)
\end{aligned}$$

$(II)-3\cdot(III)$:

$$-3\delta-3(3-x^2)\delta=0$$

$$(-12+3x^2)\delta=0$$

kako je $\delta\neq 0$ to je

$$-12+3x^2=0$$

$$3x^2=12$$

$$x^2=4$$

$$x_1=-2 \quad x_2=2$$

Za $x=\pm 2$ početni sistem jednačina ima beskonačno mnogo rešenja tj. za $x=\pm 2$ vektori $\vec{a}, \vec{b}, \vec{c}$ i $\vec{d}$ su linearno zavisni.

Ispitati f-ju i nacrtati joj grafik $y=\frac{x^3+1}{2x^2-2}$.

Rj: $y=\frac{x^3+1}{2x^2-2}=\frac{\cancel{(x+1)}(x^2-x+1)}{2(x-1)\cancel{(x+1)}}=\frac{x^2-x+1}{2(x-1)}=\frac{1}{2}\cdot\frac{x^2-x+1}{x-1}$

definiciono područje

$x-1\neq 0$

$x\neq 1$

$D:\ x\in(-\infty,1)\cup(1,+\infty)$

parnost (neparnost), periodičnost

D nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $x^2-x+1=0$

Kako $x^2-x+1>0\ \forall x\in\mathbb{R}$ to f-ja nema nulu.

$f(0)=\frac{1}{2}\cdot\frac{1}{-1}=-\frac{1}{2}$ $\left(0,-\frac{1}{2}\right)$ je presjek sa y-osom

$y>0$ akko $x-1>0$ tj. $x>1$

$y<0$ akko $x<1$ ← znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ f-ja ima prekid

$$\lim_{x\to 1-0}f(x)=\frac{1}{2}\lim_{x\to 1-0}\frac{x^2-x+1}{x-1}=\frac{1}{2}\cdot\frac{(1-0)^2-(1-0)+1}{1-0+1}=\frac{1}{2}\cdot\frac{1-0}{-0}=-\infty$$

$$\lim_{x\to 1+0}f(x)=\frac{1}{2}\lim_{x\to 1+0}\frac{x^2-x+1}{x-1}=\frac{1}{2}\cdot\frac{(1+0)^2-(1+0)+1}{1+0-1}=\frac{1}{2}\cdot\frac{1+0}{+0}=+\infty$$

$\Rightarrow$ $x=1$ je vertikalna asimptota

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\frac{1}{2}\frac{x^2-x+1}{x-1}\ \begin{matrix}/:x\\/:x\end{matrix}=\lim_{x\to\pm\infty}\frac{1}{2}\frac{x-1+\cancelto{0}{\frac{1}{x}}}{1-\cancelto{0}{\frac{1}{x}}}=\pm\infty\ \Rightarrow \text{f-ja nema H.A.}$$

tražimo kosu asimptotu u obliku $y=kx+n$

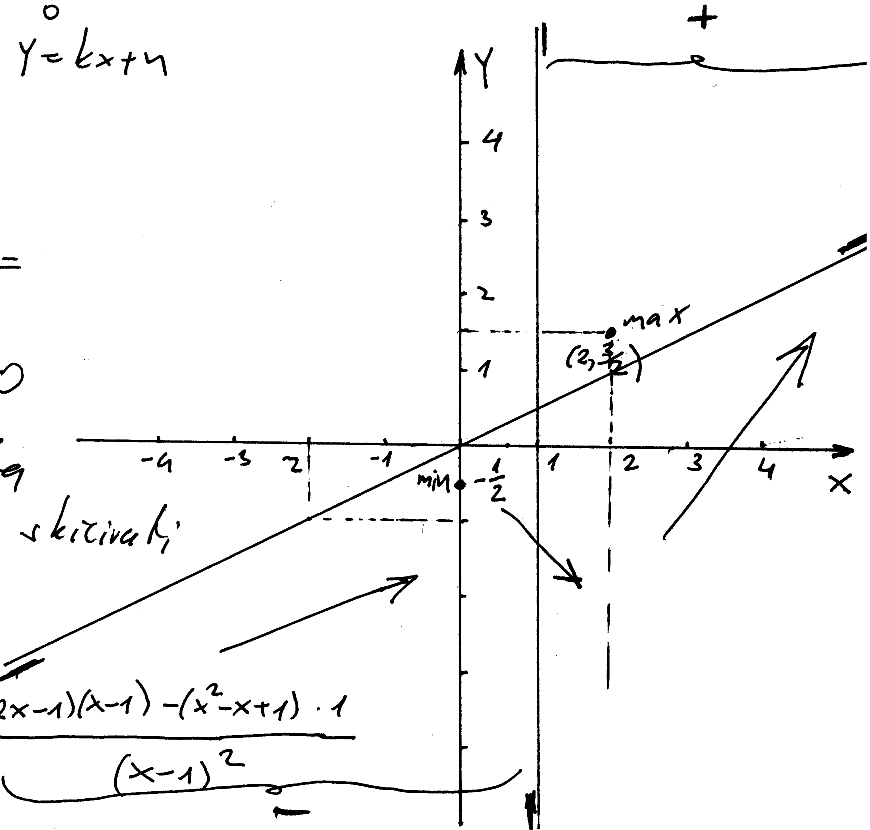
$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2-x+1}{2x^2-2x}\ \begin{matrix}/:x^2\\/:x^2\end{matrix}=\frac{1}{2}$$

$$n=\lim_{x\to\infty}\left[f(x)-kx\right]=\lim_{x\to\infty}\left[\frac{x^2-x+1}{2x-2}-\frac{x}{2}\right]=$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-\cancel{x}+1-\cancel{x^2}+\cancel{x}}{2x-2}=\lim_{x\to\infty}\frac{1}{2x-2}=0$$

$y=\frac{1}{2}x$ je kosa asimptota

Nakon ovog koraka počinjemo skicirati grafik f-je.



rast i opadanje

$$y'=\left(\frac{x^2-x+1}{2x-2}\right)'=\frac{1}{2}\left(\frac{x^2-x+1}{x-1}\right)'=\frac{1}{2}\cdot\frac{(2x-1)(x-1)-(x^2-x+1)\cdot 1}{(x-1)^2}$$

$$y' = \frac{1}{2}\cdot\frac{2x^2 - 2x - \cancel{x} + 1 - x^2 + \cancel{x} - 1}{(x-1)^2} = \frac{1}{2}\cdot\frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{2(x-1)^2}$$

$0 \quad 1 \quad 2$

prekidi f-je y + nule y'

$y' = 0$ akko $x(x-2) = 0$

tj. $x = 0$ ili $x = 2$

x	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, +\infty)$
y'	$+$	$-$	$-$	$+$
y	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

max min

rast i opadanje

$f(0) = -\frac{1}{2}$

$f(2) = \frac{1}{2}\cdot\frac{3}{1} = \frac{3}{2}$

ekstremi f-je

Na osnovu tabele rasta i opadanja $(0, -\frac{1}{2})$ je tačka lokalnog maksimuma, a tačka $(2, \frac{3}{2})$ je tačka lokalnog minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{1}{2}\cdot\frac{x^2-2x}{(x-1)^2}\right)' = \frac{1}{2}\left(\frac{x^2-2x}{(x-1)^2}\right)' = \frac{1}{2}\cdot\frac{(2x-2)(x-1)^{\cancel{2}\,1} - (x^2-2x)\,2\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} =$$

$$= \frac{1}{\cancel{2}}\cdot\frac{2x^2 - 2x - 2x + \cancel{2} - 2x^2 + 4x}{(x-1)^3} = \frac{1}{(x-1)^3}$$

1

$y'' \neq 0 \quad \forall x \in D$

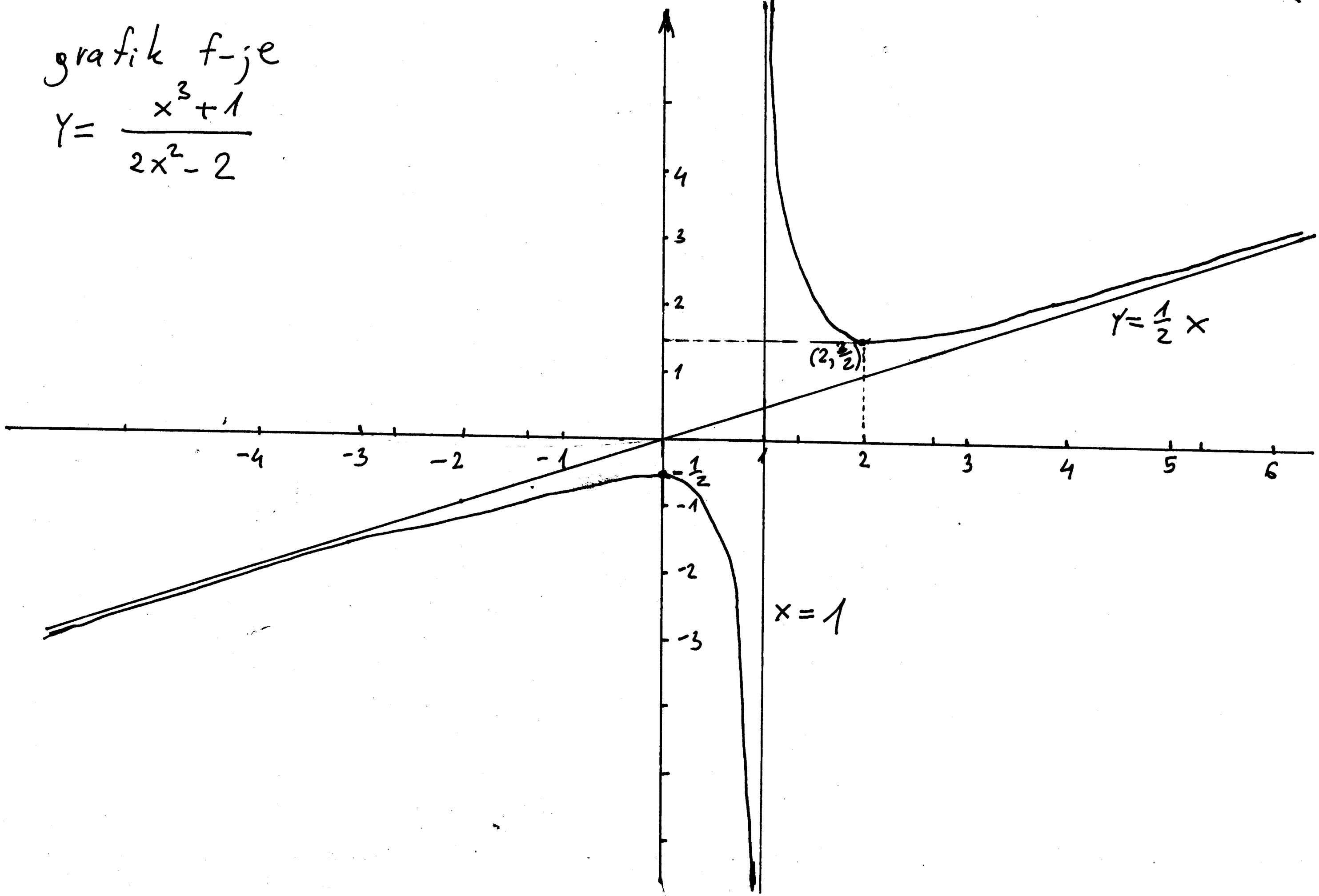
f-ja nema prevojne tačke

x	$(-\infty, 1)$	$(1, +\infty)$
y''	$-$	$+$
y	$\cap$	$\cup$

konveksnost i konkavnost

grafik f-je

$$y = \frac{x^3+1}{2x^2-2}$$



(#) Ispitati f-ju i nacrtati joj grafik: $y=\frac{x^2}{2}+8x^{-2}$.

