Univerzitet u Zenici
Ekonomski fakultet

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Pismeni ispit iz predmeta Matematika

1. Izračunati x ako se zna da treći član razvoja $(2 \cdot \sqrt[x]{2^{-1}} + \frac{4}{\sqrt[4-x]{4}})^6$ ima vrijednost 240.

2. Napisati sva rješenja jednačine $x^4 + x^2 + 1 = 0$ u trigonometriskom obliku.

3. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x + y - z &= 0 \\ x - y + \lambda z &= 1 \\ -x - 3y + (\lambda + 2)z &= \lambda^2 \quad . \end{aligned}$$

4. Dati su vektori $\vec{a} = (m^2 + 1, m, -2)$, $\vec{b} = (m^2, 2, -m)$, $\vec{c} = (-2m - 1, 0, m + 2)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

5. Ispitati i grafički predstaviti funkciju $y = \frac{(2x-1)^3}{(x+2)^2}$.

6. Ispitati i grafički predstaviti funkciju $y = (6 + x)e^{\frac{1}{x}}$.

7. Ispitati i grafički predstaviti funkciju $y = \frac{lnx - 1}{x^3}$.

8. Odrediti parametar a i b tako da funkcija $y = \frac{x}{x^2 + ax + b}$ ima ekstrem u tački $T(2, \frac{1}{7})$. Zatim ispitati tako dobijenu funkciju i nacrtati joj grafik.

9. Izračunati integral $I = \int \frac{x}{9x^2 + 24x + 17} dx$.

10. Izračunati integral $I = \int x^3 e^{\frac{x}{2}} dx$.

11. Izračunati integral $I = \int_0^1 arcsin\frac{x}{2} dx$.

12. Izračunati površinu koju gradi kriva $y = x^2 + x - 6$ zajedno sa svojim tangentama povučenim na tu krivu u nul - tačkama krive.

13. Naći sve ekstreme funkcije $z = y^2 e^{-x^2 - y^2}$, $y \neq 0$.

14. Naći uslovne ekstreme funkcije $z = 2x^4 + 8y^4 + 24$ ako je $8x + 4y = 1$.

15. Riješiti diferencijalnu jednačinu $y' = 2^{2x+y}$.

16. Riješiti diferencijalnu jednačinu $y'cosx - ysinx = x^3 e^{x^2}$ uz početni uslov $y(0) = 1$.

(Za uočene greške pisati na infoarrt@gmail.com)

(#) Izračunati x ako se zna da treći član razvoja $\left(2\cdot\sqrt[x]{2^{-1}}+\frac{4}{\sqrt[4-x]{4}}\right)^6$ ima vrijednost 240.

$$\boxed{\sqrt[4-x]{4}=4^{\frac{1}{4-x}}}$$

Rj.
$$\left(2\sqrt[x]{2^{-1}}+\frac{4}{\sqrt[4-x]{4}}\right)^6=\sum_{k=0}^{6}\binom{6}{k}\left(2\sqrt[x]{2^{-1}}\right)^{6-k}\left(\frac{4}{\sqrt[4-x]{4}}\right)^k=$$

$$=\sum_{k=0}^{6}\binom{6}{k}\left(2\cdot 2^{-\frac{1}{x}}\right)^{6-k}\left(4\cdot 4^{-\frac{1}{4-x}}\right)^k=\sum_{k=0}^{6}\binom{6}{k}\left(2^{1-\frac{1}{x}}\right)^{6-k}\left(4^{1-\frac{1}{4-x}}\right)^k$$

$k=0$ dobijemo prvi član
$k=1$ drugi član
$k=2$ treći član

$$\binom{6}{2}\left(2^{1-\frac{1}{x}}\right)^4\left(4^{1-\frac{1}{2}}\right)^2=240$$

$$\frac{6\cdot 5}{2}\cdot\left(2^{\frac{x-1}{x}}\right)^4\cdot\left(4^{\frac{1}{2}}\right)^2=240$$

$$3\cdot 5\cdot 2^{\frac{4(x-1)}{x}}\cdot 4=240 \quad /:(4\cdot 5)$$

$$3\cdot 2^{\frac{4(x-1)}{x}}=12 \quad /:3$$

$$2^{\frac{4(x-1)}{x}}=4$$

$$2^{\frac{4(x-1)}{x}}=2^2$$

$$\frac{4(x-1)}{x}=2 \quad /\cdot x\ (x\neq 0)$$

$$4x-4=2x$$

$$2x=4$$

$$x=2$$

Za $x=2$ treći član razvoja binoma ima vrijednost 240.

$$\boxed{(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k}$$

(#) Napisati sva rješenja jednačine $x^4+x^2+1=0$ u trigonometriskom obliku.

