Univerzitet u Zenici
Ekonomski fakultet

Zadaci sa pismenog ispita rađenog 20.04.2009. iz predmeta MATEMATIKA, sve četri grupe

1. Odrediti realni i imaginarni dio broja $z = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^{17}(cos\frac{5\pi}{12} + isin\frac{5\pi}{12})$.

2. Riješiti matričnu jednačinu $(A + XB)^{-1} = BX^{-1}$, ako je $A = \begin{bmatrix} 1 & 8 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}$.

3. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} (\lambda-1)x + y - 4z &= 2 \\ 6x + 2y - (\lambda+1)z &= 7 \\ x + y - z &= 3 \end{aligned}.$$

4. Date su dvije trojke vektora u vektorskom prostoru $V_3 : \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ i $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, pri čemu je $\vec{a}_1 = (3, 5, 2)$, $\vec{a}_2 = (-2, 3, 4)$, $\vec{a}_3 = (-2, 4, -3)$, $\vec{b}_1 = (1, -3, 0)$, $\vec{b}_2 = (2, 4, 1)$, $\vec{b}_3 = (5, 1, 2)$. Odrediti koja od ovih trojki predstavlja bazu prostora V_3 i razložiti vektor $\vec{c} = (-1, 15, -18)$ u toj bazi.

5. Ispitati funkciju $y = \dfrac{x^3}{(x^2-4)^2}$ i nacrtati njen grafik.

6. Ispitati funkciju $y = \dfrac{x^2 + bx + c}{x^2 - 9}$ i nacrtati njen grafik ako se zna da ona ima ekstrem u tački $T(0, -\frac{2}{9})$.

7. Ispitati funkciju $y = (x-6)e^{-\frac{1}{x}}$ i nacrtati njen grafik.

8. Ispitati funkciju $y = \dfrac{x\,lnx}{lnx - 1}$ i nacrtati njen grafik.

9. Odrediti integral $I = \displaystyle\int \frac{dx}{x(\sqrt{x} + \sqrt[5]{x^2})}$.

10. Izračunati integral $I = \displaystyle\int_0^{\frac{\pi}{4}} \frac{x\,sinx}{cos^3x}\,dx$.

11. Izračunati površinu elipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

12. Izračunati površinu figure koja je ograničena kružnicom $x^2 + y^2 = 2$ i parabolom $y^2 = 2x - 1$ ako je $x \geq \frac{1}{2}$.

13. Odrediti ekstreme funkcije $z(x, y) = x^2 - 6xy + y^3 + 3x + 6y$.

14. Naći sve ekstreme funkcije $z(x, y) = 8xy + \dfrac{1}{x} + \dfrac{1}{y}$.

15. Riješiti diferencijalnu jednačinu $(x - 3y + 3)y' = 6y - 2x - 5$.

16. Riješiti diferencijalnu jednačinu $(xy^2 + 3x)dx + (2x^2y - 5y)dy = 0$.

Pismeni ispit iz predmeta Matematika, 20.04.2009.
Neki zadaci nisu detaljno urađeni.
Za uočene greške pisati na infoarrt@gmail.com

1. Odrediti realni i imaginarni dio broja

$$z=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{17}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right).$$

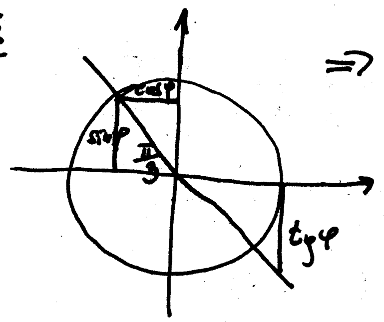
Rj.

$$z_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \qquad \sin\varphi=\frac{\frac{\sqrt{3}}{2}}{1}=\frac{\sqrt{3}}{2}$$

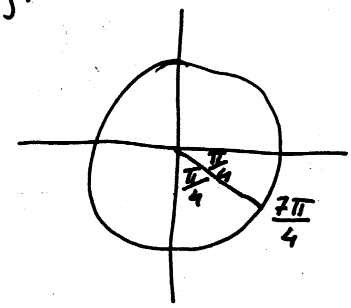
$$|z_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1 \qquad \cos\varphi=-\frac{1}{2}$$

$$z_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} \qquad tg\,\varphi=-\sqrt{3}$$

$$tg\,60^\circ=\sqrt{3}$$



$$\Rightarrow \varphi=\frac{2\pi}{3}$$



$$z=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{17}\cdot\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)=$$

$$=\left(\cos\frac{34\pi}{3}+i\sin\frac{34\pi}{3}\right)\cdot\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)=\cos\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)+i\sin\left(\frac{34\pi}{3}+\frac{5\pi}{12}\right)$$

$$=\cos\frac{141\pi}{12}+i\sin\frac{141\pi}{12}=\cos\frac{47\pi}{4}+i\sin\frac{47\pi}{4}=\cos\left(10\pi+\frac{7\pi}{4}\right)+i\sin\left(10\pi+\frac{7\pi}{4}\right)$$

$$=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$$

$$Re(z)=\frac{\sqrt{2}}{2} \qquad Im(z)=-\frac{\sqrt{2}}{2}$$

2. Riješiti matričnu jednačinu $(A+XB)^{-1}=BX^{-1}$, ako je $A=\begin{bmatrix}1 & 8\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}7 & 5\\ 8 & 6\end{bmatrix}$.