Rj. $y=\frac{x^2}{2}+8x^{-2}$

$$y=\frac{x^2}{2}+\frac{8}{x^2}=\frac{x^4+16}{2x^2}$$

definiciono područje

$x\neq 0 \quad \mathcal{D}: x\in(-\infty,0)\cup(0,+\infty)$

parnost (neparnost), periodičnost

$$f(-x)=\frac{(-x)^4+16}{2(-x)^2}=\frac{x^4+16}{2x^2}=f(x)$$

f-ja je parna (simetrična u odnosu na Y-osu – dovoljno ju je ispitati za $x>0$)

f-ja nije periodična.

nule, presjek sa y-osom, znak f-je

$y=0$ akko $x^4+16=0$

odavde vidimo da f-ja nema nulu

$f(0)=\ldots \leftarrow$ nije definisano

f-ja ne siječe Y-osu

$x^4+16>0 \quad \forall x\in\mathcal{D}$

$2x^2>0 \quad \forall x\in\mathcal{D}$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{x^4+16}{2x^2}=\frac{(0-)^4+16}{2(0-)^2}=\frac{16+0}{0+}=+\infty \Rightarrow x=0 \text{ je V.A.}$$

(sa lijeve strane)

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{x^4+16}{2x^2}=\frac{(0+)^4+16}{2(0+)^2}=\frac{16+0}{0+}=+\infty \Rightarrow x=0 \text{ je V.A.}$$

(sa desne strane)

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^4+16}{2x^2}\,\frac{/:x^2}{/:x^2}=\lim_{x\to\pm\infty}\frac{x^2+\frac{16}{x^2}\to 0}{2}=+\infty \Rightarrow \text{f-ja nema H.A.}$$

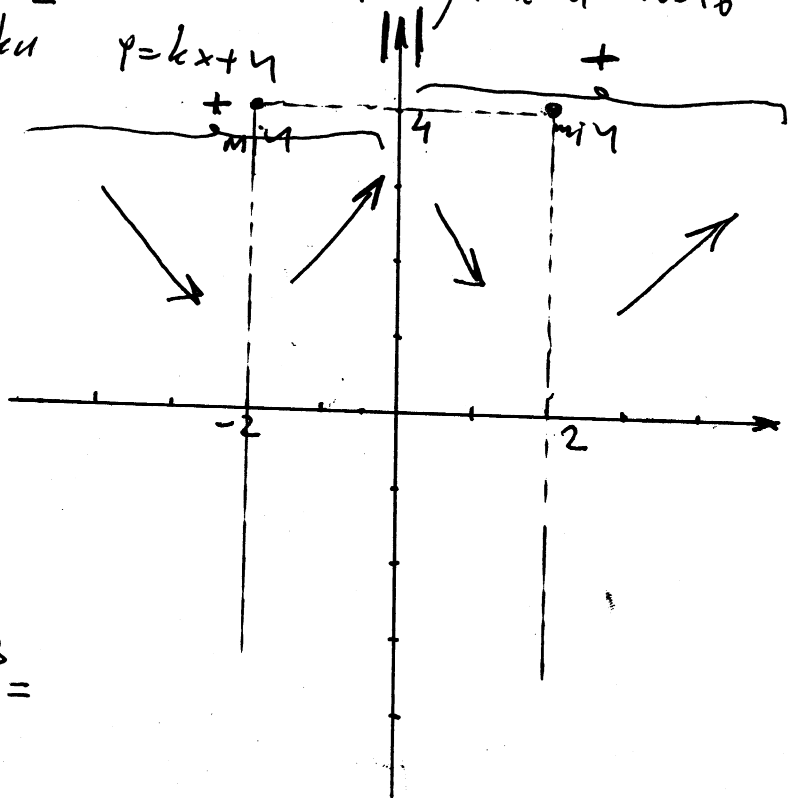
Tražimo kosu asimptotu u obliku $y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^4+16}{2x^3}\,\frac{/:x^3}{/:x^3}=$$

$$=\lim_{x\to\infty}\frac{x+\frac{16}{x^3}\to 0}{2}=\infty$$

f-ja nema kosu asimptotu

Nakon ovog koraka počinjemo sa skiciranjem grafa f-je



rast i opadanje

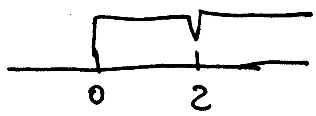
$$y'=\left(\frac{x^2}{2}+8x^{-2}\right)'=\frac{1}{2}\cdot 2x+8\cdot(-2)x^{-3}=$$

$$=x+(-16)\frac{1}{x^3}=\frac{x^4-16}{x^3}$$

$y' = 0$ akko $y^4 - 16 = 0$

$y^4 = 16$

$y_{1,2} = \pm 2$



x	$(0,2)$	$(2,+\infty)$
y'	$-$	$+$
y	$\searrow$	$\nearrow$
		min

rast i opadanje

$$f(2) = \frac{4}{2} + \frac{8}{4} = 2 + 2 = 4$$

ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da je $(2,4)$ minimum f-je (također, zbog simetričnosti, $(-2,4)$).

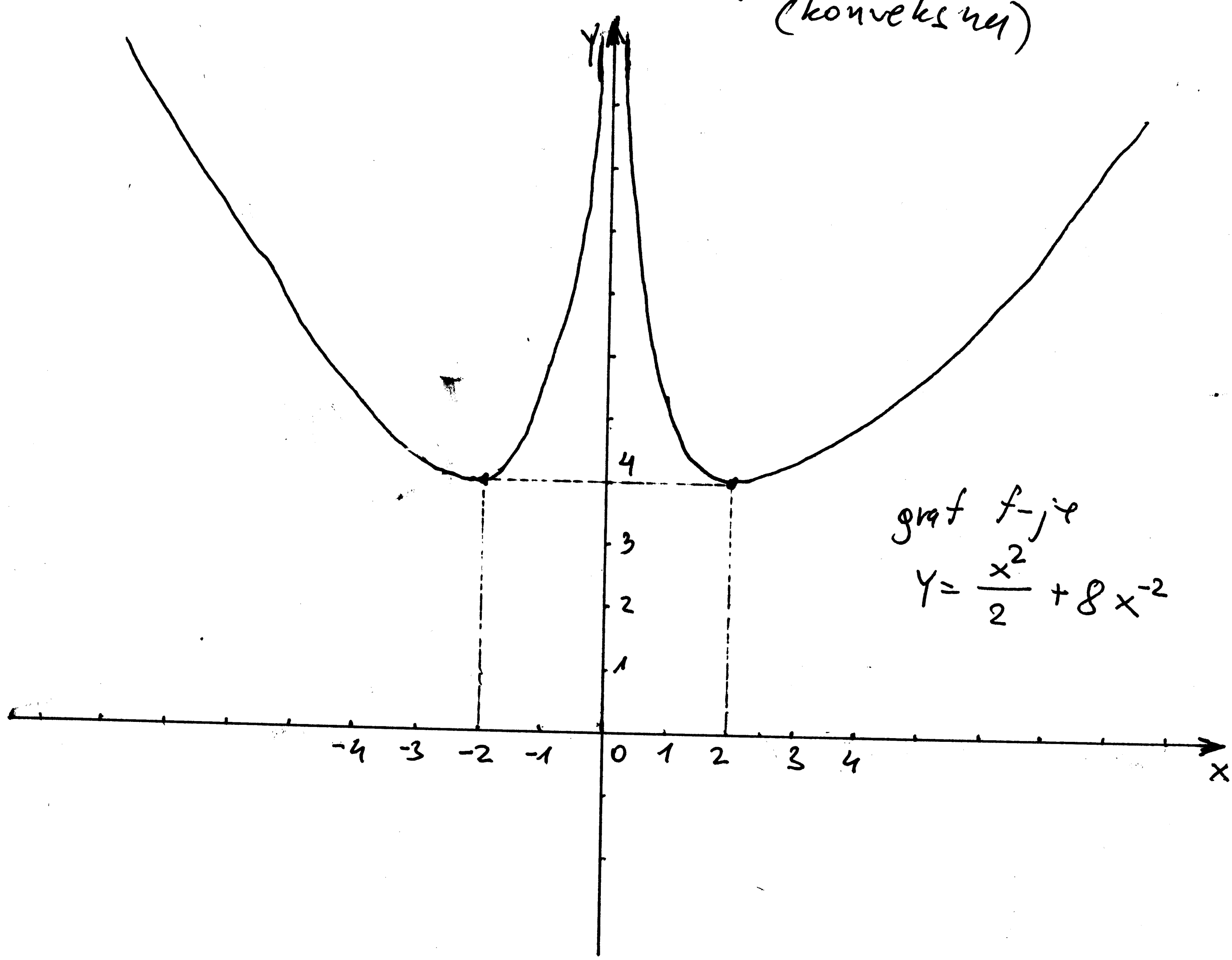
prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(x - 16 \cdot \frac{1}{x^3}\right)' = \left(x - 16x^{-3}\right)' = \left(1 - 16 \cdot (-3)x^{-4}\right) = 1 + \frac{48}{x^4}$$

$$y'' = \frac{x^4 + 48}{x^4}$$

$y'' \neq 0$ za $\forall x \in D$ f-ja nema prevojnih tački

$y'' > 0$ za $\forall x \in D$ f-ja je uvijek $\cup$ (konveksna)



graf f-je $y = \frac{x^2}{2} + 8x^{-2}$

(#) Ispitati f-ju i nacrtati joj grafik $y = e^{\frac{1}{x^2-4x+3}}$.

Rj. definiciono područje

$x^2-4x+3 \neq 0$

$(x-1)(x-3) \neq 0$

$x_1 \neq 1$

$x_2 \neq 3$

$\mathcal{D}: x \in (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0 \Leftrightarrow e^{\frac{1}{x^2-4x+3}} = 0$ #

$e^{\frac{1}{x^2-4x+3}} \neq 0$ za $\forall x \in \mathcal{D}$

tačnije $e^{\frac{1}{x^2-4x+3}} > 0 \ \forall (x \in \mathcal{D}) \Rightarrow$ f-ja je uvijek pozitivna

f-ja nema nulu

$x=0 \Rightarrow y = e^{\frac{1}{3}} = \sqrt[3]{e} \approx 1{,}3956$

$(0, \sqrt[3]{e})$ presjek sa y-osom

ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ i $x=3$ f-ja ima prekid

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} e^{\frac{1}{(x-1)(x-3)}} = e^{\frac{1}{-0(1-0-3)}} = e^{\frac{1}{+0}} = \infty \Rightarrow x=1 \text{ je V.A.}$$

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0} e^{\frac{1}{(x-1)(x-3)}} = e^{\frac{1}{+0(1+0-3)}} = e^{\frac{1}{-0}} = e^{-\infty} = 0$$

$$\lim_{x\to 3-0} f(x) = \lim_{x\to 3-0} e^{\frac{1}{(x-1)(x-3)}} = e^{\frac{1}{(2-0)(-0)}} = e^{\frac{1}{-0}} = e^{-\infty} = 0$$

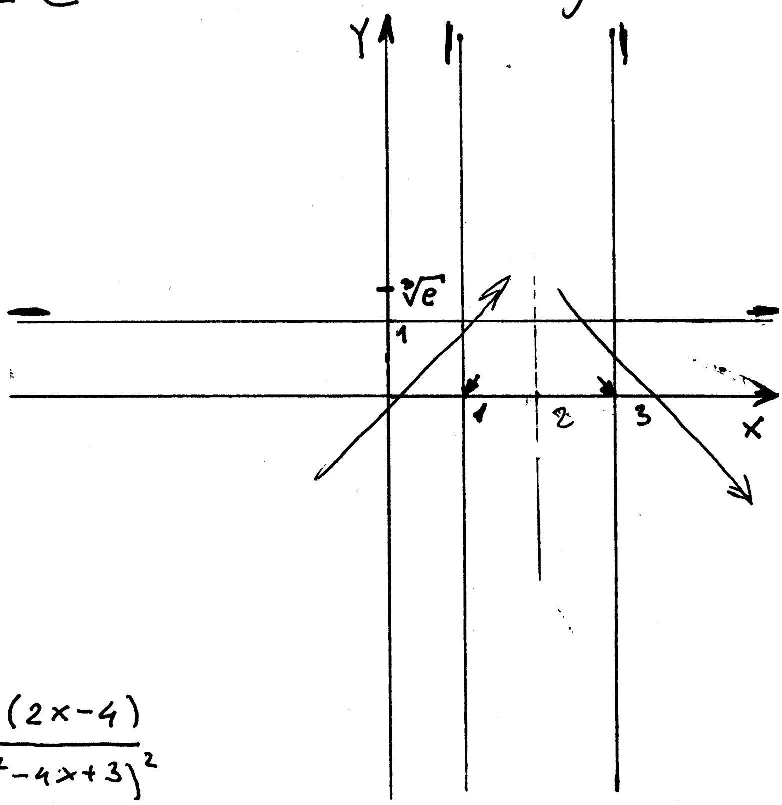
$$\lim_{x\to 3+0} f(x) = \lim_{x\to 3+0} e^{\frac{1}{(x-1)(x-3)}} = e^{\frac{1}{(2+0)(+0)}} = e^{\frac{1}{+0}} = e^{+\infty} = \infty \Rightarrow x=3 \text{ je V.A.}$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to\infty} e^{\frac{1}{x^2-4x+3}} = e^0 = 1$$

$\Rightarrow y=1$ je H.A.

$$\lim_{x\to -\infty} f(x) = \lim_{x\to-\infty} e^{\frac{1}{x^2-4x+3}} = e^0 = 1$$

$\Rightarrow y=1$ je H.A.