Rj uvodimo smjenu $x^2=t$

$$t^2+t+1=0$$

$$D=1-4=-3=3i^2$$

$$t_{1,2}=\frac{-1\pm i\sqrt{3}}{2}$$

$$t_1=\frac{-1-i\sqrt{3}}{2}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$t_2=\frac{-1+i\sqrt{3}}{2}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$t_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$t_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

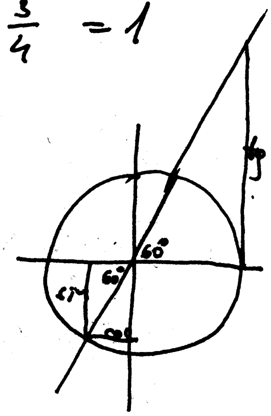
$$|t_1|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\cos\varphi_1=-\frac{1}{2}$$

$$\sin\varphi_1=-\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_1=\sqrt{3}$$

$$tg\,60^\circ=\sqrt{3}$$



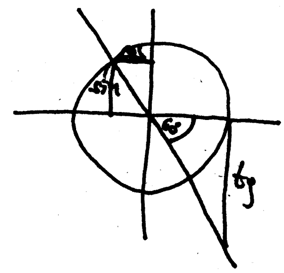
$$\varphi_1=240^\circ=\frac{4\pi}{3}$$

$$|t_2|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\cos\varphi_2=-\frac{1}{2}$$

$$\sin\varphi_2=\frac{\sqrt{3}}{2}$$

$$tg\,\varphi_2=-\sqrt{3}$$



$$\varphi_2=120^\circ=\frac{2\pi}{3}$$

$$x^2=t$$

$$z=\sqrt{t_1},\quad z_k=\sqrt{|t_1|}\left(\cos\frac{\varphi+2k\pi}{2}+i\sin\frac{\varphi+2k\pi}{2}\right),\quad k=0,1$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}}{2}+i\sin\frac{\frac{4\pi}{3}}{2}\right)=\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{4\pi}{3}+2\pi}{2}+i\sin\frac{\frac{4\pi}{3}+2\pi}{2}\right)=\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$$

$$z=\sqrt{t_2}$$

$$z_0=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}}{2}+i\sin\frac{\frac{2\pi}{3}}{2}\right)=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$z_1=\sqrt{1}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{2}+i\sin\frac{\frac{2\pi}{3}+2\pi}{2}\right)=\cos\frac{8\pi}{6}+i\sin\frac{8\pi}{6}=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$

Sva rješenja jednačine $x^4+x^2+1=0$ napisana u trigonometriskom obliku su:

$$x_1=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\quad x_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\quad x_3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

i $x_4=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra a:

$$x+y-z=0$$
$$x-y+az=1$$
$$-x-3y+(a+2)z=a^2 .$$

Rj.

$$D=\begin{vmatrix}1&1&-1\\1&-1&a\\-1&-3&a+2\end{vmatrix}\overset{I_k+III_k}{\underset{II_k+III_k}{=}}\begin{vmatrix}0&0&-1\\a+1&a-1&a\\a+1&a-1&a+2\end{vmatrix}=(-1)\begin{vmatrix}a+1&a-1\\a+1&a-1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}0&1&-1\\1&-1&a\\a^2&-3&a+2\end{vmatrix}\overset{II_k+III_k}{=}\begin{vmatrix}0&0&-1\\1&a-1&a\\a^2&a-1&a+2\end{vmatrix}=(-1)\begin{vmatrix}1&a-1\\a^2&a-1\end{vmatrix}=(-1)(a-1)\begin{vmatrix}1&1\\a^2&1\end{vmatrix}$$
$$=(-1)(a-1)(1-a^2)=(a-1)(a^2-1)$$
$$=(a-1)^2(a+1)$$

$$D_y=\begin{vmatrix}1&0&-1\\1&1&a\\-1&a^2&a+2\end{vmatrix}\overset{I_k+III_k}{=}\begin{vmatrix}0&0&-1\\a+1&1&a\\a+1&a^2&a+2\end{vmatrix}=(-1)\begin{vmatrix}a+1&1\\a+1&a^2\end{vmatrix}=(-1)(a+1)\begin{vmatrix}1&1\\1&a^2\end{vmatrix}=(-1)(a+1)(a^2-1)$$
$$=(-1)(a-1)(a+1)^2$$

$$D_z=\begin{vmatrix}1&1&0\\1&-1&1\\-1&-3&a^2\end{vmatrix}\overset{I_k-II_k}{=}\begin{vmatrix}0&1&0\\2&-1&1\\2&-3&a^2\end{vmatrix}=(-1)\begin{vmatrix}2&1\\2&a^2\end{vmatrix}=(-1)(2a^2-2)=(-2)(a+1)(a-1)$$