Rj.

$$(A+XB)^{-1}=BX^{-1} \quad /(A+XB) \text{ sa desne}$$

$$(A+XB)^{-1}\cdot(A+XB)=BX^{-1}\cdot(A+XB)$$

$$I=BX^{-1}\cdot A+BX^{-1}\cdot X\cdot B$$

$$I=BX^{-1}\cdot A+B^2$$

$BX^{-1}A = I - B^2 \quad / \cdot B^{-1}$ sa lijeve strane

$X^{-1}A = B^{-1}\cdot(I-B^2) \quad / \cdot A^{-1}$ sa desne strane

$X^{-1} = B^{-1}\cdot(I-B^2)\cdot A^{-1}$

$X = A\cdot(I-B^2)^{-1}\cdot B$

$C = I - B^2 = \begin{bmatrix} -88 & -65 \\ -104 & -75 \end{bmatrix}$

$B^2 = B\cdot B = \begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 89 & 65 \\ 104 & 76 \end{bmatrix}$

$C^{-1} = \frac{1}{\det C}\cdot C_{adj}$

$C_{adj} = (C_{kof})^T$

$C_{adj} = \begin{bmatrix} -75 & 65 \\ 104 & -88 \end{bmatrix}$, $\det(C) = 160$, $C^{-1} = \begin{bmatrix} \frac{15}{32} & -\frac{13}{32} \\ -\frac{13}{20} & \frac{11}{20} \end{bmatrix}$

$A\cdot C^{-1} = \begin{bmatrix} -\frac{757}{160} & \frac{639}{160} \\ -\frac{131}{160} & \frac{157}{160} \end{bmatrix}$ $\quad X = \begin{bmatrix} -\frac{187}{160} & \frac{43}{160} \\ -\frac{81}{160} & -\frac{13}{160} \end{bmatrix}$

3. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$(\lambda-1)x + y - 4z = 2$$
$$6x + 2y - (\lambda+1)z = 7$$
$$x + y - z = 3.$$

Rj.

$D = \begin{vmatrix} \lambda-1 & 1 & -4 \\ 6 & 2 & -\lambda-1 \\ 1 & 1 & -1 \end{vmatrix} \underset{III_k + II_k}{\overset{I_k - II_k}{=}} \begin{vmatrix} \lambda-2 & 1 & -3 \\ 4 & 2 & -\lambda+1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \lambda-2 & 3 \\ 4 & \lambda-1 \end{vmatrix} = \underbrace{(\lambda-2)(\lambda-1)-12}_{}$

$\overline{\lambda^2 - 3\lambda - 10}$

$= (\lambda-5)(\lambda+2)$

$D_x = \begin{vmatrix} 2 & 1 & -4 \\ 7 & 2 & -\lambda-1 \\ 3 & 1 & -1 \end{vmatrix} = -(\lambda+2)$ $\quad D_y = \begin{vmatrix} \lambda-1 & 2 & -4 \\ 6 & 7 & -\lambda-1 \\ 1 & 3 & -1 \end{vmatrix} = 3(\lambda-5)(\lambda+2)$

$D_z = \begin{vmatrix} \lambda-1 & 1 & 2 \\ 6 & 2 & 7 \\ 1 & 1 & 3 \end{vmatrix} = (-1)(\lambda+2)$

Diskusija

1° $\lambda \neq 5$; $\lambda \neq -2$

$D \neq 0$ sistem ima jedinstveno rješenje

$$x = \frac{D_x}{D} = -\frac{1}{\lambda - 5}, \quad y = \frac{D_y}{D} = 3, \quad z = \frac{D_z}{D} = -\frac{1}{\lambda - 5}$$

2° $\lambda = -2$

$D = D_x = D_y = D_z = 0$, sistem postaje

$$\begin{aligned} -3x + y - 4z &= 2 \\ 6x + 2y + z &= 7 \\ x + y - z &= 3 \end{aligned}$$

sistem ima ∞ mnogo rješenja $\left(\frac{1-3t}{4}, \frac{7t+11}{4}, t\right)$, $t \in \mathbb{R}$

3° $\lambda = 5$

$D = 0$, $D_x \neq 0 \Rightarrow$ sistem nema ni jedno rješenje

4. Date su dvije trojke vektora u vektorskom prostoru V_3: $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ i $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, pri čemu je $\vec{a}_1 = (3, 5, 2)$, $\vec{a}_2 = (-2, 3, 4)$, $\vec{a}_3 = (-2, 4, -3)$, $\vec{b}_1 = (1, -3, 0)$, $\vec{b}_2 = (2, 4, 1)$, $\vec{b}_3 = (5, 1, 2)$.
Odrediti koja od ovih trojki predstavlja bazu prostora V_3 i razložiti vektor $\vec{c} = (-1, 15, -18)$ u toj bazi.

Rj: Vektori $\vec{a}_1$, $\vec{a}_2$ i $\vec{a}_3$ su linearno nezavisni ako ne postoje skalari α, β, γ različiti od nule takvi da važi $\alpha\vec{a}_1 + \beta\vec{a}_2 + \gamma\vec{a}_3 = \vec{0}$.

$$\alpha(3,5,2) + \beta(-2,3,4) + \gamma(-2,4,-3) = (0,0,0)$$

$$\begin{aligned} 3\alpha - 2\beta - 2\gamma &= 0 \\ 5\alpha + 3\beta + 4\gamma &= 0 \\ 2\alpha + 4\beta - 3\gamma &= 0 \end{aligned}$$

$$D = \begin{vmatrix} 3 & -2 & -2 \\ 5 & 3 & 4 \\ 2 & 4 & -3 \end{vmatrix} = -149 \neq 0$$

sistem nema netrivijalnih rješenja pa vektori $\vec{a}_1$, $\vec{a}_2$ i $\vec{a}_3$ su linearno nezavisni. Vektori $\vec{a}_1$, $\vec{a}_2$ i $\vec{a}_3$ mogu predstavljati bazu vektorskog prostora V_3.

$\alpha \vec{a}_1 + \beta \vec{a}_2 + \gamma \vec{a}_3 = \vec{c}$

$$\begin{aligned} 3\alpha - 2\beta - 2\gamma &= -1 \\ 5\alpha + 3\beta + 4\gamma &= 15 \\ 2\alpha + 4\beta - 3\gamma &= -18 \end{aligned}$$

$\alpha = 1, \ \beta = -2, \ \gamma = 4$

$\vec{c} = \vec{a}_1 - 2\vec{a}_2 + 4\vec{a}_3$

Slično bi uradili sa bazom $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ da su vektori $\vec{a}_1$, $\vec{a}_2$ i $\vec{a}_3$ bili linearno zavisni.

5. Ispitati f-ju i nacrtati njen graf $y = \dfrac{x^3}{(x^2-4)^2}$

Rj.

