Zadaci sa ispita, 27. 01. 2009.

Neki zadaci nisu detaljno urađeni.

Za uočene greške pisati na infoarrt@gmail.com

1. Koliko racionalnih članova ima u razvoju $(\sqrt[3]{2}+\sqrt[4]{3})^{100}$?

Rj. $(\sqrt[3]{2}+\sqrt[4]{3})^{100}=\sum_{k=0}^{100}\binom{100}{k}(\sqrt[3]{2})^{100-k}\cdot(\sqrt[4]{3})^{k}=\sum_{k=0}^{100}\binom{100}{k}2^{\frac{100-k}{3}}\cdot 3^{\frac{k}{4}}$

Da bi član bio racionalan (pripadao skupu $\mathbb{Q}$) potrebno je i dovoljno da $\frac{100-k}{3}\in\mathbb{Z}$ i $\frac{k}{4}\in\mathbb{Z}$

Iz $\frac{k}{4}\in\mathbb{Z}$ vidimo da je k djeljivo sa 4 ($k\in\{0,4,8,\dots,80,\dots,100\}$)

Iz $\frac{100-k}{3}\in\mathbb{Z}$ vidimo da je $100-k$ djeljivo sa 3.

tj. $100-k\in\{0,3,6,9,\dots,90,93,96,99\}$

$\Downarrow$

$k\in\{1,4,7,10,\dots,91,94,97,100\}$

Ako stavimo da je

$A=\{0,4,8,\dots,80,\dots,100\}$

$B=\{1,4,7,10,\dots,91,94,97,100\}$

$A\cap B=\{4,16,28,40,52,64,76,88,100\}$

9 racionalnih članova ima u razvoju binoma.

2. Naći sve vrijednosti $\sqrt[3]{z}$ ako je $z=(\sqrt{3}-i)^9$.

Rj.

$(\sqrt{3}-i)^9=(\sqrt{3}-i)^4\cdot(\sqrt{3}-i)^4\cdot(\sqrt{3}-i)$

$(\sqrt{3}-i)^2=3-2i\sqrt{3}+i^2=2-2i\sqrt{3}$

$(\sqrt{3}-i)^4=(2-2i\sqrt{3})^2=4-8i\sqrt{3}+12i^2=-8-8i\sqrt{3}$

$(\sqrt{3}-i)^8=(-8-8i\sqrt{3})^2=64+128i\sqrt{3}+192i^2=-128+128i\sqrt{3}$

$(\sqrt{3}-i)^9=(-128+128i\sqrt{3})(\sqrt{3}-i)=\underline{-128\sqrt{3}}+128i+3\cdot128i+\underline{128\sqrt{3}}$

$(\sqrt{3}-i)^9=4\cdot128i=2^2\cdot2^7i=2^9i$

$z=2^9i=2^9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ $k=0,1,2$

$\sqrt[3]{z}$ računamo po formuli $z_k=\sqrt[3]{|z|}\left[\cos\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)+i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)\right]$

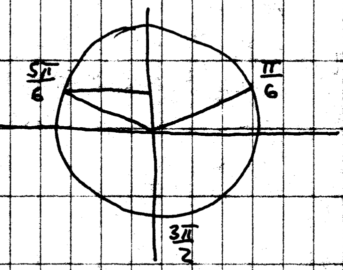
$z_0 = 8\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 8\cdot\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = 4\sqrt{3} + 4i = 4(\sqrt{3} + i)$

$z_1 = 8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 8\cdot\left(-\cos\frac{\pi}{6} + i\sin\frac{5\pi}{6}\right) = 8\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) =$

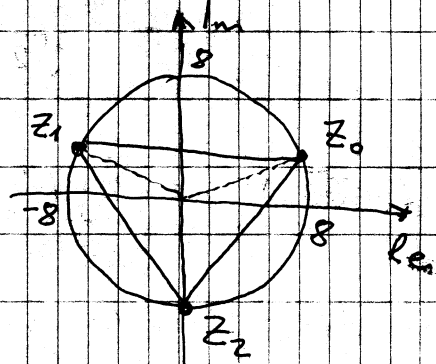
$= 4(-\sqrt{3} + i)$

$\dfrac{\frac{\pi}{2} + 2\pi}{3} = \dfrac{\frac{5\pi}{2}}{3} = \dfrac{5\pi}{6}$

$\dfrac{\frac{\pi}{2} + 4\pi}{3} = \dfrac{\frac{9\pi}{2}}{3} = \dfrac{9\pi}{6} = \dfrac{3\pi}{2}$



$z_2 = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -8i$



Sve vrijednosti $\sqrt[3]{z}$ su

$4(\sqrt{3} + i)$, $4(-\sqrt{3} + i)$ i $-8i$.

3. Diskutovati o rangu matrice $M = \begin{bmatrix} a & b & 1 \\ 1 & ab & 1 \\ 1 & b & a \end{bmatrix}$ u zavisnosti od parametara a i b.

Rj.

$$M = \begin{bmatrix} a & b & 1 \\ 1 & ab & 1 \\ 1 & b & a \end{bmatrix} \underset{I_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & b & a \\ 1 & ab & 1 \\ a & b & 1 \end{bmatrix} \underset{\substack{II_v - I_v \\ III_v - I_v \cdot a \\ (a \neq 0)}}{\sim} \begin{bmatrix} 1 & b & a \\ 0 & ab-b & 1-a \\ 0 & b-ab & 1-a^2 \end{bmatrix} \sim$$

$$\underset{III_v + II_v}{\sim} \begin{bmatrix} 1 & b & a \\ 0 & ab-b & 1-a \\ 0 & 0 & -a^2-a+2 \end{bmatrix} \sim \begin{bmatrix} 1 & b & a \\ 0 & b(a-1) & 1-a \\ 0 & 0 & -(a-1)(a+2) \end{bmatrix}$$

Kako sam množio sa a, razmatram još slučaj kad je $a = 0$

$$\begin{bmatrix} 0 & b & 1 \\ 1 & 0 & 1 \\ 1 & b & 0 \end{bmatrix} \underset{I_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & b & 0 \\ 1 & 0 & 1 \\ 0 & b & 1 \end{bmatrix} \underset{II_v - I_v}{\sim} \begin{bmatrix} 1 & b & 0 \\ 0 & -b & 1 \\ 0 & b & 1 \end{bmatrix} \sim$$

$$\underset{III_k \; II_k}{\sim} \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -b \\ 0 & 1 & b \end{bmatrix} \underset{III_v - II_v}{\sim} \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -b \\ 0 & 0 & 2b \end{bmatrix}$$

Diskusija:

1° za $a \neq 0$, $a \neq 1$, $a \neq -2$, $b \in R$ rang $M = 3$

2° za $a = 0$, $b \neq 0$ rang $M = 3$

3° za $a = 0$, $b = 0$ rang $M = 2$

4° za $a = 1$, $b \in R$ rang $M = 1$

5° za $a = -2$, $b \in R$ rang $M = 2$

4. Riješiti sistem jednačina i diskutovati rješenja u zavisnosti od parametra:

$$\begin{aligned} x + y + z &= 0 \\ x + (\lambda+1)y + 2z &= 1 \\ x + 2y + (4\lambda+1)z &= 2. \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda+1 & 2 \\ 1 & 2 & 4\lambda+1 \end{vmatrix} \;\begin{matrix} II_v - I_v \\ \overline{\overline{}} \\ III_v - I_v \end{matrix}\; \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 1 & 4\lambda \end{vmatrix} = 4\lambda^2 - 1 = (2\lambda-1)(2\lambda+1)$$

$$D_x = \begin{vmatrix} 0 & 1 & 1 \\ 1 & \lambda+1 & 2 \\ 2 & 2 & 4\lambda+1 \end{vmatrix} \;\underline{\underline{II_k - III_k}}\; \begin{vmatrix} 0 & 0 & 1 \\ 1 & \lambda-1 & 2 \\ 2 & -4\lambda+1 & 4\lambda+1 \end{vmatrix} = -4\lambda+1-2\lambda+2 =$$

$$= -6\lambda+3$$

$$= 3(-2\lambda+1)$$

$$= -3(2\lambda-1)$$

$$D_y = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4\lambda+1 \end{vmatrix} \;\begin{matrix} II_v - I_v \\ \overline{\overline{}} \\ III_v - I_v \end{matrix}\; \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 4\lambda \end{vmatrix} = 4\lambda - 2 = 2(2\lambda-1)$$

$$D_z = \begin{vmatrix} 1 & 1 & 0 \\ 1 & \lambda+1 & 1 \\ 1 & 2 & 2 \end{vmatrix} \;\begin{matrix} II_v - I_v \\ \overline{\overline{}} \\ III_v - I_v \end{matrix}\; \begin{vmatrix} 1 & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\lambda - 1$$

Diskusija:

1° $\lambda \neq \frac{1}{2}$ i $\lambda \neq -\frac{1}{2}$ rješenje sistema je $\left(-\frac{3}{2\lambda+1}, \frac{2}{2\lambda+1}, \frac{1}{2\lambda+1}\right)$

2° $\lambda = -\frac{1}{2}$, $D_x \neq 0$ sistem nema rješenja

3° $\lambda = \frac{1}{2}$, $D = D_x = D_y = D_z = 0$

sistem postaje:

$$\begin{aligned} x + y + z &= 0 \quad (1) \\ x + \tfrac{3}{2}y + 2z &= 1 \quad (2) \\ x + 2y + 3z &= 2 \quad (3) \end{aligned}$$

$(1)-(3)$: $-y - 2z = -2 \quad / \cdot (-1)$

$(2)-(3)$: $-\frac{1}{2}y - z = -1 \quad / \cdot (-2)$

$$y + 2z = 2$$

$$y + 2z = 2$$

$$y = 2 - 2z$$

$$x + 2 - 2z + z = 0$$

$$x = z - 2$$

Sistem ima beskonačno mnogo rješenja:

$(z-2,\ 2-2z,\ z)$, $z \in R$

5. Ispitati f-ju $y = \frac{x^4}{(1+x)^3}$ i nacrtati njen grafik.

Rj. $\mathcal{D}: (-\infty, -1) \cup (-1, +\infty)$

x	$(-\infty, -1)$	$(-1, +\infty)$
y	−	+

znak f-je

f-ja nije ni parna ni neparna
nije periodična
(0, 0) je nula f-je i presjek sa y-osom

$\lim_{x \to -1-0} f(x) = -\infty \Rightarrow x = -1$ je vertikalna asimptota

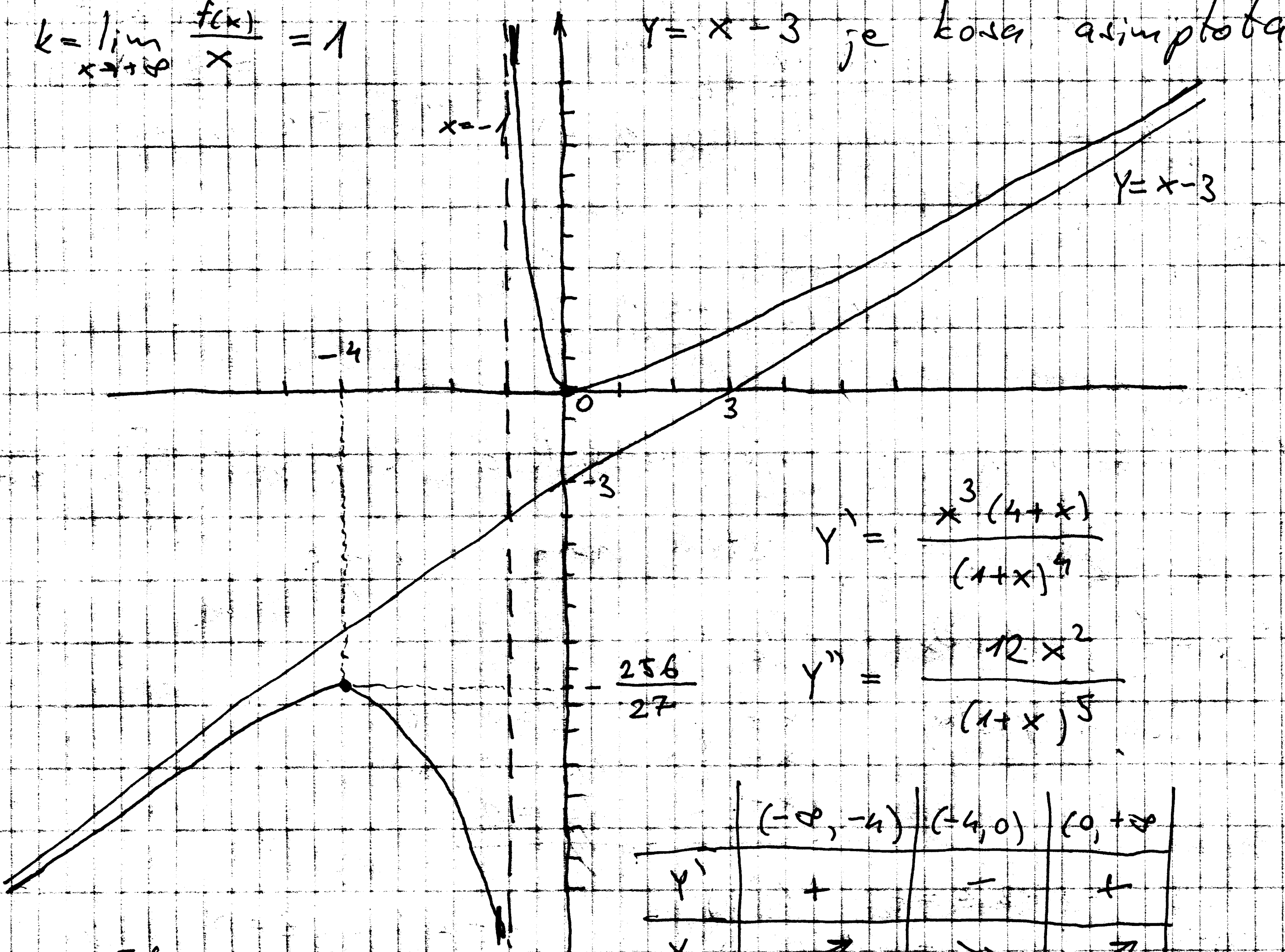
$\lim_{x \to -1+0} f(x) = +\infty \Rightarrow x = -1$ je vertikalna asimptota