Diskusija

$D=0 \quad \forall a\in R$

1° $a\neq 1$; $a\neq -1$

$D=0$; $D_x\neq 0$ sistem nema rješenja

2° $a=1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$x+y-z=0 \quad (1)$$
$$x-y+z=1 \quad (2)$$
$$-x-3y+3z=1 \quad (3)$$

$(1)+(3)$: $-2y+2z=1$
$(2)+(3)$: $-4y+4z=2$

$2z=2y+1$

$z=y+\frac{1}{2}$

$x=z-y$

$x=\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(\frac{1}{2}, t, t+\frac{1}{2})$ gdje je $t\in R$.

3° $a=-1$

$D=D_x=D_y=D_z=0$, sistem postaje

$$x+y-z=0 \quad I$$
$$x-y-z=1 \quad II$$
$$-x-3y+z=1 \quad III$$

$(I)+(III)$: $-2y=1$
$(II)+(III)$: $-4y=2$

$y=-\frac{1}{2}$

$(I)+(II)$: $2x-2z=1$
$(III)-3\cdot(II)$: $-4x+4z=2$

$2x=2z+1$

$x=z+\frac{1}{2}$

Sistem ima ∞ mnogo rješenja oblika $(t+\frac{1}{2}, -\frac{1}{2}, t)$, $t\in Z$

#) Dati su vektori $\vec{a}=(m^2+1, m, -2)$, $\vec{b}=(m^2, 2, -m)$, $\vec{c}=(-2m-1, 0, m+2)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

R: Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni ako postoji bar jedan nenula skalar α, β ili γ takav da je

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=\vec{0}$ tj.

$$
\begin{aligned}
(m^2+1)\alpha + m^2\beta + (-2m-1)\gamma &= 0\\
m\alpha + 2\beta + 0\gamma &= 0\\
-2\alpha + (-m)\beta + (m+2)\gamma &= 0
\end{aligned}
$$

Ovo je homogeni sistem.

Za $D=0$ sistem ima netrivijalnih rješenja.

$$
D=\begin{vmatrix} m^2+1 & m^2 & -2m-1\\ m & 2 & 0\\ -2 & -m & m+2 \end{vmatrix} = -m\begin{vmatrix} m^2 & -2m-1\\ -m & m+2\end{vmatrix} + 2\begin{vmatrix} m^2+1 & -2m-1\\ -2 & m+2\end{vmatrix} =
$$

$$
= -m\left(m^3+\underline{2m^2}-(\underline{2m^2}+m)\right)+2\left(m^3+2m^2\underline{\underline{+m}}\,\underline{+2}-(\underline{\underline{4m}}\underline{+2})\right) =
$$

$$
= -m(m^3-m)+2(m^3+2m^2-3m) = -m^2(m^2-1)+2m(m^2+2m-3) =
$$

$D=4+12=16$

$m_{1,2}=\frac{-2\pm 4}{2}$

$m_1=\frac{-6}{2}=-3$

$m_2=\frac{2}{2}=1$

$$
= m\left[-m(m-1)(m+1)+2(m-1)(m+3)\right] = m(m-1)\left[-m(m+1)+2(m+3)\right] =
$$

$$
= m(m-1)(-m^2-m+2m+6) = m(m-1)(-m+m+6) = -m(m-1)(m+2)(m-3)
$$

$D=0$ akko $m=0$ ili $m=1$ ili $m=-2$ ili $m=3$

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su linearno zavisni ako $m\in\{-2,0,1,3\}$

Za $m=3$: $\vec{a}=(10,3,-2)$, $\vec{b}=(9,2,-3)$ i $\vec{c}=(-7,0,5)$

$\vec{a}=\mu\vec{b}+\omega\vec{c}$

$(9\mu, 2\mu, -3\mu)+(-7\omega, 0, 5\omega)=(10,3,-2)$

$$
\begin{aligned}
9\mu-7\omega&=10\\
2\mu+0&=3\\
-3\mu+5\omega&=-2
\end{aligned}
$$