$D: x \in R \setminus \{2, -2\}$

f-ja je neparna

nije periodična

(0,0) je nula f-je i presjek sa Y-osom

x	$(-\infty, 0)$	$(0, +\infty)$
Y	−	+

$\lim\limits_{x \to 2 \pm 0} f(x) = \infty$

$\Rightarrow x = 2$ je V. A.

$y' = \dfrac{-x^2(x^2+12)}{(x^2-4)^3}$

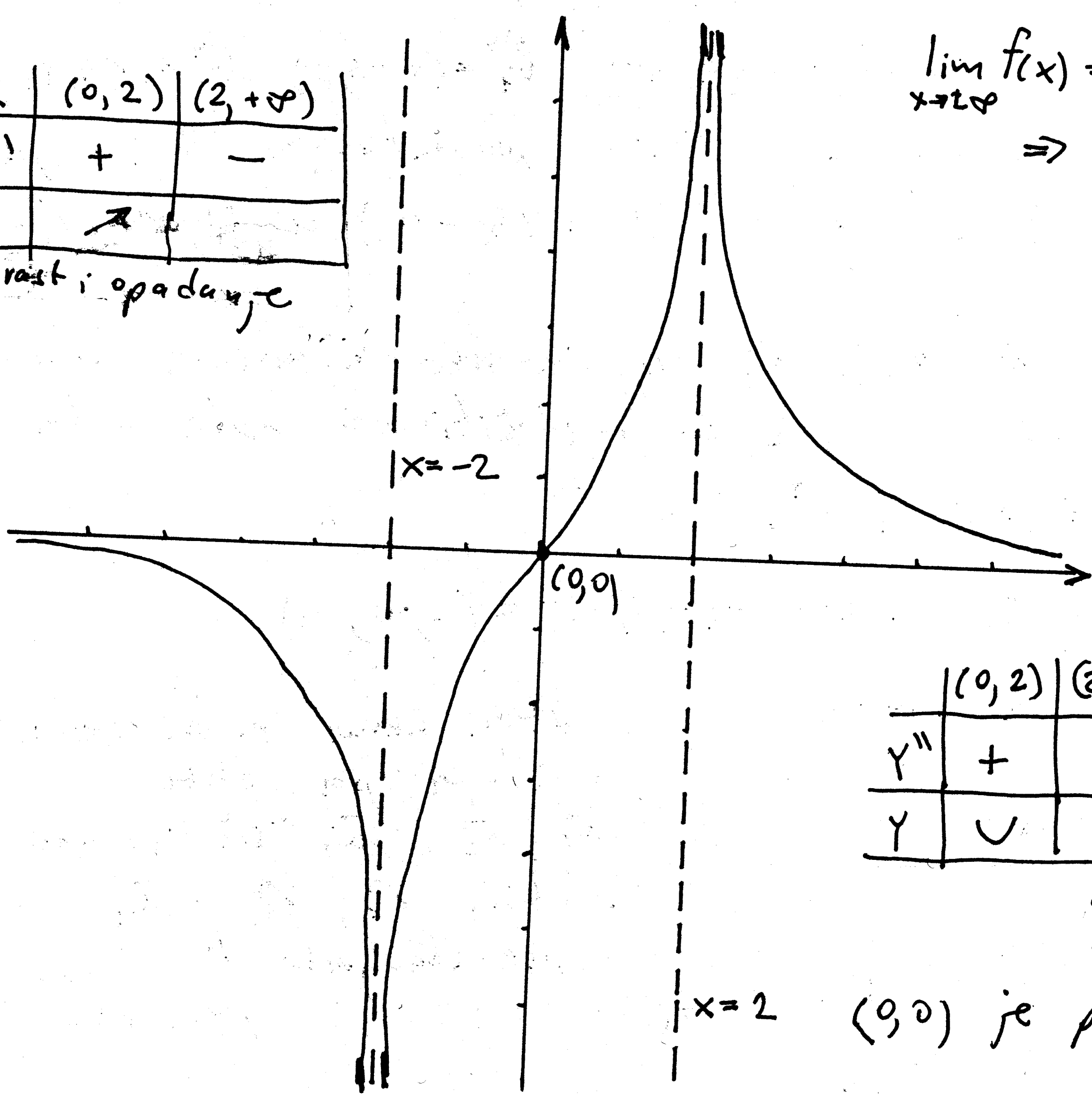
$y'' = \dfrac{2x(x^4+32x^2+48)}{(x^2-4)^4}$

X	(0, 2)	$(2, +\infty)$
Y'	+	−
Y	↗	↘

rast i opadanje

$\lim\limits_{x \to \pm\infty} f(x) = 0$

$\Rightarrow Y = 0$ je H. A.



	(0, 2)	$(2, +\infty)$
Y''	+	+
Y	∪	∪

konveksnost i konkavnost

(0,0) je prevojna tačka

6. Ispitati i grafički predstaviti f-ju $y = \frac{x^2+bx+c}{x^2-9}$ ako se zna da ona ima ekstrem u tački $T(0, -\frac{2}{9})$.