$\lim_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow$ f-ja nema horizontalnu asimptotu

kosa $y = kx + n$, $n = \lim_{x \to \infty} [f(x) - k \cdot x] = -3$

$k = \lim_{x \to +\infty} \frac{f(x)}{x} = 1$

$y = x - 3$ je kosa asimptota



$y' = \frac{x^3(4+x)}{(1+x)^4}$

$y'' = \frac{12x^2}{(1+x)^5}$

	$(-\infty, -4)$	$(-4, 0)$	$(0, +\infty)$
y'	+	−	+
y	↗	↘	↗

rast i opadanje

$\left(-4, -\frac{256}{27}\right)$ tačka maks.

(0, 0) tač. min.

	$(-\infty, -1)$	$(-1, 0)$	$(0, +\infty)$
y''	−	+	+
y	∩	∪	∪

konveksnost i konkavnost

f-ja nema prevojnih tački

(6.) Ispitati f-ju $y=(2x+1)e^{-\frac{2}{x}}$ i nacrtati njen grafik.

Rj: $D: x \in (-\infty, 0) \cup (0, +\infty)$

f-ja nije ni parna ni neparna

nije periodična

x	$(-\infty, -\frac{1}{2})$	$(-\frac{1}{2}, +\infty)$
y	$-$	$+$

znak f-je

$(-\frac{1}{2}, 0)$ nula f-je, $f(0)$ nije definisano

f-ja ne siječe y-osu

$\lim\limits_{x\to 0-} f(x) = +\infty \Rightarrow x=0$ je vertikalna asimptota

$\lim\limits_{x\to 0+} f(x) = 0$, $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty \Rightarrow$ f-ja nema horizontalnu asimptotu

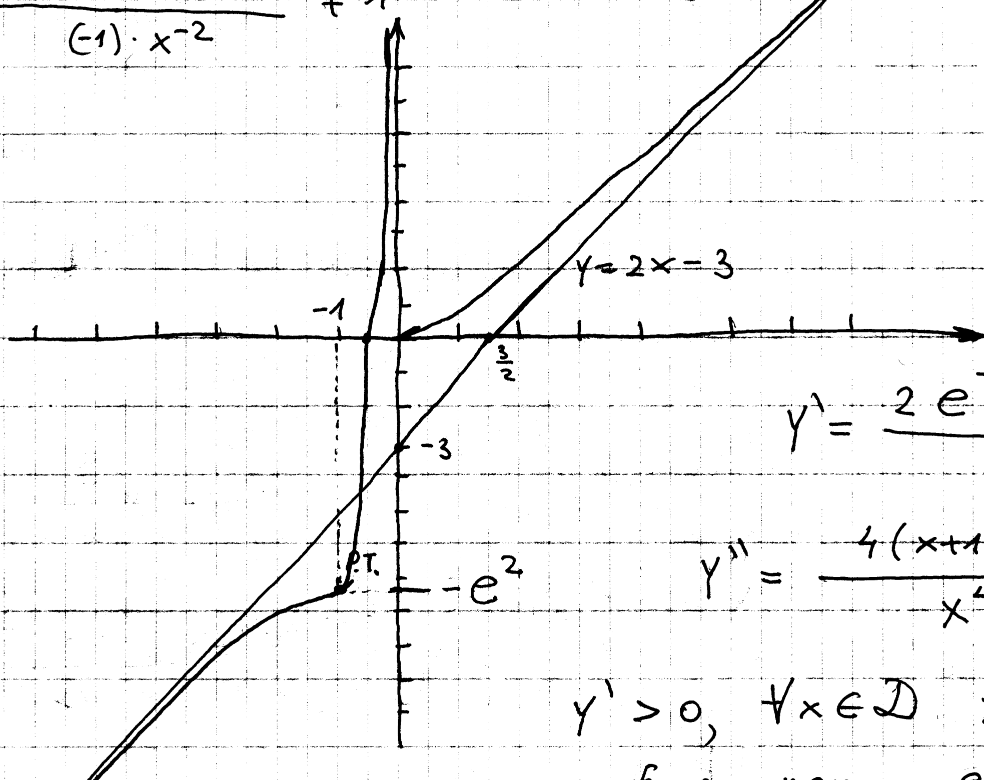
$y = kx+n$, $k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = 2$

$n = \lim\limits_{x\to\infty} [f(x) - kx] = \lim\limits_{x\to\infty} \left[(2x+1)e^{-\frac{2}{x}} - 2x\right] = \lim\limits_{x\to\infty}\left[2xe^{-\frac{2}{x}} + e^{-\frac{2}{x}} - 2x\right] =$

$= \lim\limits_{x\to\infty} \underbrace{\left[2x(e^{-\frac{2}{x}} - 1)\right]}_{\infty\cdot 0 \text{ neodređeno}} + \underbrace{\lim\limits_{x\to\infty} e^{-\frac{2}{x}}}_{1} = \lim\limits_{x\to\infty} \frac{2\cdot(e^{-\frac{2}{x}}-1)}{\frac{1}{x}} + 1 \overset{L.P.}{=}$

$= \lim\limits_{x\to\infty} \frac{2\cdot e^{-\frac{2}{x}}\cdot(-2)\cdot(-1)\cdot x^{-2}}{(-1)\cdot x^{-2}} + 1 = -4+1 = -3$

$y = 2x-3$ kosa asimptota



$$y' = \frac{2e^{-\frac{2}{x}}(x+1)^2}{x^2}$$

$$y'' = \frac{4(x+1)e^{-\frac{2}{x}}}{x^4}$$

$y' > 0, \forall x \in D$ f-ja raste

f-ja nema ekstrema

x	$(-\infty, -1)$	$(-1, 0)$	$(0, +\infty)$
y''	$-$	$+$	$+$
y	$\frown$	$\smile$	$\smile$

P.T.