$\mu=\frac{3}{2}$

$-\frac{9}{2}+5\omega=-2 \quad /\cdot 2$

$-9+10\omega=-4$

$10\omega=5$

$\omega=\frac{1}{2}$

$\vec{a}=\frac{3}{2}\vec{b}+\frac{1}{2}\vec{c}$

vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$

Razvijmo determinantu D i na drugi način:

$$D = \begin{vmatrix} m^2+1 & m^2 & -2m-1 \\ m & 2 & 0 \\ -2 & -m & m+2 \end{vmatrix} \overset{I_R + III_R}{=\!=} \begin{vmatrix} m^2-1 & m^2-m & -m+1 \\ m & 2 & 0 \\ -2 & -m & m+2 \end{vmatrix} =$$

$$= \begin{vmatrix} (m-1)(m+1) & m(m-1) & -(m-1) \\ m & 2 & 0 \\ -2 & -m & m+2 \end{vmatrix} = (m-1) \begin{vmatrix} m+1 & m & -1 \\ m & 2 & 0 \\ -2 & -m & m+2 \end{vmatrix} \overset{I_K + III_K}{=\!=}$$

$$= (m-1) \begin{vmatrix} m & m & -1 \\ m & 2 & 0 \\ m & -m & m+2 \end{vmatrix} = m(m-1) \begin{vmatrix} 1 & m & -1 \\ 1 & 2 & 0 \\ 1 & -m & m+2 \end{vmatrix} \underset{III_R - II_R}{\overset{I_R - II_R}{=\!=}}$$

$$= m(m-1) \begin{vmatrix} 0 & m-2 & -1 \\ 1 & 2 & 0 \\ 0 & -m-2 & m+2 \end{vmatrix} = -m(m-1) \begin{vmatrix} m-2 & -1 \\ -(m+2) & m+2 \end{vmatrix} = -m(m-1)(m+2) \begin{vmatrix} m-2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -m(m-1)(m+2)\big(m-2-1\big) = -m(m-1)(m+2)(m-3)$$

(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{(2x-1)^3}{(x+2)^2}$.

Rj. definiciono područje

$D: x \in \mathbb{R} \setminus \{-2\}$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $(2x-1)^3=0$

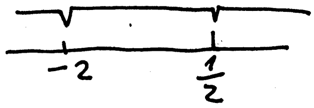
$2x-1=0$

$x=\frac{1}{2}$

$(\frac{1}{2},0)$ je nula f-je

$f(0)=\frac{(-1)^3}{2^2}=-\frac{1}{4}$

$(0,-\frac{1}{4})$ je tačka presjeka sa y-osom



x	$(-\infty,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
$(2x-1)^3$	−	−	+
y	−	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$$\lim_{x\to-2-0} f(x)=\lim_{x\to-2-0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2-0)-1)^3}{(-2-0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V.A.}$$

(sa lijeve strane)

$$\lim_{x\to-2+0} f(x)=\lim_{x\to-2+0}\frac{(2x-1)^3}{(x+2)^2}=\frac{(2\cdot(-2+0)-1)^3}{(-2+0+2)^2}=\frac{(-5-0)^3}{+0}=-\infty \Rightarrow x=-2 \text{ je V.A.}$$

(sa desne strane)

$$(2x-1)^3=(2x)^3+3\cdot(2x)^2\cdot(-1)+3\cdot 2x\cdot(-1)^2+(-1)^3=8x^3-12x^2+6x-1$$

1
1 1
1 2 1
1 3 3 1

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{(2x-1)^3}{(x+2)^2}=\lim_{x\to-\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^3}{x^2+4x+2 \quad /:x^3}=\lim_{x\to-\infty}\frac{8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{\frac{1}{x}+\frac{4}{x^2}+\frac{2}{x^3}}=-\infty$$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^2}{x^2+4x+2 \quad /:x^2}=\lim_{x\to+\infty}\frac{8x-12+\frac{6}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{2}{x^2}}=+\infty$$

f-ja nema H.A.

kosa asimptota je oblika $y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1 \quad /:x^3}{x^3+4x^2+2x \quad /:x^3}$$