Rj. $f(0) = \frac{c}{-9}$, $f(0) = -\frac{2}{9}$ $\Rightarrow c = 2$

$y' = \frac{-bx^2-22x-9b}{(x^2-9)^2}$

$y'(0) = 0 \Rightarrow b = 0$

$y = \frac{x^2+2}{x^2-9}$

$D: x \in \mathbb{R} \setminus \{-3, 3\}$

f-ja je parna

nije periodična

$(0, -\frac{2}{9})$ je presjek sa y-osom

f-ja nema nule

x	(0,3)	(3,+∞)
y	−	+

znak f-je

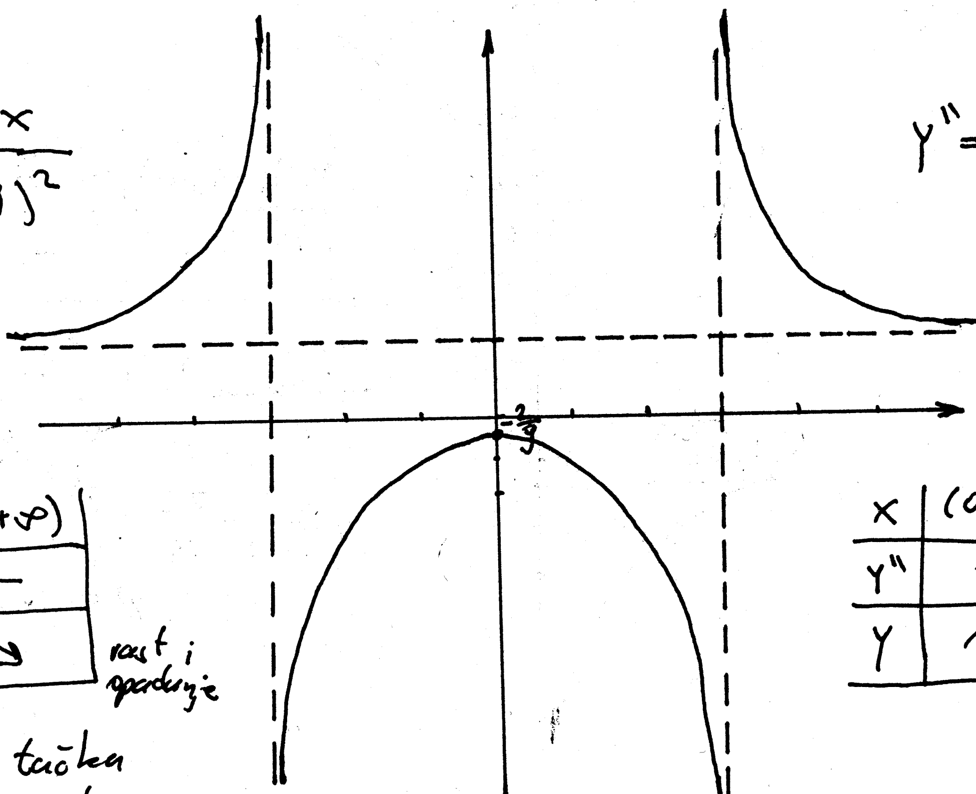
$\lim_{x\to 3-0} f(x) = -\infty$

$\lim_{x\to 3+0} f(x) = +\infty$ $\Rightarrow x=3$ je V. A.

$\lim_{x\to \pm\infty} f(x) = 1 \Rightarrow y=1$ je H. A.

$y' = \frac{-22x}{(x^2-9)^2}$

$y'' = \frac{66(x^2+3)}{(x^2-9)^3}$



x	(0,3)	(3,+∞)
y'	−	−
y	↘	↘

rast i opadanje

$(0, -\frac{2}{9})$ tačka maksimuma

x	(0,3)	(3,+∞)
y''	−	+
y	∩	∪

konveksnost i konkavnost

7. Ispitati f-ju i nacrtati graf $y = (x-6)e^{-\frac{1}{x}}$.

Rj. $D: x \in \mathbb{R} \setminus \{0\}$

f-ja nije ni parna ni neparna

f-je nije periodična

(6,0) je nula f-je

x	(−∞, 6)	(6, +∞)
y	−	+

znak f-je

$\lim_{x\to 0+} f(x) = 0$

f-ja ne sijeće y-osu

$\lim_{x\to 0-} f(x) = -\infty \Rightarrow x=0$ je V. A.

$\lim\limits_{x\to\mp\infty} f(x) = \mp\infty \Rightarrow$ f-ja nema horizontalnu asimptotu

$Y = x - 7$ je K. A.

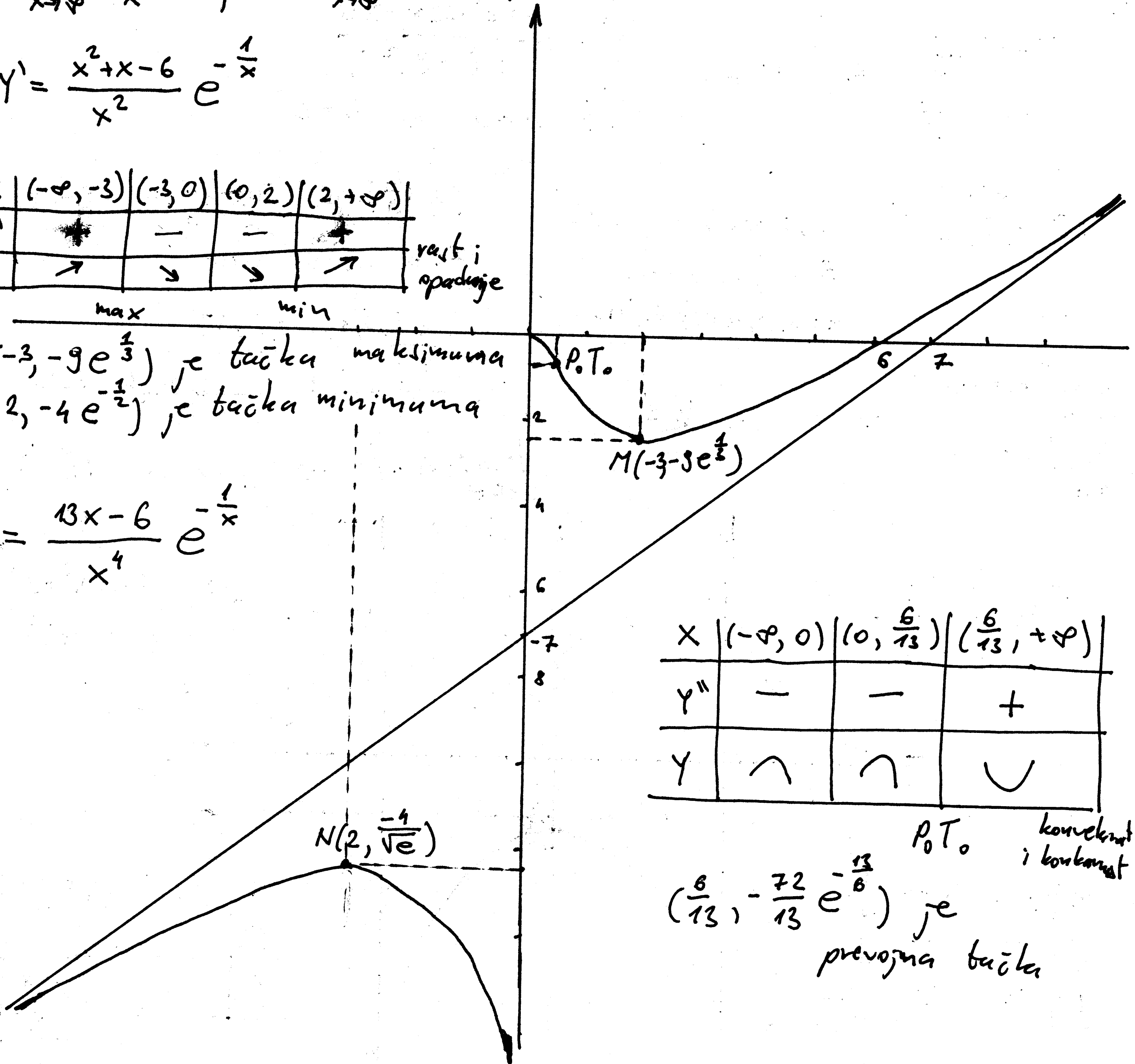
$k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = 1,\quad n = \lim\limits_{x\to\infty} [f(x) - kx] = -7$