$(-1, -e^2)$ je prevojna tačka

7. Ispitati f-ju $y = 3\ln\frac{x}{x-3} - 1$ i nacrtati njen grafik.

Rj. $\mathcal{D}: x \in (-\infty, 0) \cup (3, +\infty)$

f-ja nije ni parna ni neparna

nije periodična

x	$(-\infty, 0)$	$\left(3, \frac{3\sqrt[3]{e}}{\sqrt[3]{e}-1}\right)$	$\left(\frac{3\sqrt[3]{e}}{\sqrt[3]{e}-1}, +\infty\right)$
y	$-$	$-$	$+$

znak f-je

$\left(\frac{3\sqrt[3]{e}}{\sqrt[3]{e}-1}, 0\right)$ nula f-je,

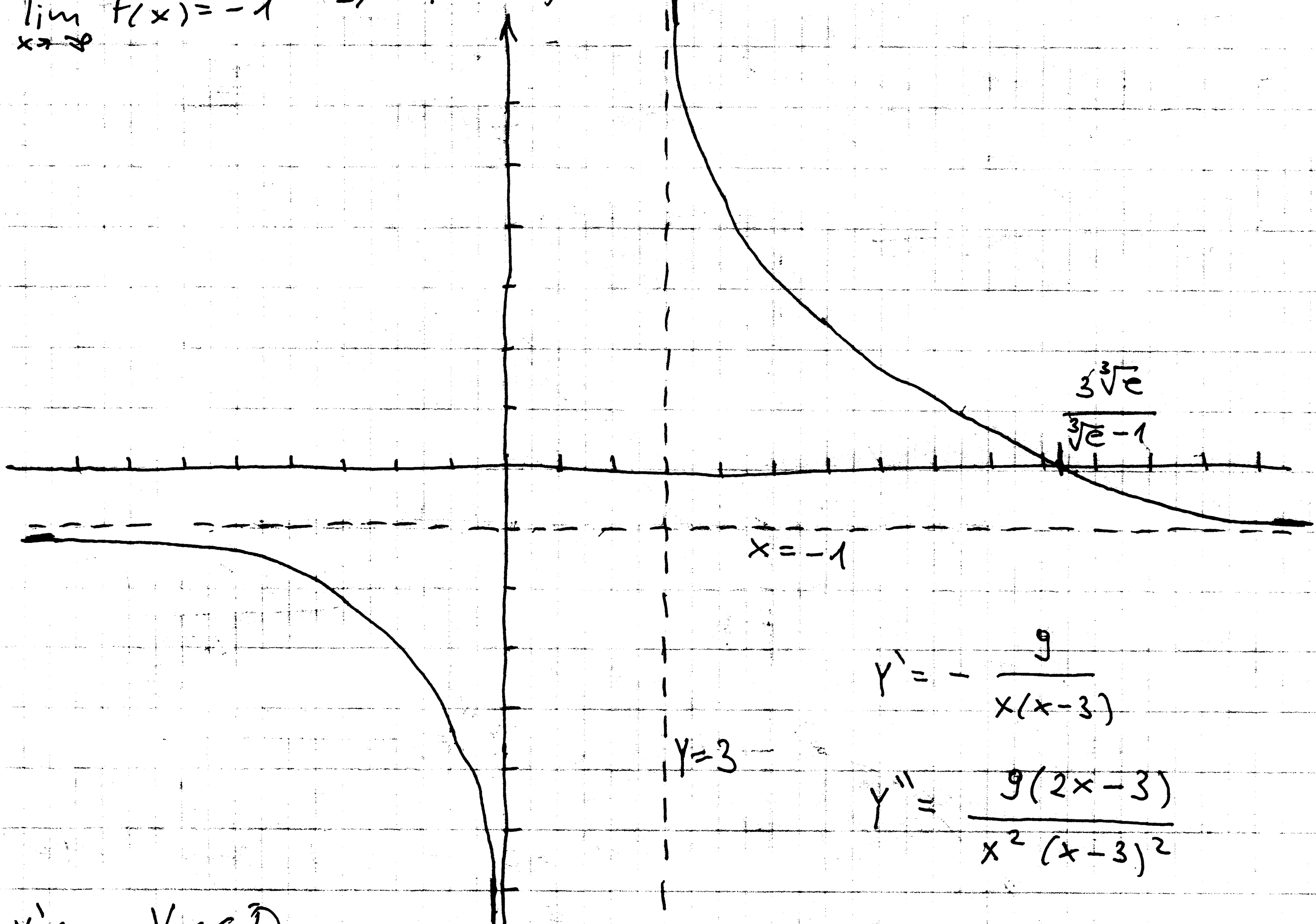
$f(0)$ nije definisano, f-ja ne siječe y-osu

$\lim\limits_{x\to 0-} f(x) = -\infty \Rightarrow x = 0$ je vertikalna asimptota

$\lim\limits_{x\to 3+0} f(x) = +\infty \Rightarrow x = 3$ je vertikalna asimptota

$\lim\limits_{x\to -\infty} f(x) = -1 \Rightarrow y = -1$ je horizontalna asimptota

$\lim\limits_{x\to \infty} f(x) = -1 \Rightarrow y = -1$ je horizontalna asimptota



$$y' = -\frac{9}{x(x-3)}$$

$$y'' = \frac{9(2x-3)}{x^2(x-3)^2}$$

$y' < 0 \quad \forall x \in \mathcal{D}$

$\Rightarrow$ f-ja uvijek opada

f-ja nema ekstrema

x	$(-\infty, 0)$	$(3, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

konveksnost i konk.

8. Ispitati f-ju $y=-1+\frac{1}{a-bx^3}$ i nacrtati njen grafik ako se zna da joj je prevojna tačka $P\left(-\frac{1}{\sqrt[3]{2}}, -\frac{1}{3}\right)$.