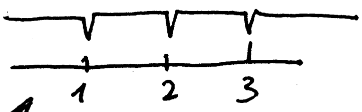
Nakon ovog koraka počinjemo skicirati grafik.

rast i opadanje

$$y' = \left(e^{\frac{1}{x^2-4x+3}}\right)' = e^{\frac{1}{x^2-4x+3}} \cdot \frac{-(2x-4)}{(x^2-4x+3)^2}$$

$$y' = (-2)\, e^{\frac{1}{x^2-4x+3}} \frac{x-2}{(x^2-4x+3)^2}$$

$y' = 0$ akko $x - 2 = 0 \Rightarrow x = 2$



prekidi y
+ nule y'

x	$(-\infty, 1)$	$(1,2)$	$(2,3)$	$(3, +\infty)$
y'	+	+	−	−
y	↗	↗	↘	↘

max

rast i opadanje

$$f(2) = e^{\frac{1}{4-8+3}} = e^{-1} = \frac{1}{e} \approx 0{,}3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da f-ja ima max u tački $(2, \frac{1}{e})$.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left[(-2)\, e^{\frac{1}{x^2-4x+3}} \cdot \frac{x-2}{(x^2-4x+3)^2}\right]' = (-2)\left[\underline{\underline{e^{\frac{1}{x^2-4x+3}}}} \cdot (-2) \cdot \frac{x-2}{(x^2-4x+3)^2} \cdot\right.$$

$$\left. \cdot \frac{x-2}{(x^2-4x+3)^2} + \underline{\underline{e^{\frac{1}{x^2-4x+3}}}} \cdot \frac{(x^2-4x+3)^{\cancel{2}\,1} - (x-2)\, 2\cancel{(x^2-4x+3)} \cdot (2x-4)}{(x^2-4x+3)^{\cancel{4}\,3}}\right] =$$

$$= (-2)(-2)\, e^{\frac{1}{x^2-4x+3}} \left[\frac{(x-2)^2}{(x^2-4x+3)^4} + \frac{x^2-4x+3 + (x-2) \cdot 2(x-2)}{(x^2-4x+3)^3}\right] =$$

$2 \cdot (x^2-4x+4)$

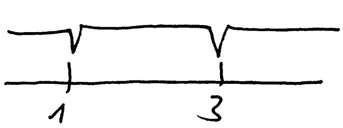
$$= \frac{4}{(x^2-4x+3)^3}\, e^{\frac{1}{x^2-4x+3}} \left[\frac{(x-2)^2}{x^2-4x+3} + 3x^2 - 12x + 11\right] =$$

$$= \frac{4\, e^{\frac{1}{x^2-4x+3}}}{(x^2-4x+3)^4} \left[(x-2)^2 (x^2-4x+3) + 3x^2 - 12x + 11\right]$$

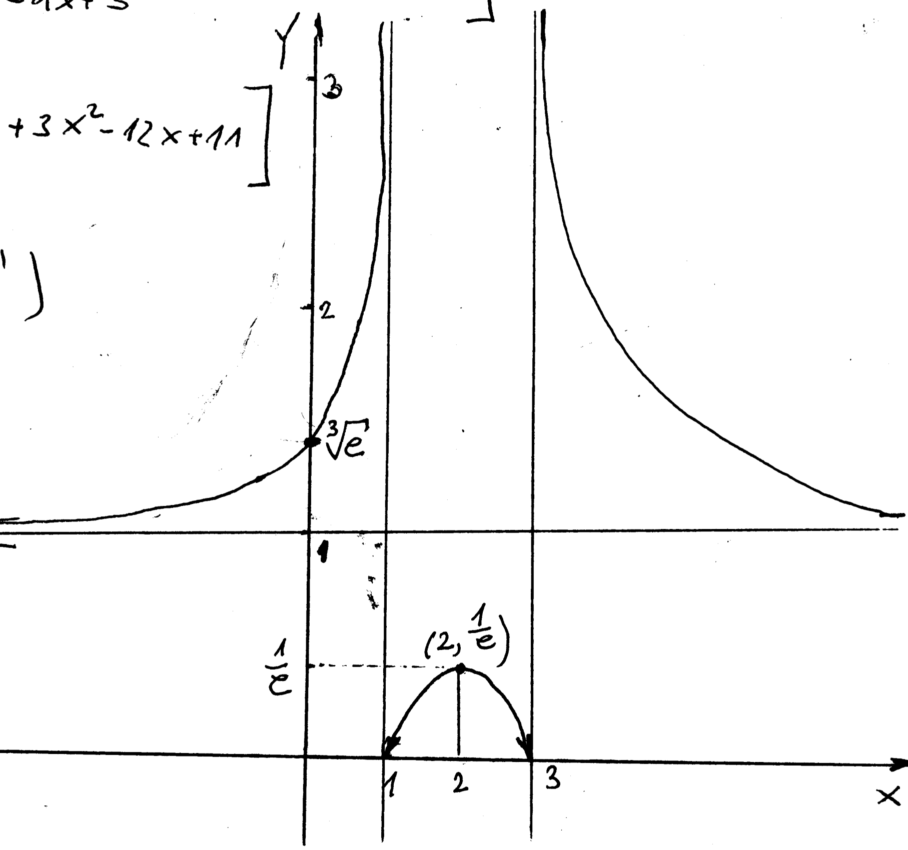
ostavićemo u ovom obliku
(nećemo tražiti nule y'')

x	$(-\infty, 1)$	$(1,3)$	$(3, +\infty)$
y''	+	−	+
y	∪	∩	∪

konveksnost
i konkavnost



prekidi y
+ nule y''



Ispitati i grafički predstaviti f-ju $y = \frac{x^2-1}{e^{x^2}}$.

Rj. definiciono područje

$e^x > 0$ za $\forall x \in \mathbb{R} \Rightarrow e^{x^2} > 0 \; \forall x \in \mathbb{R}$

$\mathcal{D}: x \in \mathbb{R}$

$x \in (-\infty, +\infty)$

parnost (neparnost) periodičnost

$f(-x) = \frac{(-x)^2-1}{e^{(-x)^2}} = \frac{x^2-1}{e^{x^2}} = f(x)$

f-ja je parna (simetrična u odnosu na y-osu)

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y = 0$ akko $x^2-1=0$

$(x-1)(x+1)=0$

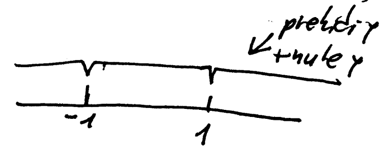
$(1,0)$ i $(-1,0)$ su nule f-je

$f(0) = \frac{0-1}{e^0} = -1$

$(0,-1)$ je presjek grafa sa y-osom

znak f-je

prekidi + nule y



x	(0,1)	(1,+∞)
x−1	−	+
x+1	+	+
e^{x^2}	+	+
y	−	+

Znak f-je

kako je f-ja simetrična dovoljno ju je ispitati za $x > 0$

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekida $\Rightarrow$ f-ja nema V. A.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-1}{e^{x^2}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{2x}{e^{x^2}\cdot 2x} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \Rightarrow$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-1}{e^{x^2}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to-\infty} \frac{1}{e^{x^2}} = 0 \Rightarrow y=0 \text{ je H. A.}$$

$\Rightarrow y=0$ je H. A.

f-ja nema K. A.

Poslije ovog koraka počinjemo skicirati graf.

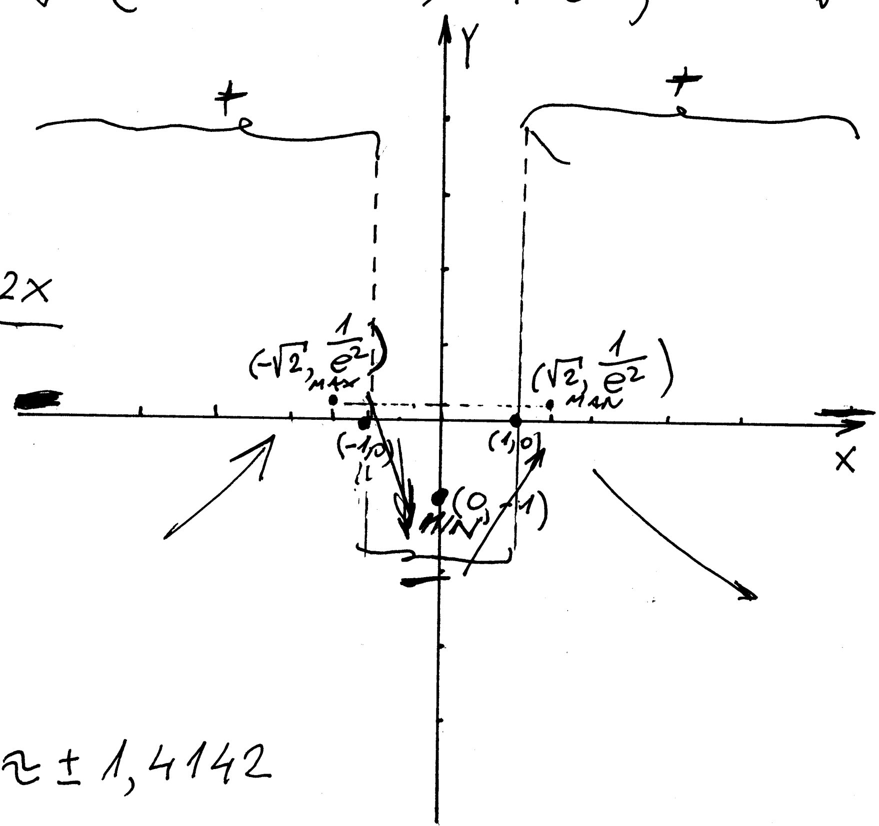
rast i opadanje

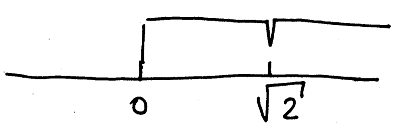
$$y' = \left(\frac{x^2-1}{e^{x^2}}\right)' = \frac{2xe^{x^2} - (x^2-1)e^{x^2}\cdot 2x}{(e^{x^2})^2}$$

$$= \frac{2x(1-x^2+1)}{e^{x^2}} =$$

$$= 2\frac{x(2-x^2)}{e^{x^2}}$$

$y'=0$ akko $x=0$ ili $x = \pm\sqrt{2} \approx \pm 1{,}4142$





prekidi Y + nule Y'

X	$(0,\sqrt{2})$	$(\sqrt{2},+\infty)$
Y'	$+$	$-$
Y	$\nearrow$	$\searrow$
	MIN	MAX

rasti i opadanje

ekstremi f-je

$Y'=0$ akko $x=0$ ili $x=\pm\sqrt{2}$

Stacionarne tačke su $x=0$ ili $x=\pm\sqrt{2}$ i u njima f-ja može imati ekstrem. Na osnovu tabele rasta i opadanja vidimo da u njima f-ja ima ekstrem.