$$Y' = \frac{x^2+x-6}{x^2} e^{-\frac{1}{x}}$$

X	$(-\infty,-3)$	$(-3,0)$	$(0,2)$	$(2,+\infty)$	
Y'	+	−	−	+	
Y	↗	↘	↘	↗	rast i opadanje
	max			min	

$M(-3, -9e^{\frac{1}{3}})$ je tačka maksimuma

$N(2, -4e^{-\frac{1}{2}})$ je tačka minimuma



$$Y'' = \frac{13x-6}{x^4} e^{-\frac{1}{x}}$$

X	$(-\infty,0)$	$(0,\frac{6}{13})$	$(\frac{6}{13},+\infty)$	
Y''	−	−	+	
Y	∩	∩	∪	konveksnost i konkavnost
			P. T.	

$(\frac{6}{13}, -\frac{72}{13} e^{-\frac{13}{6}})$ je prevojna tačka

8. Ispitati i grafički predstaviti f-ju $Y = \frac{x\ln x}{\ln x - 1}$.

Rj. $D: x \in R \setminus \{e\}$

f-ja nije ni parna ni neparna

nije periodična

$(1,0)$ je nula f-je

f-ja ne siječe Y-osu

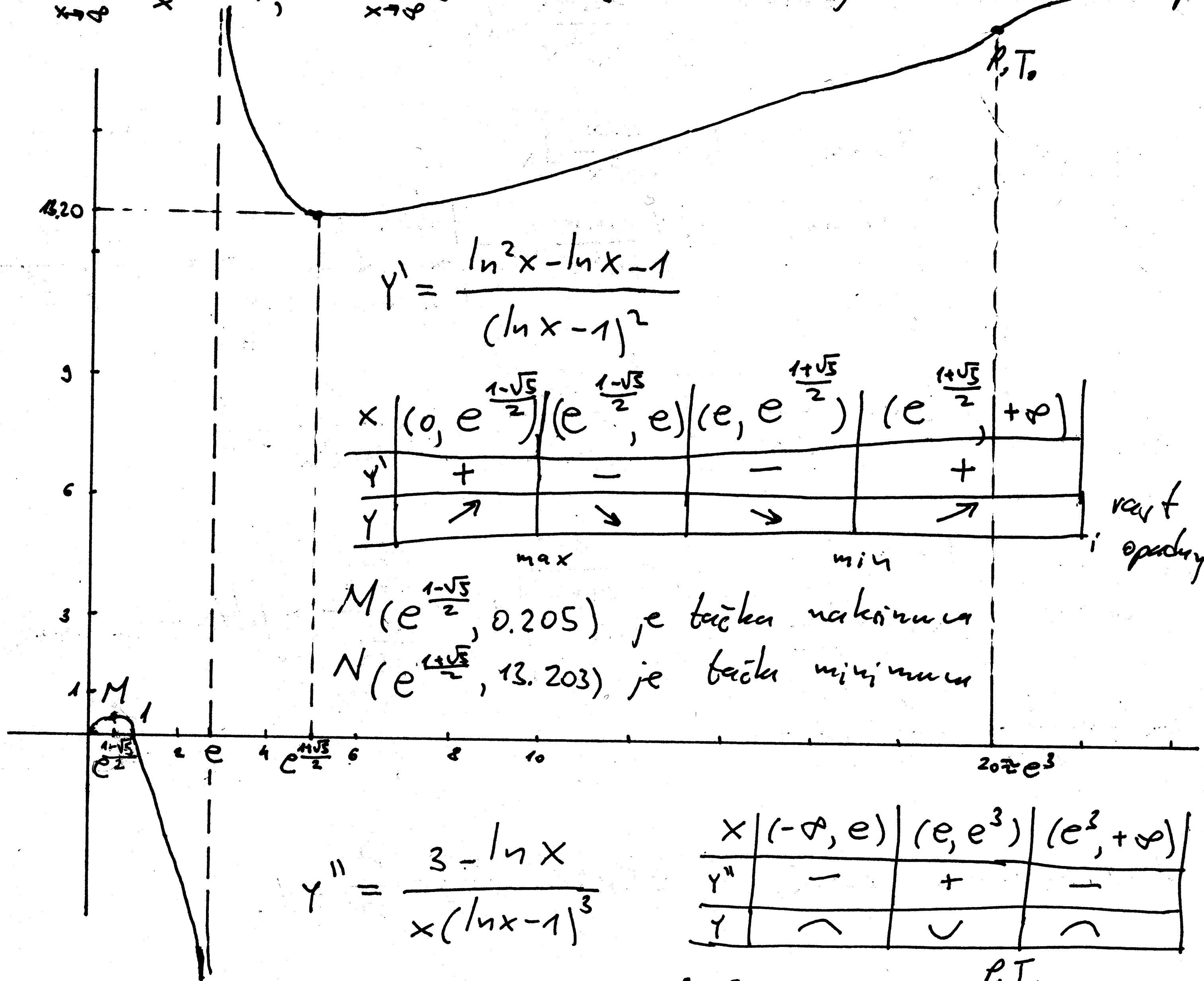
X	$(0,1)$	$(1,e)$	$(e,+\infty)$	
Y	+	−	+	znak f-je

$\lim\limits_{x\to e-0} f(x) = -\infty$

$\lim\limits_{x\to e+0} f(x) = +\infty \quad \Rightarrow \quad x = e$ je V. A.