$$y\left(-\frac{1}{\sqrt[3]{2}}\right)=-\frac{1}{3}$$

Rj. $y'=\frac{3bx^2}{(a-bx^3)^2}$, $y''=\frac{-6bx(2bx^3+a)}{(-a+bx^3)^3}$

$$y''\left(-\frac{1}{\sqrt[3]{2}}\right)=0$$

$$a=b=1$$

$y=\frac{x^3}{1-x^3}$, $D: x\in(-\infty,1)\cup(1,+\infty)$

f-ja nije ni parna ni neparna
nije periodična

(0,0) je nula f-je i presjek sa y-osom

x	$(-\infty,0)$	$(0,1)$	$(1,+\infty)$
x^3	−	+	+
$1-x^3$	+	+	−
y	−	+	−

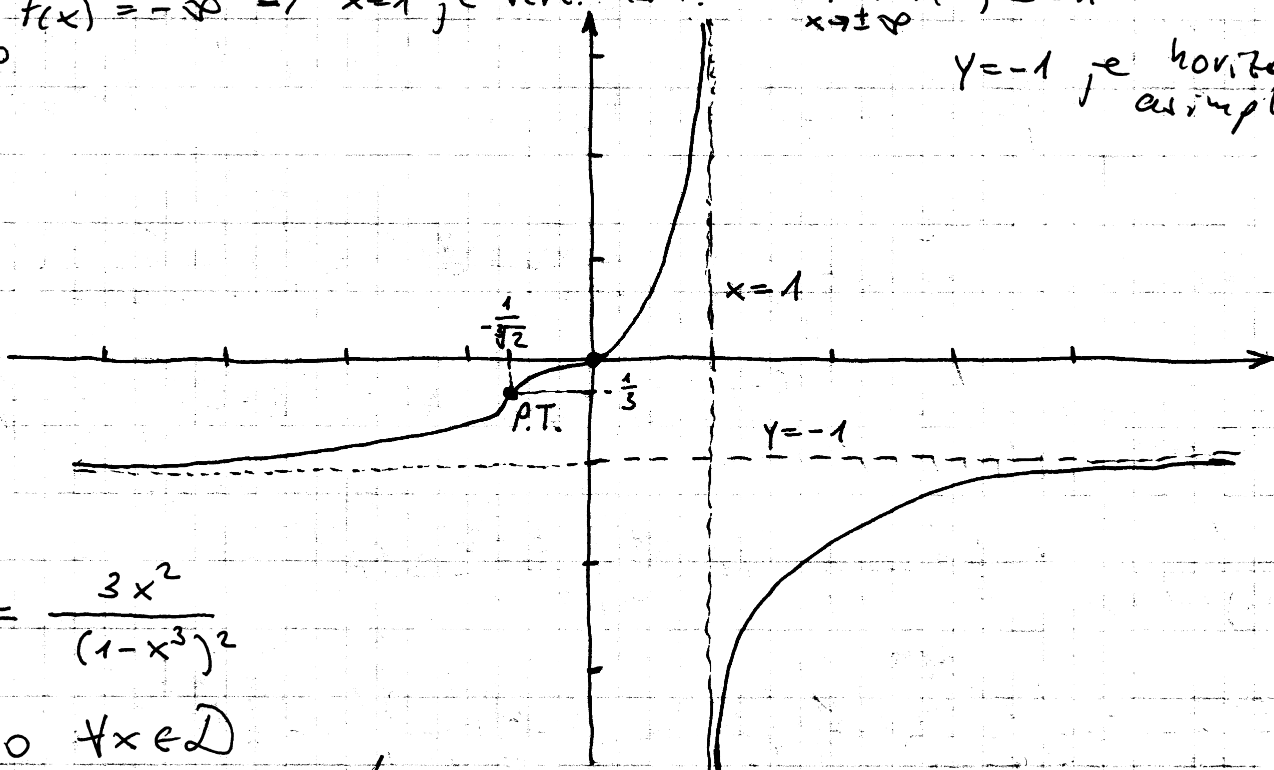
znak

$\lim_{x\to1-0} f(x)=+\infty \Rightarrow x=1$ je vert. asim.

$\lim_{x\to1+0} f(x)=-\infty \Rightarrow x=1$ je vert. asim.

$\lim_{x\to\pm\infty} f(x)=-1$

$y=-1$ je horizont. asimptota



$y'=\frac{3x^2}{(1-x^3)^2}$

$y'>0 \quad \forall x\in D$

f-ja uvijek raste, nema ekstrema

$y''=\frac{-6x(2x^3+1)}{(-1+x^3)^3}$

x	$\left(-\infty,-\frac{1}{\sqrt[3]{2}}\right)$	$\left(-\frac{1}{\sqrt[3]{2}},0\right)$	$(0,1)$	$(1,+\infty)$
y''	+	−	+	−
y	∪	∩	∪	∩
		P.T.	P.T.	

$\left(-\frac{1}{\sqrt[3]{2}},-\frac{1}{3}\right)$ i (0,0) su prevojne tačke f-je

9. Izračunati integral $A=\int\frac{\ln x\,dx}{x\sqrt{1+\ln x}}$.

Rj. $A=\int\frac{\ln x}{x\sqrt{1+\ln x}}\,dx=\left|\begin{matrix}\ln x=t\\ \frac{1}{x}dx=dt\end{matrix}\right|=\int\frac{\overset{+1\,-1}{t}}{\sqrt{1+t}}\,dt=$

$$=\int\frac{t+1}{\sqrt{1+t}}\,\frac{\cdot\sqrt{1+t}}{\cdot\sqrt{1+t}}\,dt-\int\frac{1}{\sqrt{1+t}}\,dt\doteq\int\sqrt{1+t}\,dt-\int(1+t)^{-\frac{1}{2}}dt$$

$$=\frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(1+t)^{\frac{1}{2}}}{\frac{1}{2}}+C=\frac{2}{3}\sqrt{(1+\ln x)^3}-2\sqrt{1+\ln x}+C$$

$$=\frac{2}{3}\sqrt{1+\ln x}\left(1+\ln x-3\right)+C=\frac{2}{3}(\ln x-2)\sqrt{1+\ln x}+C$$

10. Izračunati integral $I=\int\frac{\operatorname{arc\,tg} x^3}{x^2}\,dx$.

Rj. $\int\frac{\operatorname{arc\,tg} x^3}{x^2}=\left|\begin{matrix}u=\operatorname{arc\,tg} x^3 & dv=\frac{1}{x^2}\\ du=\frac{3x^2}{x^6+1} & v=-\frac{1}{x}dx\end{matrix}\right|=-\frac{1}{x}\cdot\operatorname{arc\,tg} x^3+3\int\frac{x}{x^6+1}\,dx$

$$\int\frac{x}{x^6+1}\,dx=\left|\begin{matrix}x^2=t\\ 2x\,dx=dt\end{matrix}\right|=\frac{1}{2}\int\frac{1}{t^3+1}\,dt=\frac{1}{2}\int\frac{1}{(t+1)(t^2-t+1)}\,dt$$

$$\frac{1}{(t+1)(t^2-t+1)}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1}\Longrightarrow A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}$$

$$\int\frac{1}{(t+1)(t^2-t+1)}\,dt=\frac{1}{3}\int\frac{1}{t+1}\,dt-\frac{1}{3}\int\frac{t-2}{t^2-t+1}\,dt$$

$$\int\frac{t-2}{t^2-t+1}\,dt=\frac{1}{2}\int\frac{2t\overset{-1-3}{-4}}{t^2-t+1}\,dt=\frac{1}{2}\int\frac{2t-1}{t^2-t+1}\,dt-\frac{3}{2}\int\frac{dt}{t^2-t+1}=\frac{1}{2}\ln|t^2-t+1|-\frac{3}{2}\int\frac{dt}{t^2-t+1}$$

$$\int\frac{dt}{t^2-t+1}=\left|\begin{matrix}t^2-t+1=t^2-2\cdot\frac{1}{2}t+\frac{1}{4}-\frac{1}{4}+1\\ =(t-\frac{1}{2})^2+\frac{3}{4}\end{matrix}\right|=\int\frac{dt}{(t-\frac{1}{2})^2+\frac{3}{4}}=\left|\begin{matrix}t-\frac{1}{2}=\frac{\sqrt{3}}{2}s\\ dt=\frac{\sqrt{3}}{2}ds\end{matrix}\right|=$$

$$=\frac{\sqrt{3}}{2}\int\frac{ds}{\frac{3}{4}s^2+\frac{3}{4}}=\frac{\sqrt{3}}{2}\cdot\frac{4}{3}\int\frac{ds}{s^2+1}=\frac{2\sqrt{3}}{3}\operatorname{arc\,tg} s=\frac{2\sqrt{3}}{3}\operatorname{arc\,tg}\frac{2t-1}{\sqrt{3}}$$