$f(0)=\dfrac{-1}{e^0}=-1$ $(0,-1)$ je minimum f-je

$f(\sqrt{2})=\dfrac{2-1}{e^2}=\dfrac{1}{e^2}\approx 0{,}1353$ $\left(-\sqrt{2},\dfrac{1}{e^2}\right)$ i $\left(\sqrt{2},\dfrac{1}{e^2}\right)$ su tačke maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(2\,\frac{x(2-x^2)}{e^{x^2}}\right)'=2\left(\frac{2x-x^3}{e^{x^2}}\right)'=2\cdot\frac{(2-3x^2)\cancel{e^{x^2}}-(2x-x^3)\cancel{e^{x^2}}\cdot 2x}{(e^{x^2})^{\cancel{2}}}=$$

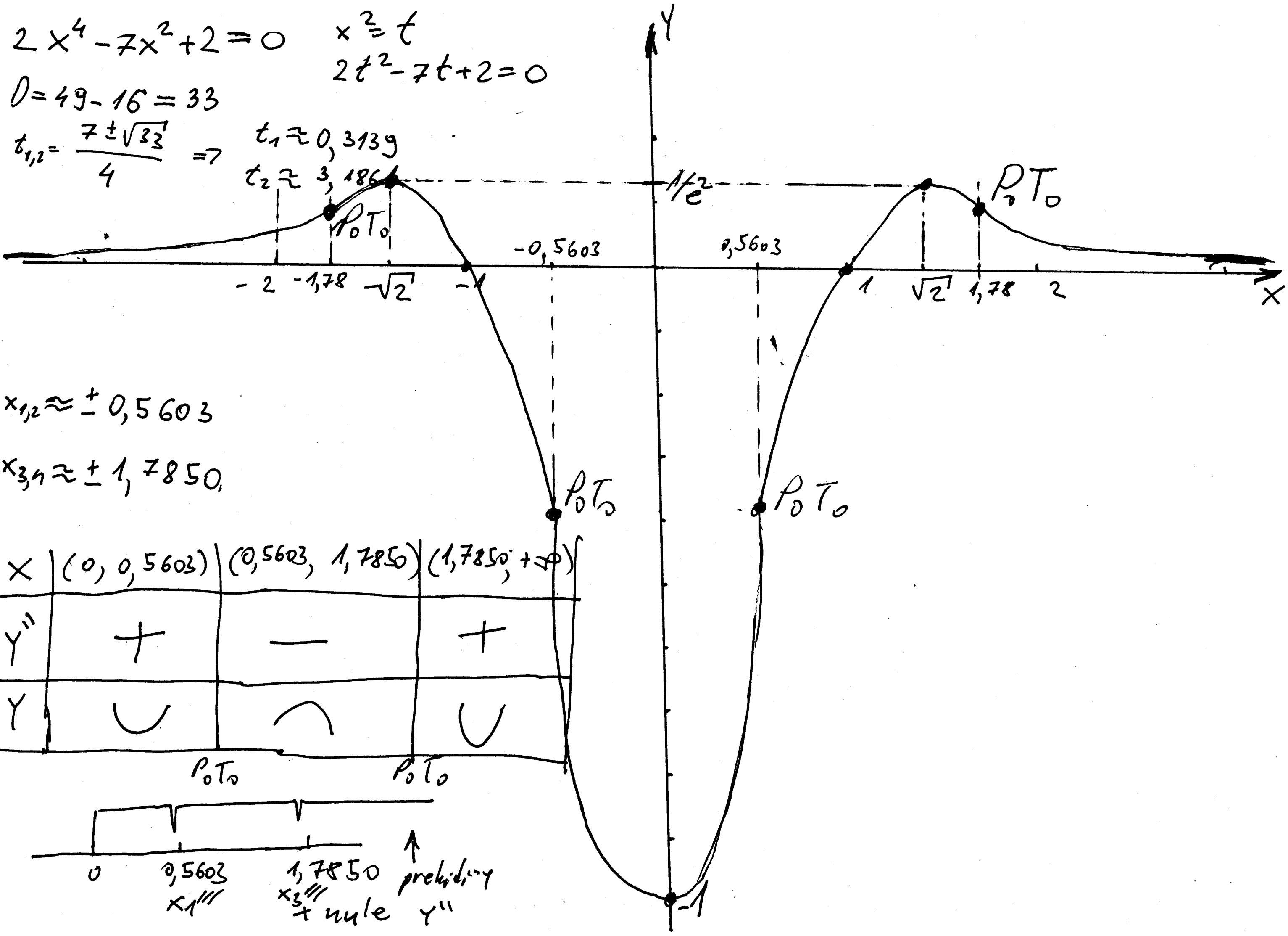
$$=2\,\frac{2-3x^2-4x^2+2x^4}{e^{x^2}}=2\,\frac{2-7x^2+2x^4}{e^{x^2}}$$

$2x^4-7x^2+2=0$ $x^2=t$

$2t^2-7t+2=0$

$D=49-16=33$

$t_{1,2}=\dfrac{7\pm\sqrt{33}}{4}$ $\Rightarrow$ $t_1\approx 0{,}3139$, $t_2\approx 3{,}186$



$x_{1,2}\approx\pm 0{,}5603$

$x_{3,4}\approx\pm 1{,}7850$

X	$(0,\ 0{,}5603)$	$(0{,}5603,\ 1{,}7850)$	$(1{,}7850,+\infty)$
Y''	$+$	$-$	$+$
Y	$\cup$	$\cap$	$\cup$
	PoTo	PoTo	

0 0,5603 x_1''' 1,7850 x_3'''

prekidi Y + nule Y''

Ispitati f-ju i nacrtati joj grafik $y=(x-1)\ln^2(x-1)$.

Rj: deficiono područje

$x-1>0$

$x>1$

$\mathcal{D}: x\in(1,+\infty)$

nule, presjek sa Y-osom, znak f-je

$y=0$ akko $(x-1)\ln^2(x-1)=0$

$x-1=0$ ili $\ln^2(x-1)=0$

$x=1\notin\mathcal{D}$ $\quad x-1=1$

$x=2$

Tačka $(2,0)$ je nula f-je

za $x=0$ f-ja nije definisana (f-ja ne siječe Y-osu)

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

(f-ja nije periodična)

$\ln^2(x-1)>0 \quad \forall x\in\mathcal{D}$

$y>0$ za $x-1>0$ $\Rightarrow$ f-ja je pozitivna za svako $x\in\mathcal{D}$

ponašanje na krajevima intervala definisanosti i asimptote

za $x\leq 1$ f-ja nije definisana

$$\lim_{x\to1+0} f(x)=\lim_{x\to1+0}(x-1)\ln^2(x-1)\ \underset{\text{neodređen izraz}}{\left(=(+0)(+\infty)\right)}=\lim_{x\to1+0}\frac{\ln^2(x-1)}{\frac{1}{x-1}}\left(=\frac{+\infty}{+\infty}\right)$$

$$\overset{LoP.}{=}\lim_{x\to1+0}\frac{2\ln(x-1)\cdot\frac{1}{x-1}\cdot1}{\frac{-1}{(x-1)^2}}=\lim_{x\to1+0}\frac{2\ln(x-1)}{\frac{-1}{x-1}}\left(=\frac{-\infty}{\infty}\right)\overset{LoP.}{=}\lim_{x\to1+0}\frac{2\cdot\frac{1}{x-1}\cdot1}{\frac{1}{(x-1)^2}}$$

$$=2\lim_{x\to1+0}(x-1)=2(1+0-1)=+0=0$$ $\Rightarrow$ f-ja nema V_0A_0

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(x-1)\ln^2(x-1)=\infty\cdot\infty=\infty$$ $\Rightarrow$ f-ja nema H_0A_0

(ne tražimo $\lim_{x\to-\infty} f(x)$ zato što f-ja za $x<1$ nije definisana)

Tražimo kosu asimptotu u obliku

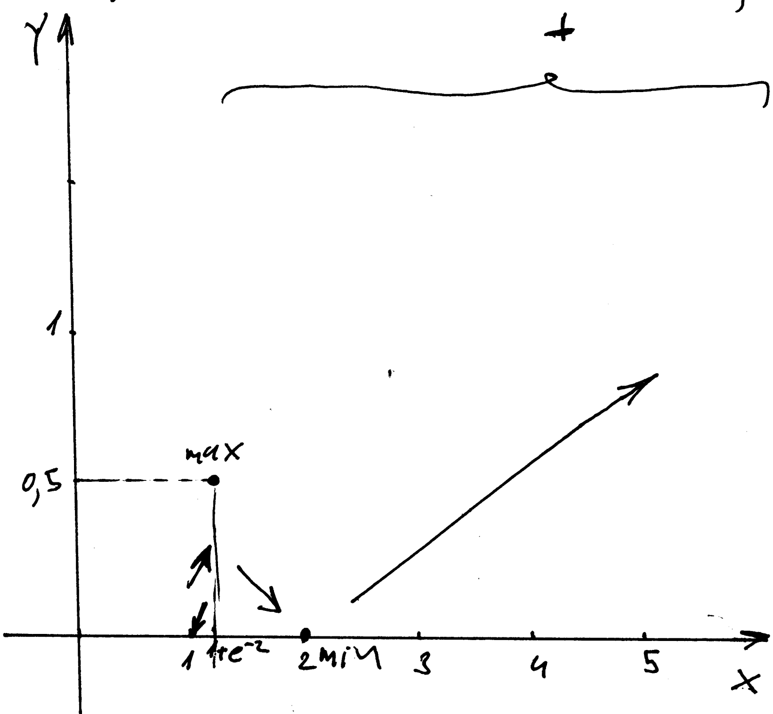
$y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left(1-\frac{1}{x}\right)\ln^2(x-1)$$

$$=1\cdot\infty=\infty$$

f-ja nema vertikalnu asimptotu

Nakon ovog koraka počinjemo skicirati graf f-je



rast i opadanje

$$y' = \left((x-1)\ln^2(x-1)\right)' = \ln^2(x-1) + \cancel{(x-1)}\, 2\ln(x-1)\cdot\frac{1}{\cancel{x-1}}\cdot 1$$

$$y' = \ln^2(x-1) + 2\ln(x-1) = \ln(x-1)\left[\ln(x-1)+2\right]$$

$y' = 0$ akko $\ln(x-1) = 0$ ili $\ln(x-1)+2 = 0$

$x-1 = 1$

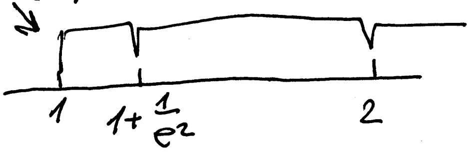
$x = 2$

$\ln(x-1) = -2$

$x-1 = e^{-2}$

$x = 1 + \frac{1}{e^2} \approx 1{,}1353$

prekidi y + nule y'



$1+\frac{1}{e^3}$ $1+e^{-1}$

x	$(1, 1+\frac{1}{e^2})$	$(1+\frac{1}{e^2}, 2)$	$(2, +\infty)$
y'	+	−	+
y	↗	↘	↗
		max	min

rast i opadanje

$$f(1+e^{-2}) = (1+e^{-2}-1)\cdot\ln^2(1+e^{-2}-1) = e^{-2}\cdot(-2)^2 = \frac{4}{e^2} \approx 0{,}5413$$

$f(2) = 0$

ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da je $\left(1+e^{-2}, \frac{4}{e^2}\right)$ maksimum i $(2,0)$ minimum f-je.

prevojne tačke i intervali konveksnosti i konkavnosti

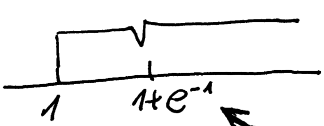
$$y'' = \left(\ln^2(x-1) + 2\ln(x-1)\right)' = 2\ln(x-1)\cdot\frac{1}{x-1} + 2\cdot\frac{1}{x-1} = \frac{2\ln(x-1)+2}{x-1}$$