$\lim\limits_{x\to\mp\infty} f(x) = \mp\infty \Rightarrow$ nema H. A.

$k=\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$, $n=\lim\limits_{x\to\infty}[f(x)-k\cdot x]=\infty$ $\Rightarrow$ f-ja nema kose asimptote



$$y'=\frac{\ln^2 x-\ln x-1}{(\ln x-1)^2}$$

x	$(0, e^{\frac{1-\sqrt5}{2}})$	$(e^{\frac{1-\sqrt5}{2}}, e)$	$(e, e^{\frac{1+\sqrt5}{2}})$	$(e^{\frac{1+\sqrt5}{2}}, +\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗
		max		min

rast i opadanje

$M(e^{\frac{1-\sqrt5}{2}}, 0.205)$ je tačka maksimuma

$N(e^{\frac{1+\sqrt5}{2}}, 13.203)$ je tačka minimuma

$$y''=\frac{3-\ln x}{x(\ln x-1)^3}$$

x	$(-\infty, e)$	(e, e^3)	$(e^3, +\infty)$
y''	−	+	−
y	$\frown$	$\smile$	$\frown$
		P.T.	

$(e^3, \frac{3}{2}e^3)$ je prevojna tačka

9. Odrediti integral $I=\int\frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})}$.

Rj.

$$I=\int\frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})}=\left|\begin{array}{l}x=t^{10}\\ dx=10t^9dt\\ t=\sqrt[10]{x}\end{array}\right|=\int\frac{10t^9dt}{t^{10}(t^5+t^4)}=10\int\frac{dt}{t^6+t^5}=10\int\frac{dt}{t^5(t+1)}$$

$$\frac{1}{t^5(t+1)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t^4}+\frac{E}{t^5}+\frac{F}{t+1}\Rightarrow A=1,\ B=-1,\ C=1,\ D=-1,\ E=1,\ F=-1$$

$$I=10\int\left(\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t^3}-\frac{1}{t^4}+\frac{1}{t^5}-\frac{1}{t+1}\right)dt=$$

$$=10\left(\ln|t|+\frac{1}{t}-\frac{1}{2t^2}+\frac{1}{3t^3}-\frac{1}{4t^4}-\ln|t+1|\right)+C=$$

$$=10\ln\sqrt[10]{x}+\frac{1}{\sqrt[10]{x}}-\frac{1}{2\sqrt[5]{x}}+\frac{1}{3\sqrt[10]{x^3}}-\frac{1}{4\sqrt[5]{x^2}}-\ln|\sqrt[10]{x}+1|+C$$

10. Izračunati integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x\sin x}{\cos^3 x}\,dx$.

Rj.

$$\int \frac{x\sin x}{\cos^3 x}\,dx = \left| \begin{array}{l} u = x \\ du = dx \end{array} \right. \qquad dv = \frac{\sin x}{\cos^3 x}\,dx$$

$$v = \int \frac{\sin x}{\cos^3 x}\,dx = \left| \begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \\ \sin x\,dx = -dt \end{array} \right| = -\int \frac{dt}{t^3} = \frac{1}{2t^2} =$$

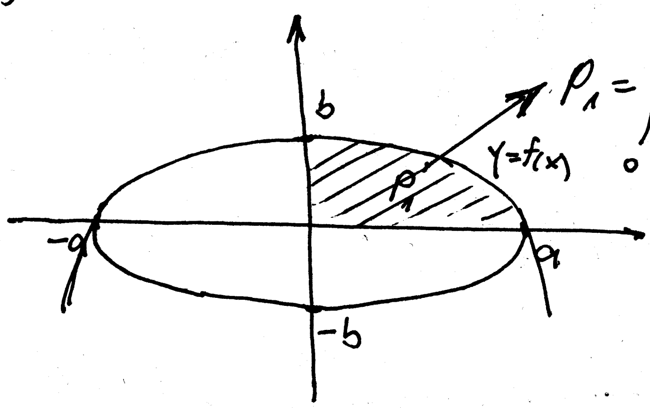
$$= \frac{1}{2\cos^2 x} \Bigg| = \frac{1}{2}x\cdot\frac{1}{\cos^2 x} - \frac{1}{2}\int\frac{dx}{\cos^2 x} = \frac{1}{2}x\cdot\frac{1}{\cos^2 x} - \frac{1}{2} I_1$$

$$I_1 = \int \frac{dx}{\cos^2 x} = \left| \begin{array}{l} \operatorname{tg} x = t \\ dx = \dfrac{dt}{1+t^2} \end{array} \quad \cos^2 x = \frac{1}{1+t^2} \right| = \int \frac{\frac{dt}{1+t^2}}{\frac{1}{1+t^2}} = \int dt = t + c_1 = \operatorname{tg} x + c_1$$

$$I = \frac{1}{2}x\cdot\frac{1}{\cos^2 x} - \frac{1}{2}\operatorname{tg} x + C$$

11. Izračunati površinu elipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Rj.

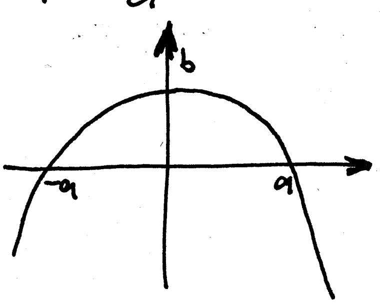


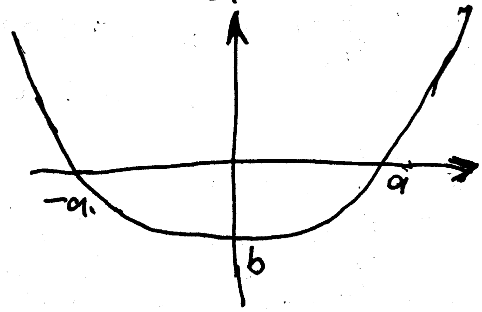
$$P_1 = \int_0^a f(x)\,dx \qquad P_{elipse} = 4P_1 = P$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \qquad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} \qquad y = \pm\sqrt{\frac{b^2}{a^2}(a^2-x^2)}$$