Prema tome: $\int\frac{\operatorname{arc\,tg} x^3}{x^2}=-\frac{1}{x}\operatorname{arc\,tg} x^3+\frac{1}{2}\ln(x^2+1)-\frac{1}{4}\ln(x^4-x^2+1)+\frac{\sqrt{3}}{2}\operatorname{arc\,tg}\frac{2x^2-1}{\sqrt{3}}+C$

11. Izračunati površinu figure koju obrazuju linije $y = 9 - x^2$, $x \geq 0$, $y = 8x$ i $y = \frac{5}{2}x$.

Rj:

$y = 9 - x^2$

$x_{1,2} = \pm 3 \Rightarrow y = 0$

$x = 0 \Rightarrow y = 9$

$$\begin{aligned} y &= 9 - x^2 \\ y &= 8x \end{aligned}$$

$8x = 9 - x^2$

$x^2 + 8x - 9 = 0$

$(1, 8)$ i $(-9, -72)$ su tačke presjeka prave $y = 8x$ i parabole $y = 9 - x^2$

$$\begin{aligned} y &= 9 - x^2 \\ y &= \frac{5}{2}x \end{aligned}$$

$\frac{5}{2}x = 9 - x^2$

$x^2 + \frac{5}{2}x - 9 = 0 \quad / \cdot 2$

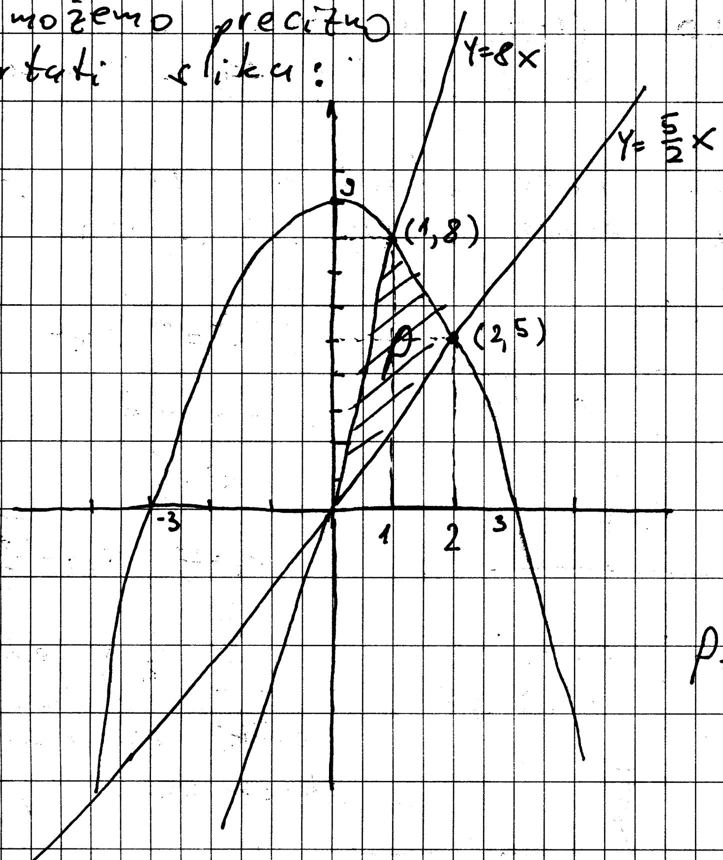
$2x^2 + 5x - 18 = 0$

$2\left(x + \frac{9}{2}\right)(x - 2) = 0$

$(2x + 9)(x - 2) = 0$

$(2, 5)$ i $\left(-\frac{9}{2}, -\frac{45}{2}\right)$ su tačke presjeka prave $y = \frac{5}{2}x$ i parabole $y = 9 - x^2$

sad možemo precizno nacrtati sliku:



$P = ?$

$$P = \int_0^1 \left(8x - \frac{5}{2}x\right) dx + \int_1^2 \left(9 - x^2 - \frac{5}{2}x\right) dx$$

$$P = \int_0^1 \frac{11}{2}x \, dx + \int_1^2 \left(-x^2 - \frac{5}{2}x + 9\right) dx$$

$$\int_0^1 \frac{11}{2}x \, dx = \frac{11}{2} \cdot \frac{x^2}{2}\Big|_0^1 = \frac{11}{4}x^2\Big|_0^1 = \frac{11}{4}(1 - 0) = \frac{11}{4}$$

$$\int_1^2 \left(-x^2 - \frac{5}{2}x + 9\right) dx = -\frac{x^3}{3}\Big|_1^2 - \frac{5}{2} \cdot \frac{x^2}{2}\Big|_1^2 + 9x\Big|_1^2 = -\frac{1}{3}\underbrace{x^3\Big|_1^2}_{8-1} - \frac{5}{4}\underbrace{x^2\Big|_1^2}_{4-1} + 9(2-1)$$

$$= -\frac{7}{3} - \frac{15}{4} + 9 = \frac{35}{12}$$

$$P = \frac{11}{4} + \frac{35}{12} = \frac{68}{12} = \frac{17}{3}$$

Površina figure koju obrazuju linije iznosi $\frac{17}{3}$.

12. Izračunati površinu dijela ravni koji se nalazi izvan kružnice $x^2+y^2=3$, a unutar elipse $x^2+\frac{y^2}{9}=1$ i u prvom kvadrantu.