$y'' = 0$ akko $2\ln(x-1)+2 = 0$

$\ln(x-1) = -1$

$x-1 = e^{-1}$

$x = 1 + e^{-1} \approx 1{,}3679$

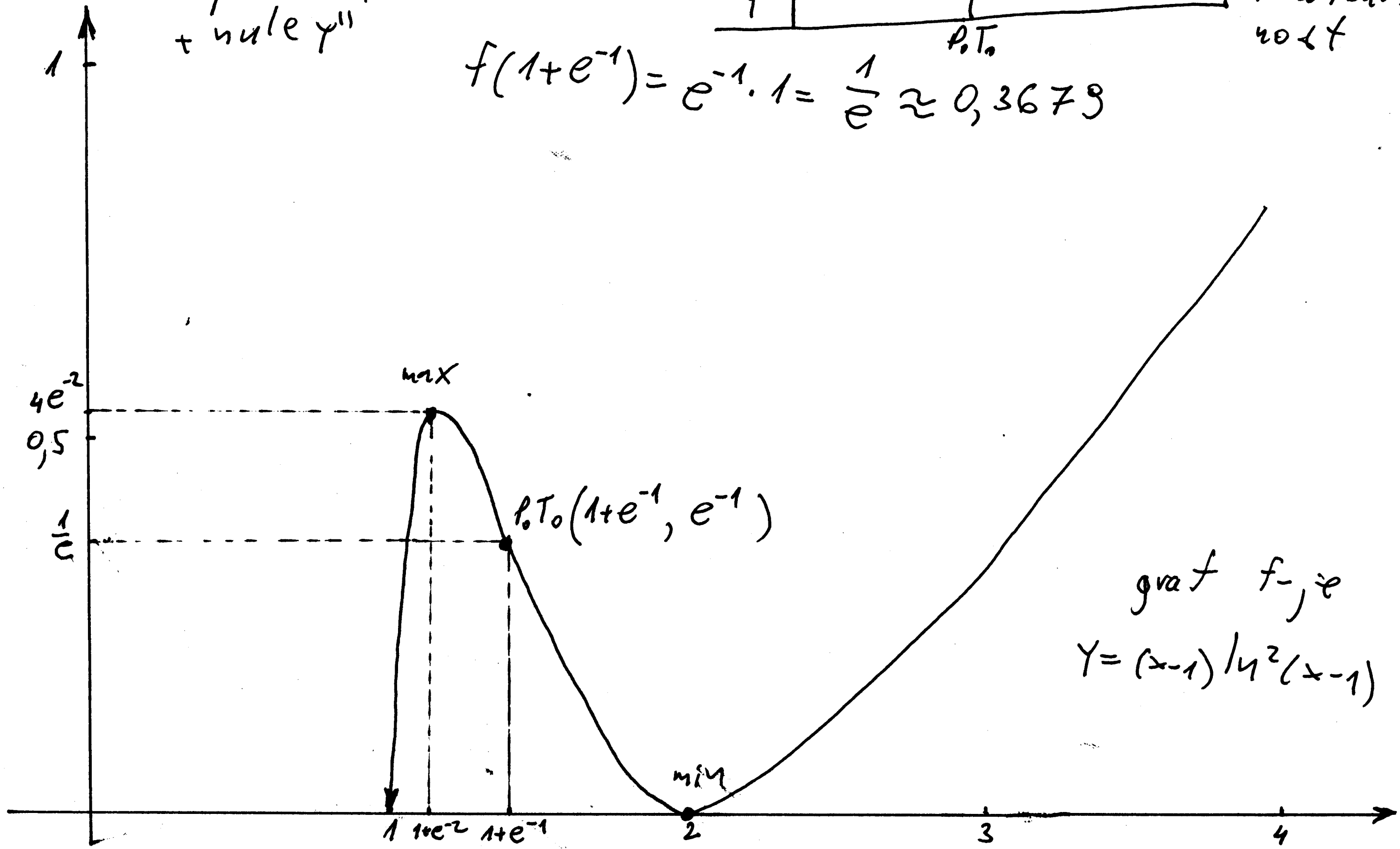


prekidi y + nule y''

x	$(1, 1+e^{-1})$	$(1+e^{-1}, +\infty)$
y''	−	+
y	⌒	⌣
	P.T.	

konveksnost i konkavnost

$$f(1+e^{-1}) = e^{-1}\cdot 1 = \frac{1}{e} \approx 0{,}3679$$



graf f-je

$y = (x-1)\ln^2(x-1)$

(#) Ispitati f-ju i nacrtati joj grafik $y = \frac{3\ln x - 5}{x^2}$.

Rj: definiciono područje

$x^2 \neq 0$ i $x > 0$

$D: x \in (0, +\infty)$

parnost (neparnost), periodičnost

D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

F-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y = 0$ akko $3\ln x - 5 = 0$

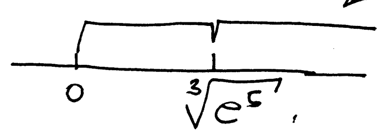
$3\ln x = 5$

$\ln x = \frac{5}{3}$

$x = e^{\frac{5}{3}} = \sqrt[3]{e^5} \approx 5{,}2945$

$(\sqrt[3]{e^5}, 0)$ je nula f-je

prekidi y + nule y



$f(0)$ nije definisano $\Rightarrow$ f-ja ne siječe y-osu

x	$(0, \sqrt[3]{e^5})$	$(\sqrt[3]{e^5}, +\infty)$
$3\ln x - 5$	$-$	$+$
x^2	$+$	$+$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

Za $x = 0$ f-ja nije definisana

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{3\ln x - 5}{x^2} = \frac{-\infty}{+0} = -\infty \Rightarrow x = 0 \text{ je V. A.}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3\ln x - 5}{x^2} \left(= \frac{\infty}{\infty}\right) \stackrel{L.P.}{=\!=} \lim_{x \to \infty} \frac{3 \cdot \frac{1}{x}}{2x} = \lim_{x \to \infty} \frac{3}{2x^2} = 0$$

f-ja nema kosu asimptotu $\Rightarrow y = 0$ je H. A.

Nakon ovog koraka počinjemo skicirati graf

rast i opadanje

$$y' = \left(\frac{3\ln x - 5}{x^2}\right)' = \frac{3 \cdot \frac{1}{\cancel{x}} \cdot x^{\cancel{2}} - (3\ln x - 5) \cdot 2x}{x^4} =$$

$$= \frac{3 - 6\ln x + 10}{x^3} = \frac{13 - 6\ln x}{x^3}$$

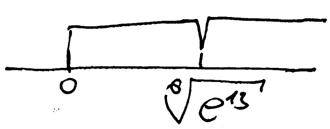
$y' = 0$ akko $13 - 6\ln x = 0$

$6\ln x = 13$

$\ln x = \frac{13}{6}$

$x = e^{\frac{13}{6}} = \sqrt[6]{e^{13}} \approx 8{,}7291$

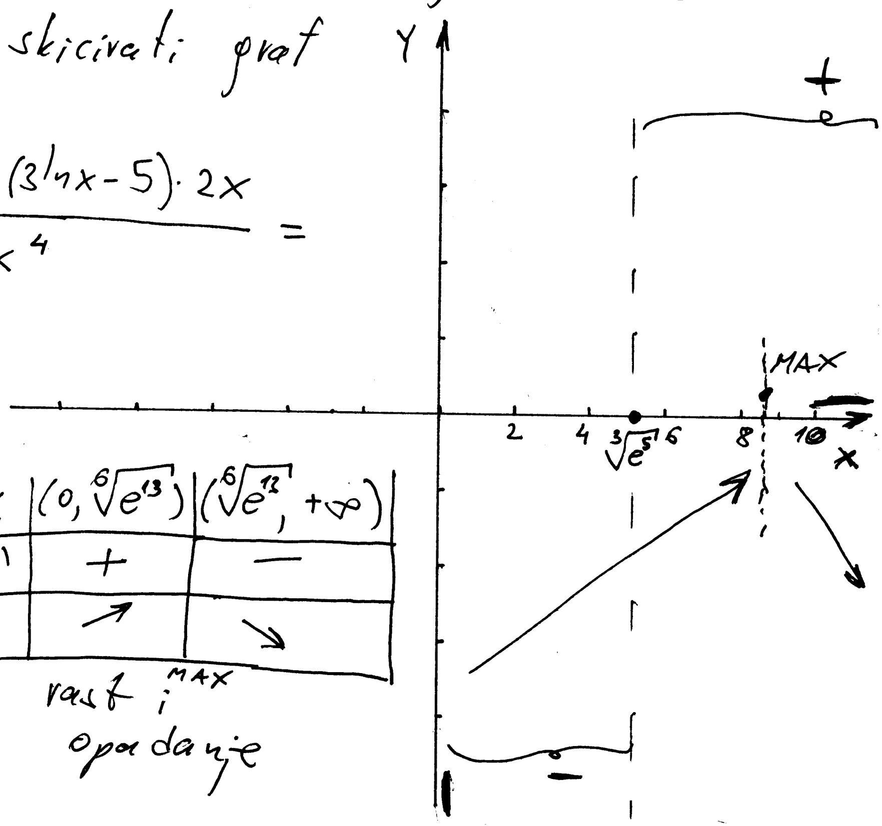
prekidi y + nule y'



X	$(0, \sqrt[6]{e^{13}})$	$(\sqrt[6]{e^{13}}, +\infty)$
y'	$+$	$-$
y	↗	↘

rast i opadanje

MAX



ekstremi f-je

Stacionarne tačke je $x=\sqrt[6]{e^{13}}$ i u njoj f-ja može imati ekstrem. Iz tabele rasta i opadanja vidimo da u njoj f-ja ima maksimum. $f(\sqrt[6]{e^{13}}) = \dfrac{3\cdot\frac{13}{6}-5}{\sqrt[6]{e^{26}}} \approx 0{,}0197$

$(\sqrt[6]{e^{13}}, 0{,}0197)$ je maksimum f-je

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{13-6\ln x}{x^3}\right)' = \frac{-6\,\frac{1}{\cancel{x}}\cdot x^{\overset{2}{\cancel{3}}} - (13-6\ln x)\cdot 3x^2}{x^6} = \frac{-6-(13-6\ln x)\cdot 3}{x^4}$$