$$y = \frac{b}{a}\sqrt{a^2-x^2} \qquad y = -\frac{b}{a}\sqrt{a^2-x^2}$$





$$P_1 = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\int\sqrt{a^2-x^2} = \int\frac{a^2-x^2}{\sqrt{a^2-x^2}} = (cx+d)\sqrt{a^2-x^2} + \lambda\int\frac{dx}{\sqrt{a^2-x^2}} \quad \Big/ \frac{d}{dx}$$

$$\frac{a^2-x^2}{\sqrt{a^2-x^2}} = c\cdot\sqrt{a^2-x^2} + (cx+d)\,\frac{-2x}{2\sqrt{a^2-x^2}} + \lambda\cdot\frac{1}{\sqrt{a^2-x^2}} \quad \Big/\cdot\sqrt{a^2-x^2}$$

$$a^2-x^2 = c\cdot(a^2-x^2) - (cx^2+dx) + \lambda$$

x^2: $\quad -c-c=-1$

$-2c=-1 \Rightarrow c=\frac{1}{2}$

x: $d=0$

x^0: $ca^2+\lambda=a^2$

$\lambda = a^2-\frac{1}{2}a^2$

$\lambda = a^2(1-\frac{1}{2})$

$\lambda=\frac{1}{2}a^2$

$$\int\sqrt{a^2-x^2} = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arcsin\frac{x}{a} + C$$

$$\int_0^a\sqrt{a^2-x^2} = \left(\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arcsin\frac{x}{a}\right)\Big|_0^a = 0+\frac{1}{2}a^2\cdot\frac{\pi}{2} - \left(0+\frac{1}{2}a^2\cdot 0\right)$$

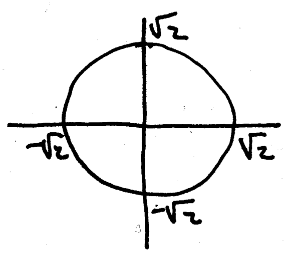
$$= \frac{\pi}{4}a^2 \qquad P_1 = \frac{b}{a}\cdot\frac{\pi}{4}a^2 = ab\frac{\pi}{4} \qquad P = ab\pi$$

tražena površina elipse

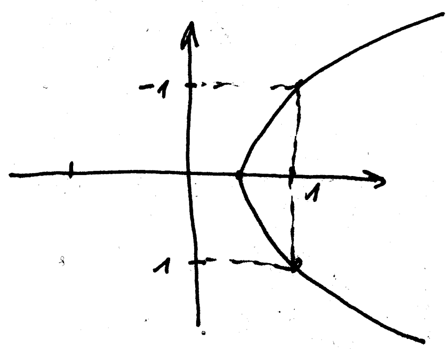
12. Izračunati površinu figure koja je ograničena kružnicom $x^2+y^2=2$ i parabolom $y^2=2x-1$ ako je $x\geq\frac{1}{2}$.

Rj:

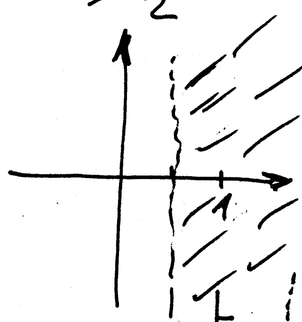
$x^2+y^2=2$

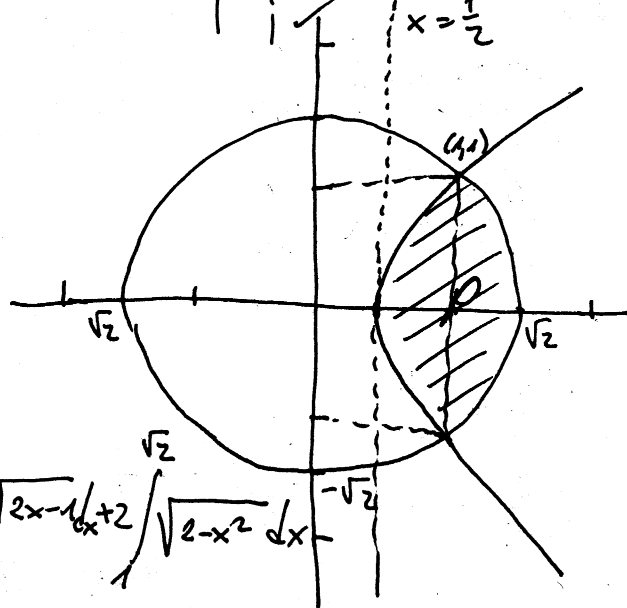


$y^2=2x-1$



$x\geq\frac{1}{2}$





$x^2+2x-1=2$

$x^2+2x-3=0$

$D=4+12=16$

$x_{1,2}=\frac{-2\pm4}{2}$

$x_1=-3 \quad x_2=1$

$(1,-1)$ i $(1,1)$ su točke presjeka kružnice i elipse

$y^2=2x-1$

za $x=\frac{1}{2}$

$y^2=0$

$$P = 2\int_{\frac{1}{2}}^{1}\sqrt{2x-1}\,dx + 2\int_{1}^{\sqrt{2}}\sqrt{2-x^2}\,dx$$

ili $P=\int_{-1}^{1}\left(\sqrt{2-y^2}-\frac{y^2+1}{2}\right)dy=\int_{-1}^{1}\sqrt{2-y^2}\,dy-\frac{1}{2}\int(y^2+1)\,dy$