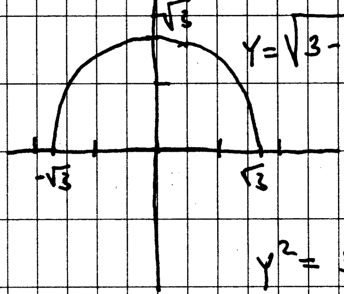
Rj: $(x-p)^2+(y-q)^2=r^2$

jednačina kruga poluprečnika r sa centrom u tački (p,q)

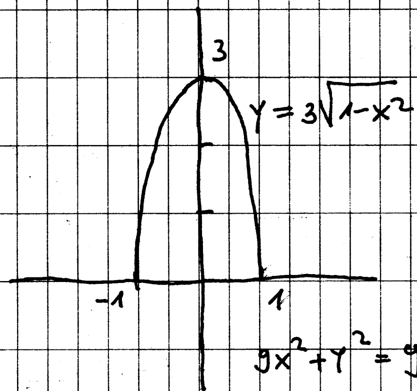
krug $x^2+y^2=3$ ima centar u tački $(0,0)$ i poluprečnik $\sqrt{3}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ jednačina elipse koja x-osu siječe u tačkama $(a,0)$ i $(-a,0)$, y-osu siječe u tačkama $(0,b)$ i $(0,-b)$, sa centrom u $(0,0)$

$x^2+\frac{y^2}{9}=1$ elepsa, x-osu siječe u tačkama $(1,0)$ i $(-1,0)$, y-osu siječe u $(0,3)$ i $(0,-3)$



$y^2=3-x^2$



$9x^2+y^2=9$

$y^2=9-9x^2$

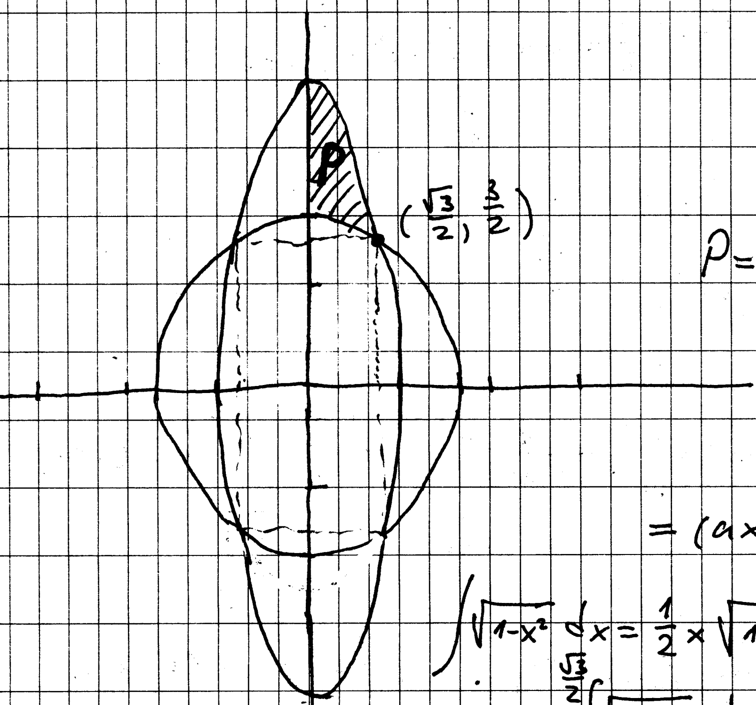
$3-x^2=9-9x^2$

$8x^2=6$

$x^2=\frac{3}{4}$

$x_1=\frac{\sqrt{3}}{2},\ x_2=-\frac{\sqrt{3}}{2}$

$\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$ jedna od tački presjeka kružnice i elipse



$$P=\int_0^{\frac{\sqrt{3}}{2}}\left(3\sqrt{1-x^2}-\sqrt{3-x^2}\right)dx$$

$$\int\sqrt{1-x^2}\,dx=\int\frac{1-x^2}{\sqrt{1-x^2}}=$$

$$=(ax+b)\sqrt{1-x^2}+\lambda\int\frac{dx}{\sqrt{1-x^2}}\Rightarrow\quad a=\frac{1}{2},\ b=0,\ \lambda=\frac{1}{2}$$

$$\int\sqrt{1-x^2}\,dx=\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\arcsin x+C$$

$$\int_0^{\frac{\sqrt{3}}{2}}\sqrt{1-x^2}\,dx=\frac{3\sqrt{3}+4\pi}{24}\Rightarrow 3\int_0^{\frac{\sqrt{3}}{2}}\sqrt{1-x^2}\,dx=\frac{3\sqrt{3}+4\pi}{8}$$

$$\int\sqrt{3-x^2}\,dx=\frac{1}{2}x\sqrt{3-x^2}+\frac{3}{2}\arcsin\frac{x}{\sqrt{3}}$$

$$P=\frac{1}{8}(3\sqrt{3}+4\pi-3\sqrt{3}-2\pi)=\frac{\pi}{4}$$ tražena površ.

13. Naći ekstreme f-je $z=-2x^2+4x^2y^2-2y^2$

Rj. $\frac{\partial z}{\partial x}=-4x+8xy^2$

$\frac{\partial z}{\partial y}=8x^2y-4y$

$$-4x+8xy^2=0$$
$$8x^2y-4y=0$$

$$-4x(1-2y^2)=0$$
$$4y(2x^2-1)=0$$

stacionarne tačke su

$M_1(0,0)$, $M_2\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$,

$M_3\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $M_4\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$M_5\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$\frac{\partial^2 z}{\partial x^2}=-4+8y^2$

$\frac{\partial^2 z}{\partial x\partial y}=16xy$

$\frac{\partial^2 z}{\partial y^2}=8x^2-4$

$M_1(0,0)$

$A=-4$ $\quad D=AC-B^2$

$B=0$ $\quad D=16>0 \Rightarrow$ f-ja ima ekstrem

$C=-4$ $\quad A<0$ f-ja ima maksimum

$z_{max}(0,0)=0$

$M_2\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$A=0$ $\quad D<0$

$B=8$ $\quad$ u tački M_2 f-ja

$C=0$ $\quad$ nema ekstrem

$M_3\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$A=0$ $\quad D<0$

$B=-8$ $\quad$ u tački M_3

$C=0$ $\quad$ f-ja nema ekstrem

$M_4\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$A=0$ $\quad D<0$

$B=-8$ $\quad$ u tački M_4

$C=0$ $\quad$ f-ja nema ekstrem

$M_5\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$A=0$ $\quad D<0$

$B=8$ $\quad$ u tački M_5

$C=0$ $\quad$ f-ja nema ekstrem

F-ja ima ekstrem u tački $M_1(0,0)$,
u toj tački f-ja z ima maksimum i on iznosi 0.