$$= \frac{-6-39+18\ln x}{x^4} = \frac{18\ln x - 45}{x^4}$$

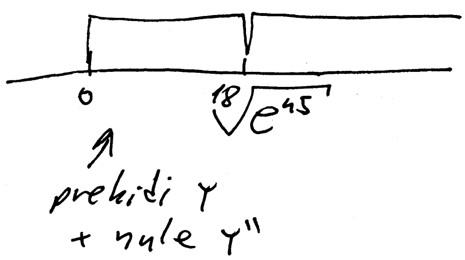
$y''=0$ akko $18\ln x - 45 = 0$

$18\ln x = 45$

$\ln x = \dfrac{45}{18}$

$x = e^{\frac{45}{18}} = \sqrt[18]{e^{45}}$

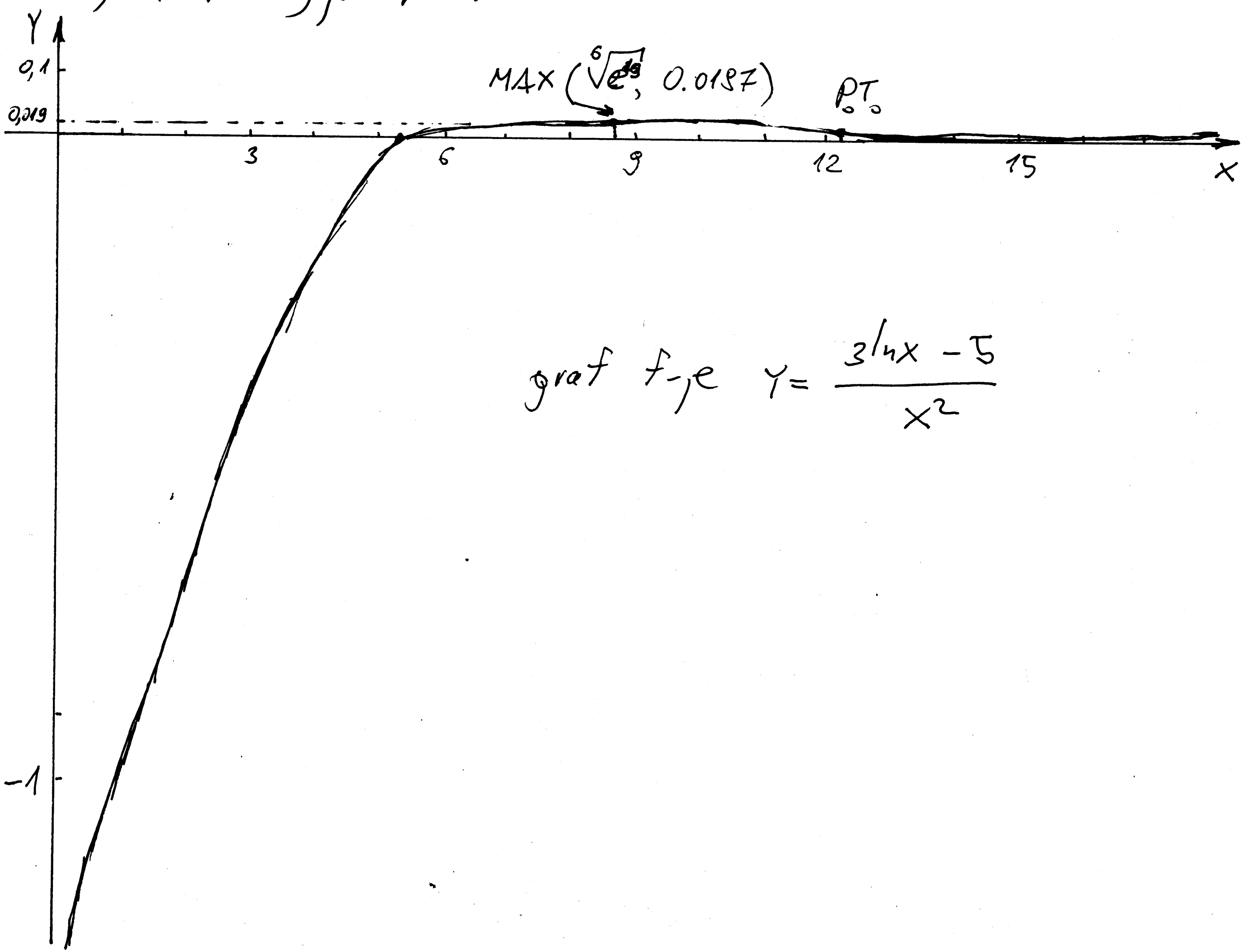
$\approx 12{,}1825$



x	$(0, \sqrt[18]{e^{45}})$	$(\sqrt[18]{e^{45}}, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

P.T.

$(\sqrt[18]{e^{45}}, f(\sqrt[18]{e^{45}}))$ je prevojna tačka



graf f-je $y=\dfrac{3\ln x-5}{x^2}$

(#) Izračunati integral $\int \sqrt{\frac{x-2}{x+2}}\,dx$.

Rj.
$$\int \sqrt{\frac{x-2}{x+2}}\,dx = \int \frac{\sqrt{x-2}}{\sqrt{x+2}}\,dx = \int \frac{\sqrt{x-2}\cdot\sqrt{x-2}}{\sqrt{x+2}\cdot\sqrt{x-2}}\,dx = \int \frac{x-2}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{dx}{\sqrt{x^2-4}}$$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \left| \begin{array}{l} x^2-4=t \\ 2x\,dx = dt \\ x\,dx = \frac{1}{2}\,dt \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt = \frac{1}{2}\cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sqrt{t} + c = \sqrt{x^2-4} + c$$

$$\int \frac{dx}{\sqrt{x^2-4}} = \left| \begin{array}{l} x = 2s \\ dx = 2\,ds \\ s = \frac{1}{2}x \end{array} \right| = \int \frac{2\,ds}{\sqrt{4s^2-4}} = \frac{2}{\sqrt{4}}\int \frac{ds}{\sqrt{s^2-1}} = \ln\left|s+\sqrt{s^2-1}\right| + C_1$$

$$= \ln\left|\frac{1}{2}x + \sqrt{\frac{1}{4}x^2-1}\right| + C_1 = \ln\left|\frac{1}{2}x + \frac{1}{2}\sqrt{x^2-4}\right| + C_1$$

$$= \ln\frac{1}{2} + \ln\left|x+\sqrt{x^2-4}\right| + C_1 = \ln\left|x+\sqrt{x^2-4}\right| + C$$

$$\int \sqrt{\frac{x-2}{x+2}}\,dx = \sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + C$$

Izračunati integral $\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx$.

Rj.

Uvodimo smjenu $tg\frac{x}{2}=t \Rightarrow \frac{x}{2}=arc\,tg\,t$

$x = 2\,arc\,tg\,t$

$dx = \frac{2}{1+t^2}$

$\sin 2x = 2\sin x\cos x$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} =$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2} \end{matrix} = \frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2} \end{matrix} =$$

$$= \frac{1-tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \left| \begin{matrix} tg\frac{x}{2}=t & \sin x = \frac{2t}{1+t^2} \\ dx=\frac{2}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \\ x = 2\,arc\,tg\,t & \end{matrix} \right| =$$

$$= \int \frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{\frac{\cancel{1}+\cancel{t^2}-2t+1-\cancel{t^2}}{\cancel{1+t^2}}}{\frac{\cancel{1}+t^2+2t-\cancel{1}+t^2}{\cancel{1+t^2}}} \cdot \frac{2}{1+t^2}\,dt$$

$$=2\int \frac{2-2t}{2t^2+2t} \cdot \frac{1}{1+t^2}\,dt = 2\int \frac{1-t}{(t^2+t)(1+t^2)}\,dt = 2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt$$

$$\frac{1-t}{t(t+1)(t^2+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1} \quad /\cdot t(t+1)(t^2+1)$$

$$-t+1 = A\underbrace{(t+1)(t^2+1)}_{t^3+t^2+t+1} + B\underbrace{(t^2+1)\cdot t}_{t^3+t} + (Ct+D)\underbrace{t(t+1)}_{t^2+t}$$

$$-t+1 = A(t^3+t^2+t+1) + B(t^3+t) + C(t^3+t^2) + D(t^2+t)$$

$$\begin{aligned}
A + B + C &= 0 \\
A + C + D &= 0 \\
A + B + D &= -1 \\
A &= 1
\end{aligned}$$

$$\begin{aligned}
B + C &= -1 \quad (a) \\
C + D &= -1 \quad (b) \\
B + D &= -2 \quad (c)
\end{aligned}$$

(a): $B + C = -1$

$(c)-(b)$: $B - C = -1$ $+$

$2B = -2$

$B = -1$

$-1 + D = -2$

$D = -1$

$C - 1 = -1$

$C = 0$

$A = 1$ $\quad$ $C = 0$

$B = -1$ $\quad$ $D = -1$

$$2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt = 2\int\left(\frac{1}{t} + \frac{(-1)}{t+1} + \frac{(-1)}{t^2+1}\right)dt =$$

$$= 2\ln|t| - 2\ln|t+1| - 2\,\mathrm{arctg}\, t + C =$$

$$= 2\ln\left|\mathrm{tg}\,\frac{x}{2}\right| - 2\ln\left|\mathrm{tg}\,\frac{x}{2} + 1\right| - 2\,\mathrm{arctg}\left|\mathrm{tg}\,\frac{x}{2}\right| + C$$

Izračunati integral $\int_0^{\frac{\pi}{4}} \sin^5 x \cos^7 x \cdot dx$

Rj.: $$\int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^7 x \, dx = \int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^6 x \cdot \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{4} \Rightarrow t = \frac{\sqrt{2}}{2} \\ \cos^6 x = (\cos^2 x)^3 = \\ = (1 - \sin^2 x)^3 = (1 - t^2)^3 \end{array} \right| =$$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \end{array}$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1 - t^2)^3 \, dt =$$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1 - 3t^2 + 3t^4 - t^6) \, dt = \int_0^{\frac{\sqrt{2}}{2}} (t^5 - 3t^7 + 3t^9 - t^{11}) \, dt =$$

$$= \frac{1}{6} t^6 \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{3}{8} t^8 \Big|_0^{\frac{\sqrt{2}}{2}} + \frac{3}{10} t^{10} \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{1}{12} t^{12} \Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{\cancel{6}_{3}} \cdot \frac{\cancel{8}^{\cancel{4}\,1}}{\cancel{64}_{16}} - \frac{3}{\cancel{8}_{1}} \cdot \frac{\cancel{16}^{\cancel{2}\,1}}{\cancel{256}_{128}} + \frac{3}{\cancel{10}_{5}} \cdot \frac{\cancel{32}^{\cancel{16}\,1}}{\cancel{1024}_{64}} - \frac{1}{\cancel{12}_{3}} \cdot \frac{\cancel{64}^{\cancel{16}\,1}}{\cancel{4096}_{256}} =$$

$$= \frac{1}{\underbrace{3 \cdot 16}_{3 \cdot 2^4}} - \frac{3}{\underbrace{128}_{2^7}} + \frac{3}{\underbrace{5 \cdot 64}_{5 \cdot 2^6}} - \frac{1}{\underbrace{3 \cdot 256}_{3 \cdot 2^8}} = \frac{5 \cdot 2^4 - 3 \cdot 3 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 2^2 - 5}{3 \cdot 5 \cdot 2^8}$$

$$= \frac{80 - 90 + 36 - 5}{3 \cdot 5 \cdot 2^8} = \frac{\cancel{21}^{7}}{\cancel{3}_{1} \cdot 5 \cdot 2^8} = \frac{7}{1280}$$ traženo rješenje

(#) Izračunati integral $\displaystyle\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$

Rj.

Metoda Ostrogradskog:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = (ax^2+bx+c)\sqrt{x^2+3} + \lambda \int \frac{dx}{\sqrt{x^2+3}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^3-7x+4}{\sqrt{x^2+3}} = (2ax+b)\sqrt{x^2+3} + (ax^2+bx+c)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}} + \frac{\lambda}{\sqrt{x^2+3}} \quad \Big/ \cdot \sqrt{x^2+3}$$

$$2x^3-7x+4 = (2ax+b)(x^2+3) + (ax^2+bx+c)x + \lambda$$

$$\underline{2x^3} - \underline{\underline{7x}} + 4 = \underline{2ax^3} + \underline{\underline{bx^2}} + \underline{\underline{6ax}} + 3b + \underline{ax^3} + \underline{\underline{bx^2}} + \underline{\underline{cx}} + \lambda$$

x^3: $2a + a = 2 \Rightarrow 3a = 2$, $a = \frac{2}{3}$

x: $6a + c = -7 \Rightarrow 6 \cdot \frac{2}{3} + c = -7$

$4 + c = -7$

$c = -11$

x^2: $b + b = 0 \Rightarrow b = 0$

x^0: $3b + \lambda = 4$

$\lambda = 4$

Prema tome:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \left(\frac{2}{3}x^2 - 11\right)\sqrt{x^2+3} + 4\int \frac{dx}{\sqrt{x^2+3}} =$$

$$= \frac{2}{3}x^2\sqrt{x^2+3} - 11\sqrt{x^2+3} + 4\ln\left|x + \sqrt{x^2+3}\right| + C$$

Prema tome

$$\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \frac{2}{3}x^2\sqrt{x^2+3}\Big|_{-1}^{1} - 11\sqrt{x^2+3}\Big|_{-1}^{1} + 4\ln\left|x+\sqrt{x^2+3}\right|\Big|_{-1}^{1} =$$

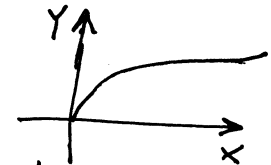
$$= \frac{2}{3}(2-2) - 11(2-2) + 4\left(\ln|1+2| - \ln|-1+2|\right) =$$

$$= 4(\ln 3 - \ln 1) = 4\ln 3$$

traženi rezultat

Izračunati površinu figure koja je određena linijama $y=\sqrt{x}$, $y=1$, $y=10-2x$.