$=1+\frac{\pi}{2}-\frac{4}{3}=\frac{\pi}{2}-\frac{1}{3}$ površina figure

13. Odrediti ekstreme f-je $z(x,y)=x^2-6xy+y^3+3x+6y$.

Rj. $\frac{\partial z}{\partial x}=2x-6y+3$ $\qquad$ $2x-6y+3=0$ $\qquad$ $M_1(\frac{3}{2},1)$

$\frac{\partial z}{\partial y}=-6x+3y^2+6$ $\qquad$ $-6x+3y^2+6=0$ $\qquad$ $M_2(\frac{27}{2},5)$

$y^2-6y+5=0$

stacionarne tačke (kandidati za ekstrem)

$\frac{\partial^2 z}{\partial x^2}=2$ $\qquad$ $M_1(\frac{3}{2},1)$

$\frac{\partial^2 z}{\partial x\partial y}=-6$ $\qquad$ $A=2$, $B=-6$, $C=6$ $\qquad$ $D=AC-B^2$

$\frac{\partial^2 z}{\partial y^2}=6y$ $\qquad$ $D=-24<0$

f-ja u tački M_1 nema ekstrem

$M_2(\frac{27}{2},5)$

$A=2$, $B=-6$, $C=30$ $\qquad$ $D=60-36=24>0$

f-ja u tački M_2 ima ekstrem

$A>0$ f-ja z u tački $M_2(\frac{27}{2},5)$ ima minimum

$z_{min}=-27,25$

14. Nađi sve ekstreme f-je $z(x,y)=8xy+\frac{1}{x}+\frac{1}{y}$.

Rj. $\frac{\partial z}{\partial x}=8y-\frac{1}{x^2}$ $\qquad$ $8y-\frac{1}{x^2}=0$ $\quad$ $/\cdot x^2\ (x\neq 0)$

$\frac{\partial z}{\partial y}=8x-\frac{1}{y^2}$ $\qquad$ $8x-\frac{1}{y^2}=0$ $\quad$ $/\cdot y^2\ (y\neq 0)$

$8x^2y-1=0$

$8xy^2-1=0$ $\quad \xRightarrow{y=\frac{1}{8x^2}}$ $M(\frac{1}{2},\frac{1}{2})$ stacionarna tačka

$\frac{\partial^2 z}{\partial x^2}=\frac{2}{x^3}$ $\qquad$ $\frac{\partial^2 z}{\partial y^2}=\frac{2}{y^3}$ $\qquad$ $M_1(\frac{1}{2},\frac{1}{2})$ $\qquad$ $D>0$

$\frac{\partial^2 z}{\partial x\partial y}=8$ $\qquad$ $A=16$, $B=8$, $C=16$ $\qquad$ f-ja u tački ima ekstrem, $A>0$

$\Rightarrow z_{min}=6$ $\qquad$ f-ja ima minimum

15. Riješiti diferencijalnu jednačinu

$$(x-3y+3)y' = 6y-2x-5$$

Rj.

$$y' = \frac{-2x+6y-5}{x-3y+3}$$

Ovo je diferencijalna jednačina prvog reda koja se svodi na homogenu

$a_1b_2 - a_2b_1 =$

$= (-2)(-3) - 1\cdot 6 = 0 \quad \Rightarrow$ uvodimo smjenu $-2x+6y=u$

$$y' = \frac{(-2)(-2x+6y-5)}{(-2)(x-3y+3)} = \frac{(-2)(-2x+6y-5)}{-2x+6y-6} = \frac{(-2)(u-5)}{u-6}$$

$-2x+6y=u$

$6y = u+2x$

$y = \frac{u+2x}{6}$

$y' = \frac{u'+2}{6}$

$$\frac{u'+2}{6} = \frac{-2u+10}{u-6} \quad /\cdot 6$$

$$u' = \frac{-12u+60}{u-6} - 2$$

$$u' = \frac{-14u+72}{u-6}$$

$$\frac{du}{dx} = \frac{-14u+72}{u-6}$$

$$\frac{u-6}{-14u+72}du = dx \quad / \int$$

$$\frac{1}{2}\int \frac{u-6}{-7u+36}du = \int dx$$

$\Downarrow (*)$

$$\int \frac{u-6}{-7u+36}du = \int \frac{7u-42}{7\cdot(-7u+36)}du =$$

$$= \int \frac{(-1)(-7u+36)-6}{7\cdot(-7u+36)}du =$$

$$= \int -\frac{1}{7}du - \frac{6}{7}\int \frac{du}{-7u+36} =$$

$$= -\frac{1}{7}u + \frac{6}{49}\ln|36-7u| \quad \ldots (*)$$

$$-\frac{1}{14}u + \frac{3}{49}\ln|36-7u| = x + C_1 \quad /\cdot 7\cdot 7\cdot 2$$

$$-7u + 6\ln|36+14x-42y| = 98x + C_1$$

$$14x-42y+6\ln|36+14x-42y|=98$$

$$\ln|14x-42y+36|=14x+7y+C_2$$

$$\ln|2\cdot(7x-21y+18)|=14x+7y+C_2$$

$$\ln 2+\ln|7x-21y+18|=14x+7y+C_2$$

$$\ln|7x-21y+18|=14x+7y+C$$ rješenje diferencijalne jednačine

(16.) Riješiti diferencijalnu jednačinu

$$(xy^2+3x)\,dx+(2x^2y-5y)\,dy=0.$$

Rj.

$$(2x^2y-5y)\,dy=-(xy^2+3x)\,dx$$

$$y(2x^2-5)\,dy=-x(y^2+3)\,dx$$

ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$\frac{y}{y^2+3}\,dy=\frac{-x}{2x^2-5}\,dx \quad \Big/\!\!\int$$

$$\frac{1}{2}\ln|y^2+3|=-\frac{1}{4}\ln|2x^2-5|+\ln C_1$$

$$\ln|y^2+3|^2=\ln|C\cdot(2x^2-5)^{-1}|$$

$$(y^2+3)^2=\frac{C}{2x^2-5}$$

rješenje diferencijalne jednačine

$$2x^2-5=\frac{C}{(y^2+3)^2}$$