14. Naći uslovne ekstreme f-je $z=\ln(x+y)$ ako je $x^2+2y^2=4$.

Rj.
$$F(x,y)=\ln(x+y)+\lambda(x^2+2y^2-4)$$

$$\frac{\partial F}{\partial x}=\frac{1}{x+y}+2\lambda x \qquad \frac{1}{x+y}+2\lambda x=0$$

$$\frac{\partial F}{\partial y}=\frac{1}{x+y}+4\lambda y \qquad \frac{1}{x+y}+4\lambda y=0$$

$$\frac{\partial F}{\partial \lambda}=x^2+2y^2-4 \qquad x^2+2y^2-4=0$$

$$2\lambda x=4\lambda y \quad /:2\lambda$$
$$x=2y$$
$$x^2=4y^2$$

$$4y^2+2y^2=4$$
$$6y^2=4$$
$$y^2=\frac{2}{3}$$
$$y_1=-\sqrt{\frac{2}{3}}$$
$$y_2=\sqrt{\frac{2}{3}}$$

$$y_1=-\sqrt{\frac{2}{3}} \Rightarrow x_1=-2\sqrt{\frac{2}{3}}$$

$(x_1,y_1)\notin \mathcal{D}$ za $z=\ln(x+y)$ (definic. podr.)

$$y=\sqrt{\frac{2}{3}} \Rightarrow x=2\sqrt{\frac{2}{3}}$$

$$\frac{1}{3\sqrt{\frac{2}{3}}}+4\sqrt{\frac{2}{3}}\,\lambda=0 \quad /\cdot 3\sqrt{\frac{2}{3}}$$

$$12\cdot\frac{2}{3}\lambda=-1 \Rightarrow \lambda=-\frac{1}{8}$$

$M\left(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$ je stacionarna tačka (za $\lambda=-\frac{1}{8}$)

$$\frac{\partial^2 F}{\partial x^2}=-\frac{1}{(x+y)^2}+2\lambda$$

$$\frac{\partial^2 F}{\partial x\partial y}=-\frac{1}{(x+y)^2}$$

$$\frac{\partial^2 F}{\partial y^2}=-\frac{1}{(x+y)^2}+4\lambda$$

$M\left(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$, $\lambda=-\frac{1}{8}$

$$A=-\frac{1}{9\cdot\frac{2}{3}}+2\cdot\left(-\frac{1}{8}\right)=-\frac{1}{6}-\frac{1}{4}=-\frac{5}{12}$$

$$B=-\frac{1}{6}$$

$$C=-\frac{1}{6}-\frac{1}{2}=\frac{-4}{6}=-\frac{2}{3}$$

$$D=AC-B^2$$

$$D=\frac{10}{36}-\frac{1}{36}=\frac{9}{36}>0$$ f-ja ima ekstrem

$A<0 \Rightarrow$ maksimum

$$z_{max}\left(2\sqrt{\tfrac{2}{3}},\sqrt{\tfrac{2}{3}}\right)=\ln\left(3\sqrt{\tfrac{2}{3}}\right)$$

vrijednost f-je u tački $M\left(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$.

15. Riješiti diferencijalnu jednačinu

$$(2x^2 - y^2)\,y' = 6xy.$$

Rj. $(2x^2-y^2)\,y' = 6xy \quad /: x^2 \ (x \neq 0$

$\left(2-\frac{y^2}{x^2}\right)y' = 6\frac{y}{x} \quad /: 2-\frac{y^2}{x^2} \quad \left(\frac{y^2}{x^2} \neq 2\right)$

$y' = \frac{6\frac{y}{x}}{2-\left(\frac{y}{x}\right)^2}$, ovo je homogena dif. jedn. $y' = f\left(\frac{y}{x}\right)$

smjena $u = \frac{y}{x}$, $y = ux$, $y' = u'x + u$

$u'x + u = \frac{6u}{2-u^2}$

$u'x = \frac{6u}{2-u^2} - u$, $u'x = \frac{6u - 2u + u^3}{2-u^2}$

$u'x = \frac{u^3+4u}{2-u^2}$, $u' = \frac{du}{dx}$, $\frac{du}{dx}x = \frac{u(u^2+4)}{2-u^2}$

$\frac{2-u^2}{u(u^2+4)}du = \frac{1}{x}dx$

$\frac{2-u^2}{u(u^2+4)} = \frac{A}{u} + \frac{Bu+C}{u^2+4} \implies A = \frac{1}{2}$, $B = -\frac{3}{2}$, $C = 0$

$\int \frac{2-u^2}{u(u^2+4)} = \frac{1}{2}\int \frac{1}{u}du - \frac{3}{2}\int \frac{u}{u^2+4}du \overset{(*)}{=} \frac{1}{2}\ln u - \frac{3}{4}\ln(u^2+4)$

$\int \frac{u}{u^2+4}du = \left| \begin{array}{l} u^2+4 = t \\ 2u\,du = dt \\ u\,du = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|u^2+4| \quad \ldots(*)$

$\int \frac{2-u^2}{u(u^2+4)}du = \int \frac{1}{x}dx$

$\ln\sqrt{u} - \ln\sqrt[4]{(u^2+4)^3} = \ln x + \ln C$

$\ln \frac{\sqrt{u}}{\sqrt[4]{(u^2+4)^3}} = \ln Cx$

$\sqrt{u} = Cx\sqrt[4]{(u^2+4)^3}$

$\sqrt{\frac{y}{x}} = Cx\sqrt[4]{\left(\frac{y^2+4x^2}{x^2}\right)^3}$

$y^2 = C(y^2+4x^2)^3$

rješenje diferenc. jednačine

(16.) Riješiti diferencijalnu jednačinu

$$3xy' + y + x^2y^4 = 0.$$

Rj: $3xy' + y = -x^2y^4 \quad / : 3x \quad (x \neq 0)$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4$$ ovo je Bernulijeva dif. jedn.

$$y' + p(x)y = q(x)\cdot y^4$$

smjena $\quad y = uv$

$$y' = u'v + uv'$$

$$u'v + u.v' + \frac{1}{3x}uv = -\frac{x}{3}(uv)^4$$

$$u'v + \underbrace{u\left(v' + \frac{1}{3x}v\right)}_{=0} = -\frac{x}{3}u^4v^4$$

$$v' + \frac{1}{3x}v = 0$$

$$\frac{dv}{dx} = -\frac{1}{3}\cdot\frac{1}{x}\cdot v$$

$$\frac{dv}{v} = -\frac{1}{3}\frac{dx}{x} \quad \Big/ \int$$

$$\ln|v| = -\frac{1}{3}\ln|x|$$

$$v = x^{-\frac{1}{3}}$$

$$v^3 = x^{-1}$$

$$v^3 = \frac{1}{x}$$

$$u'\cdot x^{-\frac{1}{3}} = -\frac{1}{3}x\cdot u^4\cdot x^{-\frac{4}{3}} \quad \Big/ \cdot x^{\frac{1}{3}}$$

$$u' = -\frac{1}{3}x^{-\frac{1}{3}}\cdot x^{\frac{1}{3}}\cdot u^4$$

$$\frac{du}{dx} = -\frac{1}{3}u^4$$

$$u^{-4}du = -\frac{1}{3}dx \quad \Big/ \int$$

$$\frac{u^{-3}}{-3} = -\frac{1}{3}x + C_1 \quad \Big/ \cdot(-3)$$

$$u^{-3} = x + C$$

$$u^3 = \frac{1}{x+C}$$

$$y = uv$$

$$y^3 = u^3\cdot v^3$$

$$y^3 = \frac{1}{x}\cdot\frac{1}{x+C}$$

$$y^3 = \frac{1}{x^2 + Cx}$$ rješenje diferencijalne jednačine