Rj. Linija $y=\sqrt{x}$ izgleda ovako



Prave $y=1$ i $y=10-2x$ nije teško nacrtati.
Pronađimo presječne tačke ovih linija

$$\frac{\begin{matrix} y=1 \\ y=\sqrt{x} \end{matrix}}{\begin{matrix} \sqrt{x}=1 \\ x=1 \end{matrix}}$$

$(1,1)$ je presječna tačka

$$\frac{\begin{matrix} y=1 \\ y=10-2x \end{matrix}}{\begin{matrix} 10-2x=1 \\ 2x=9 \\ x=\frac{9}{2} \end{matrix}}$$

$(\frac{9}{2},1)$ je presječna tačka

$$\frac{\begin{matrix} y=\sqrt{x} \\ y=10-2x \end{matrix}}{\begin{matrix} y^2=x \\ y=10-2x \end{matrix}}$$

$$y=10-2y^2$$

$$2y^2+y-10=0$$

$$D=1+80=81$$

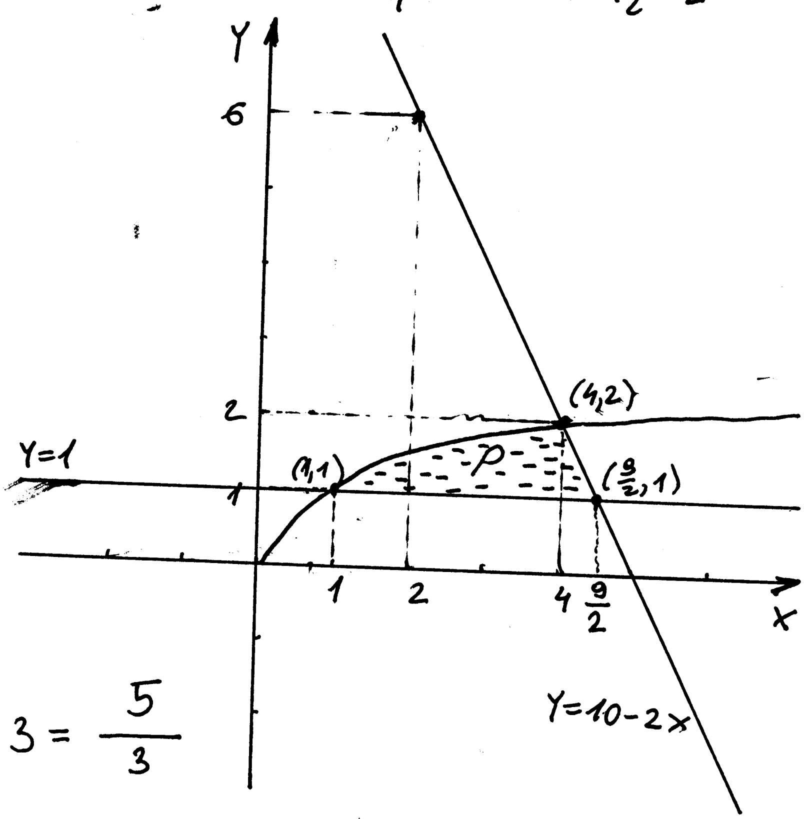
$$y_{1,2}=\frac{-1\pm 9}{4} \Rightarrow y_1=\frac{-10}{4}=-\frac{5}{2}, \quad y_2=2$$

$y_1=-\frac{5}{2} \Rightarrow \sqrt{x}=-\frac{5}{2}$ jednačina nema rješenja

$y_2=2 \Rightarrow \sqrt{x}=2$

$x=4$

$(4,2)$ presječna tačka



$$P=\int_1^4(\sqrt{x}-1)\,dx+\int_4^{9/2}[(10-2x)-1]\,dx$$

$$\int_1^4(\sqrt{x}-1)\,dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^4-x\Big|_1^4=$$

$$=\frac{2}{3}\left(\sqrt{4^3}-\sqrt{1}\right)-3=\frac{2}{3}\cdot 7-3=\frac{5}{3}$$

$$\int_4^{9/2}(9-2x)\,dx=9x\Big|_4^{\frac{9}{2}}-2\cdot\frac{1}{2}x^2\Big|_4^{\frac{9}{2}}=9\left(\frac{9}{2}-\frac{8}{2}\right)-\left(\frac{81}{4}-16\right)=\frac{9}{2}-\frac{17}{4}=$$

$$=\frac{1}{4}$$

$$P=\frac{5}{3}+\frac{1}{4}=\frac{20+3}{12}=\frac{23}{12}$$

Izračunati površinu figure koja je određena linijama $y=\dfrac{3}{x-2}$, $x+y=6$.

Rj: Nacrtajmo sliku. Pravu $x+y=6$ nije teško nacrtati. Problem predstavlja f-ja $y=\dfrac{3}{x-2}$. Ispitajmo nabrzinu ovu f-ju:

$D: x\neq 2$, f-ja nije ni parna ni neparna

$(0,-\frac{3}{2})$ je nula f-je

f-ja ne siječe y-osu

$y>0$ za $x>2$

$y<0$ za $x<2$

$$\left.\begin{aligned}\lim_{x\to 2+0}\frac{3}{x-2}&=\frac{3}{+0}=+\infty\\ \lim_{x\to 2-0}\frac{3}{x-2}&=\frac{3}{-0}=-\infty\end{aligned}\right\}\Rightarrow x=2 \text{ je V. A.}$$

$$\lim_{x\to\pm\infty}\frac{3}{x-2}=0 \Rightarrow y=0 \text{ je H. A.}$$

$$y'=\left(3(x-2)^{-1}\right)'=\frac{-3}{(x-2)^2}$$

$y'<0$ za $\forall x\in D \Rightarrow$ f-ja $\searrow$ za svako x

$y'\neq 0$ za $\forall x\in D \Rightarrow$ f-ja nema ekstrema

$$y''=\left(-3(x-2)^{-2}\right)'=\frac{6}{(x-2)^3}$$

$y''\neq 0$ za $\forall x \Rightarrow$ f-ja nema prevojnih tački

x	$(-\infty,2)$	$(2,+\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

Nađimo presječne tačke prave $x+y=6$ i f-je $y=\dfrac{3}{x-2}$; $y=6-x$

$$6-x=\frac{3}{x-2} \quad /\cdot(x-2)$$

$$(-x+6)(x-2)=3$$

$$-x^2+2x+6x-12=3$$

$$-x^2+8x-15=0 \quad /(-1)$$

$$x^2-8x+15=0 \qquad D=64-60=4$$

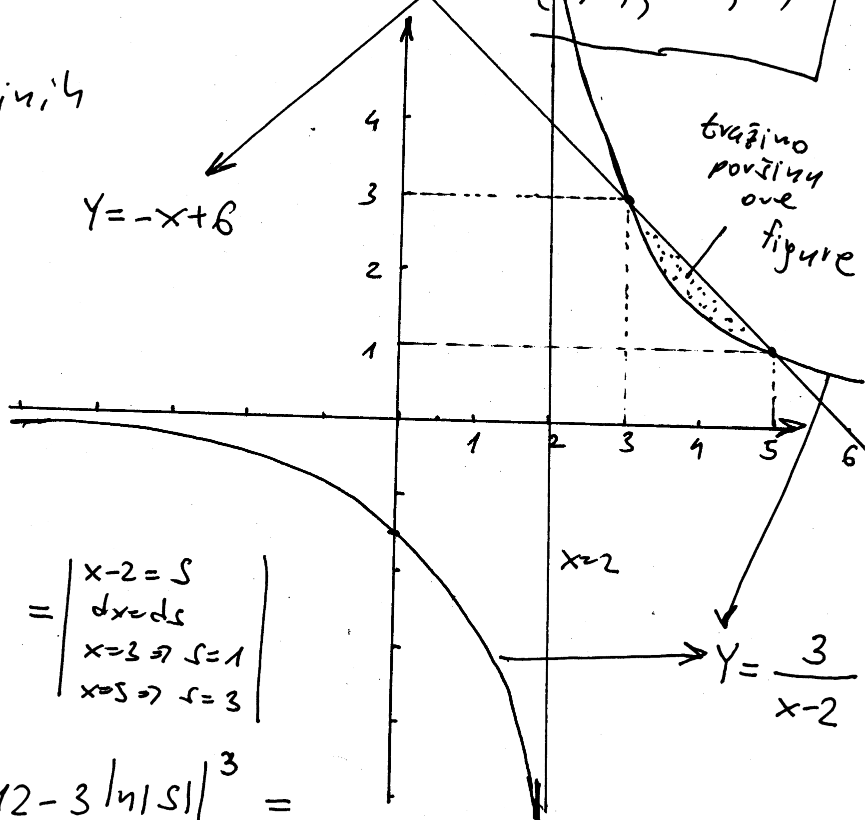
$$(x-3)(x-5)=0 \qquad x_{1,2}=\frac{8\pm 2}{2}$$

$x_1=3 \Rightarrow y_1=3$

$x_2=5 \Rightarrow y_2=1$

presječne tačke prave i krive su $(3,3)$, $(5,1)$

Nacrtajmo grafik.



$$P=\int_3^5\left[(-x+6)-\frac{3}{x-2}\right]dx=$$

$$=-\frac{1}{2}\underbrace{x^2\Big|_3^5}_{25-9}+6\cdot x\Big|_3^5-3\int_3^5\frac{1}{x-2}\,dx=\left|\begin{array}{l}x-2=s\\ dx=ds\\ x=3\Rightarrow s=1\\ x=5\Rightarrow s=3\end{array}\right|$$

$$=-\frac{1}{2}\cdot 16+6\cdot 2-3\int_1^3\frac{ds}{s}=-8+12-3\underbrace{\ln|s|\Big|_1^3}_{\ln 3-\underbrace{\ln 1}_{=0}}=$$

$=4-3\ln 3$ tražena površina

(#) Naći ekstreme f-je $z=x^2-2x-y-\ln(2-y)+4$.

Rj.

$D: 2-y>0$

$$\frac{\partial z}{\partial x}=2x-2$$

$$\frac{\partial z}{\partial y}=-1-\frac{1}{2-y}\cdot(-1)=\frac{1}{2-y}-1$$

$$2x-2=0$$

$$\frac{1}{2-y}-1=0$$

$$x=1, \quad y=1$$

Tačka $M(1,1)$ je stacionarna tačka (kandidat za ekstrem)

$(2-y)^{-1}$

$$\frac{\partial^2 z}{\partial x^2}=2$$

$$\frac{\partial^2 z}{\partial x \partial y}=0$$

$$\frac{\partial^2 z}{\partial y^2}=(-1)(2-y)^{-2}\cdot(-1)=\frac{1}{(2-y)^2}$$

$M(1,1)$

$A=2, \; B=0, \; C=1$

$D=AC-B^2=2>0$

F-ja ima ekstrem.

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(1,1)=1-2-1-\ln 1+4=-2+4=2$$

(#) Naći ekstreme f-je $z=(x^2+y)\sqrt{e^y}$.

Rj: $\frac{\partial z}{\partial x}=2x\sqrt{e^y}$

$$\frac{\partial z}{\partial y}=\sqrt{e^y}+(x^2+y)\frac{1}{2\sqrt{e^y}}\cdot e^y=\sqrt{e^y}+(x^2+y)\cdot\frac{1}{2}\sqrt{e^y}$$

$$=\left(\frac{1}{2}x^2+\frac{1}{2}y+1\right)\sqrt{e^y}=\frac{1}{2}(x^2+y+2)\sqrt{e^y}$$

$$2x\sqrt{e^y}=0$$

$$\frac{1}{2}(x^2+y^2+1)\sqrt{e^y}=0$$

$e^y>0 \ \forall y\in\mathbb{R}$

prema tome $x=0$

$\sqrt{e^y}>0 \ \forall x\in\mathbb{R}$

$x^2+y+2=0$

$x=0 \Rightarrow y+2=0$

$y=-2$

$M(0,-2)$ je stacionarna tačka (kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2}=2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x\partial y}=2x\frac{1}{2\sqrt{e^y}}\cdot e^y=x\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{1}{2}\sqrt{e^y}+\left(\frac{1}{2}x^2+\frac{1}{2}y+1\right)\frac{e^y\cdot\sqrt{e^y}}{2\sqrt{e^y}\cdot\sqrt{e^y}}=\frac{1}{2}\sqrt{e^y}\left(\frac{1}{2}x^2+\frac{1}{2}y+2\right)$$

$M(0,-2)$

$A=2\sqrt{e^{-2}}=2\frac{1}{\sqrt{e^2}}$

$B=0$

$C=\frac{1}{2}\sqrt{e^{-2}}\left(\frac{1}{2}0+\underbrace{\frac{1}{2}(-2)}_{-1}+2\right)=\frac{1}{2}\sqrt{\frac{1}{e^2}}$

$D=AC-B^2=\frac{2}{\sqrt{e^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{e^2}}=\frac{1}{e^2}$

$D>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(0,-2)=(0-2)\sqrt{e^{-2}}=(-2)\cdot\frac{1}{\sqrt{e^2}}\approx-0.7358$$

Nađi uslovne ekstreme f-je $z=xy$ ako je $x^2+y^2=2ax$, $a>0$.

Rj: Posmatramo f-ju $F(x,y,\lambda)=xy+\lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x}=y+2\lambda x-2a\lambda$$

$$\frac{\partial F}{\partial y}=x+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-2ax$$

$$\begin{aligned} y+2\lambda x-2a\lambda&=0\\ x+2\lambda y&=0\\ x^2+y^2-2a\cdot x&=0 \end{aligned}$$

(1) $y+2\lambda(x-a)=0 \Rightarrow x-a=\frac{-y}{2\lambda}$ ...(*)

(2) $x=-2\lambda y$

(3) $x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

(2) u (1): $y+2\lambda(-2\lambda y-a)=0$

(3): $(x-a)^2+y^2=a^2$

$$y-4\lambda^2y-2a\lambda=0$$

$$y(1-4\lambda^2)=2a\lambda$$

$$y=\frac{2a\lambda}{1-4\lambda^2}$$

(*) u (3):

$$\frac{y^2}{4\lambda^2}+y^2=a^2$$

$$y^2\left(\frac{1}{4\lambda^2}+1\right)=a^2$$

$$y^2\left(\frac{1+4\lambda^2}{4\lambda^2}\right)=a^2$$

$$y^2=\frac{4a^2\lambda^2}{1+4\lambda^2}$$

$$y=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}}$$

$$y=\frac{2a\lambda}{1-4\lambda^2}=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1-4\lambda^2=\pm\sqrt{1+4\lambda^2}$$

$$(1-4\lambda^2)^2=1+4\lambda^2$$

$$16\lambda^4-8\lambda^2+1=1+4\lambda^2$$

$$16\lambda^4-12\lambda^2=0$$

$$\lambda^2(16\lambda^2-12)=0$$

$\lambda_1=0$

$\lambda_{2,3}=\pm\sqrt{\frac{12}{16}}=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$

$\lambda_1=0$: $y=0$, $x=0$

$\lambda_2=\frac{\sqrt{3}}{2}$:

$$\begin{aligned} y+\sqrt{3}x-a\sqrt{3}&=0\\ x+y\sqrt{3}&=0 \quad /\cdot\sqrt{3}\\ x^2+y^2-2ax&=0 \end{aligned}$$

$$\begin{aligned} \sqrt{3}x+y&=a\sqrt{3}\\ -\quad \sqrt{3}x+3y&=0\\ \hline -2y&=a\sqrt{3} \end{aligned}$$

$y=\frac{-a}{2}\sqrt{3}$, $x=\frac{3}{2}a$

$\lambda_3=-\frac{\sqrt{3}}{2}$:

$$\begin{aligned} y-x\sqrt{3}+a\sqrt{3}&=0\\ x-y\sqrt{3}&=0 \quad /\cdot\sqrt{3}\\ x^2+y^2-2ax&=0 \end{aligned}$$

$$\begin{aligned} -x\sqrt{3}+y&=-a\sqrt{3}\\ +\quad x\sqrt{3}-3y&=0\\ \hline -2y&=-a\sqrt{3} \end{aligned}$$

$y=\frac{\sqrt{3}}{2}a \Rightarrow x=\frac{3}{2}a$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2\left(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a\right)$ za $\lambda=\frac{\sqrt{3}}{2}$ i $M_3\left(\frac{3}{2}a, \frac{\sqrt{3}}{2}a\right)$ za $\lambda=-\frac{\sqrt{3}}{2}$.

$$\frac{\partial^2 F}{\partial x^2}=2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y}=1$$

$$\frac{\partial^2 F}{\partial y^2}=2\lambda$$

$M_1(0,0)$, $\lambda=0$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$M_2\left(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a\right)$, $\lambda=\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A=\sqrt{3}>0 \Rightarrow$ f-ja ima minimum

$$Z_{min}\left(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a\right)=-\frac{3\sqrt{3}}{4}a^2$$

$M_3\left(\frac{3}{2}a, \frac{\sqrt{3}}{2}a\right)$ za $\lambda=-\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt{3}<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

$$Z_{max}\left(\frac{3}{2}a, \frac{\sqrt{3}}{2}a\right)=\frac{3\sqrt{3}}{4}a^2$$

(#) Riješiti diferencijalnu jednačinu $(x^2+2x-2y)dx - dy = 0$.

Rj.

$$(x^2+2x-2y)\,dx - dy = 0 \quad /: dx$$

$$x^2+2x-2y-y' = 0$$

$$y'+2y = x^2+2x$$ Ovo je linearna diferencijalna jednačina

Uvodimo smjenu $y = uv \quad / \frac{d}{dx}$

$$y' = u'v + uv'$$

$$u'v+uv'+2uv = x^2+2x$$

$$u'v + \underbrace{u(v'+2v)}_{=0} = x^2+2x$$

a) $v'+2v = 0$

$$\frac{dv}{dx} = -2v$$

$$\frac{dv}{v} = -2\,dx \quad /\int$$

$$\ln v = -2x$$

$$v = e^{-2x}$$

b) $u'v + u\cdot 0 = x^2+2x$

$$u'v = x^2+2x$$

$$u'e^{-2x} = x^2+2x$$

$$\frac{du}{dx} = \frac{x^2+2x}{e^{-2x}}$$

$$du = \frac{x^2+2x}{e^{-2x}}dx$$

$$du = (x^2+2x)e^{2x}dx \quad \ldots(*)$$

$2x=t$

$2dx=dt$

$dx=\frac{1}{2}dt$

$$\int (x^2+2x)e^{2x}dx = \begin{vmatrix} u=x^2+2x & dv=e^{2x}dx \\ du=2x+2 & v=\frac{1}{2}e^{2x}\end{vmatrix} = \frac{1}{2}e^{2x}(x^2+2x) - \int (x+1)e^{2x}dx$$

$$\int (x+1)e^{2x}dx = \begin{vmatrix} u=x+1 & dv=e^{2x}dx \\ du=dx & v=\frac{1}{2}e^{2x}\end{vmatrix} = \frac{1}{2}(x+1)e^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$\int (x^2+2x)e^{2x}dx = \frac{1}{2}e^{2x}(x^2+2x) - \frac{1}{2}e^{2x}(x+1) + \frac{1}{4}e^{2x} + C$$

$$= \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$(*) \Rightarrow \quad u = \frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$y = uv = \left(\frac{1}{2}x^2e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right)e^{-2x} =$$

$$= \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$$ opšte rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu

$$y' - \frac{xy}{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

Rj. $y' - \frac{x}{1+x^2}y = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$ ovo je linearna diferencijalna jednačina uvodimo smjenu $y=uv$

$y=uv, \quad y'=u'v+uv'$

$$u'v+uv' - \frac{x}{1+x^2}uv = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$u'v + u\underbrace{\left(v' - \frac{x}{1+x^2}v\right)}_{=0} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

a)

$$v' - \frac{x}{1+x^2}v = 0$$

$$\frac{dv}{dx} = \frac{x}{1+x^2}v \quad /:v$$

$$\frac{dv}{v} = \frac{x}{1+x^2}dx \quad /\int$$

$$\int\frac{dv}{v} = \int\frac{x}{1+x^2}dx$$

$$\int\frac{x}{1+x^2}dx = \left|\begin{array}{l} 1+x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{array}\right| = \frac{1}{2}\int\frac{dt}{t}$$

$$= \frac{1}{2}\ln|t| + c = \ln|1+x^2|^{\frac{1}{2}} + c$$

$$\ln|v| = \ln\sqrt{1+x^2}$$

$$v = \sqrt{1+x^2}$$

b)

$$u'v = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$\frac{du}{dx}\sqrt{1+x^2} = \frac{x\sqrt{1+x^2}}{x^2-2x+2}$$

$$du = \frac{x}{x^2-2x+2}dx \quad /\int$$

$$\int\frac{x^{-1+1}}{x^2-2x+2}dx = \int\frac{x-1}{x^2-2x+2}dx + \int\frac{dx}{x^2-2x+2}$$

$$= \left|\begin{array}{ll} x^2-2x+2=t & x^2-2x+2= \\ (2x-2)dx=dt & x^2-2x+1+1= \\ (x-1)dx=\frac{1}{2}dt & =(x-1)^2+1 \end{array}\right|$$

$$= \frac{1}{2}\int\frac{dt}{t} + \int\frac{dx}{(x-1)^2+1} =$$

$$= \frac{1}{2}\ln|t| + \operatorname{arctg}(x-1) + c$$

$$= \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1) + c$$

$$u = \frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1) + c$$

$$y = uv = \left(\frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1) + c\right)\sqrt{1+x^2} =$$

$$= c\sqrt{1+x^2} + \sqrt{1+x^2}\left(\frac{1}{2}\ln|x^2-2x+2| + \operatorname{arctg}(x-1)\right)$$

rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $2y-2xy'=a(\sqrt{1+(y')^2}-y')$.

Rj: Lagranžova diferencijalna jednačina je oblika $y=xf(y')+g(y')$

$$2y-2xy'=a(\sqrt{1+(y')^2}-y')$$

$$2y=2xy'+a(\sqrt{1+(y')^2}-y') \quad /:2$$

$$y=xy'+\frac{a}{2}(\sqrt{1+(y')^2}-y')$$

Ovo je Klerova diferencijalna jednačina

Uvodimo smjenu $y'=p$

$$y=xp+\frac{a}{2}(\sqrt{1+p^2}-p) \quad \Big/ \frac{d}{dx}$$

$$y'=p+xp'+\frac{a}{2}\left(\frac{\cancel{2}pp'}{\cancel{2}\sqrt{1+p^2}}-p'\right)$$

$$y'=p$$

$$p=p+xp'+\frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}}-1\right)$$

$$-xp'=\frac{a}{2}p'\left(\frac{p}{\sqrt{1+p^2}}-1\right)$$

$$\left[x+\frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}}-1\right)\right]p'=0$$

a) Ako je $p'=0$ imamo da je $p=c$

tj. $y'=c$ pa iz $y=xy'+\frac{a}{2}(\sqrt{1+(y')^2}-y')$

$$\Rightarrow y=xc+\frac{a}{2}(\sqrt{1+c^2}-c)$$

$y=xC_1+\frac{a}{2}C_2$ opšte rješenje diferencijalne jednačine

b) Ako je

$$x+\frac{a}{2}\left(\frac{p}{\sqrt{1+p^2}}-1\right)=0$$

$$\frac{p}{\sqrt{1+p^2}}-1=-\frac{2}{a}x$$

$$\frac{p}{\sqrt{1+p^2}}=1-\frac{2x}{a}$$

$$p^2=\left(1-\frac{2x}{a}\right)^2(1+p^2)$$

$$p^2-\left(1-\frac{2x}{a}\right)^2p^2=\left(1-\frac{2x}{a}\right)^2$$

$$p^2=\frac{\left(1-\frac{2x}{a}\right)^2}{1-\left(1-\frac{2x}{a}\right)^2}$$

$$p=\frac{1-\frac{2x}{a}}{\sqrt{1-\left(1-\frac{2x}{a}\right)^2}}$$

$$y=xy'+\frac{a}{2}\left(\sqrt{1+(y')^2}-y'\right)\Rightarrow$$

$$\Rightarrow y=\frac{x-\frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}+\frac{a}{2}\left(\sqrt{1+\frac{\left(1-\frac{2}{a}x\right)^2}{1-\left(1-\frac{2x}{a}\right)^2}}-\frac{1-\frac{2x}{a}}{\sqrt{1-\left(1-\frac{2x}{a}\right)^2}}\right)$$

kako se ovo rješenje ne može dobiti iz opšteg rješenja ovo je singularno rješenje

zadnji izraz se može pojednostaviti

$$Y = \frac{x - \frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}} + \frac{a}{2}\left(\sqrt{\frac{1-\cancel{\left(1-\frac{2}{a}x\right)^2}+\cancel{\left(1-\frac{2}{a}x\right)^2}}{1-\left(1-\frac{2}{a}x\right)^2}} - \frac{1-\frac{2}{a}x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}\right)$$

$$Y = \frac{x - \frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}} + \frac{\frac{a}{2}}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}} - \frac{\frac{a}{2} - x}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}$$

$$Y = \frac{2x - \frac{2}{a}x^2}{\sqrt{1-\left(1-\frac{2}{a}x\right)^2}}$$

singularno rješenje dif. jedn.