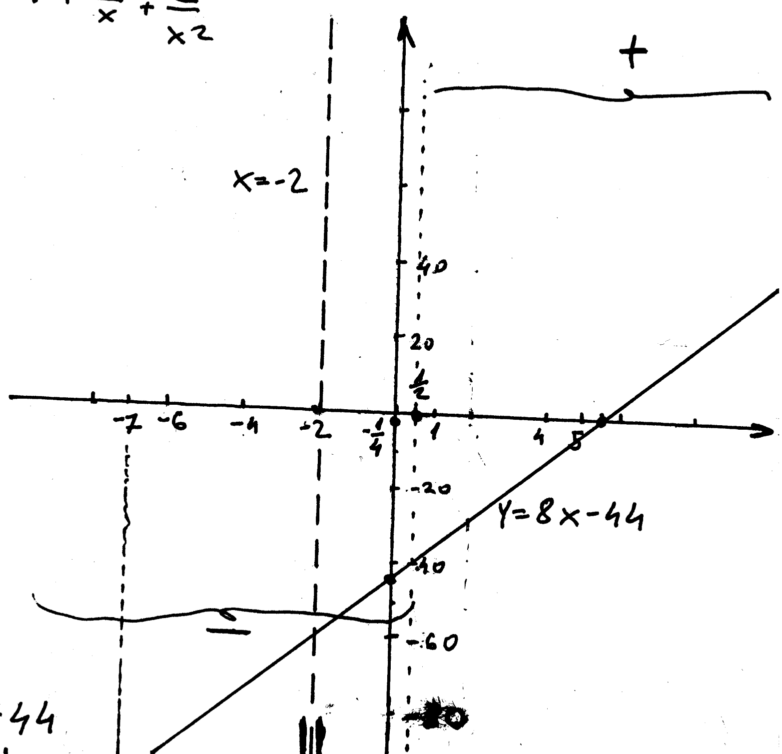
$$=\lim_{x\to\infty}\frac{8-\frac{12}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{1+\frac{4}{x}+\frac{2}{x^2}}=8$$

$$n=\lim_{x\to\infty}[f(x)-k\cdot x]=\lim_{x\to\infty}\left(\frac{(2x-1)^3}{(x+2)^2}-8x\right)=$$

$$=\lim_{x\to\infty}\frac{8x^3-12x^2+6x-1-8x(x^2+4x+2)}{(x+2)^2}=$$

$$=\lim_{x\to\infty}\frac{\cancel{8x^3}-12x^2+6x-1-\cancel{8x^3}-32x^2-16x}{x^2+4x+4}$$

$$=\lim_{x\to\infty}\frac{-44x^2-10x-1 \quad :x^2}{x^2+4x+4 \quad :x^2}=\lim_{x\to\infty}\frac{-44-\frac{10}{x}-\frac{1}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}=-44$$



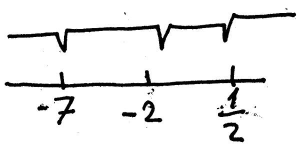
$Y = 8x - 44$ je K.A. (počinjemo sa skiciranjem grafa)

$(Y = 8x-44,\quad Y=0 \Rightarrow 8x=44,\quad x=\frac{44}{8}=\frac{11}{2}=5{,}5 \qquad x=0 \Rightarrow Y=-44\ )$

rast i opadanje

$$Y' = \left(\frac{(2x-1)^3}{(x+2)^2}\right)' = \frac{3(2x-1)^2\cdot 2\,(x+2)^{\cancel{2}\,1} - (2x-1)^3\,2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}} = \frac{2(2x-1)^2\,(3x+6-2x+1)}{(x+2)^3} = \frac{2(2x-1)^2(x+7)}{(x+2)^3}$$

$Y'=0$ akko $x=\frac{1}{2}$ i $x=-7$



$\leftarrow$ nule Y' + prekidi od Y

X	$(-\infty,-7)$	$(-7,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
Y'	+	−	+	+
Y	↗	↘	↗	↗
	max			

rast i opadanje

$$f(-7) = \frac{(-15)^3}{(-5)^2} = \frac{-3375}{25} = -135$$

ekstremi f-je Na osnovu tabele rasta i opadanja $M(-7,-135)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y'' = \left(2\,\frac{(2x-1)^2(x+7)}{(x+2)^3}\right)' = 2\cdot\frac{\left[2(2x-1)\cdot 2\cdot(x+7) + (2x-1)^2\right](x+2)^{\cancel{3}\,1} - (2x-1)^2(x+7)\,3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}} =$$

$$= 2\cdot\frac{\left[(2x-1)\,(4x+28+2x-1)\right](x+2) - 3(2x-1)^2(x+7)}{(x+2)^4} = 2\cdot\frac{(2x-1)\left[\overbrace{(6x+27)(x+2)}^{6x^2+12x+27x+54} - 3\overbrace{(2x-1)(x+7)}^{2x^2+13x-7}\right]}{(x+2)^4}$$

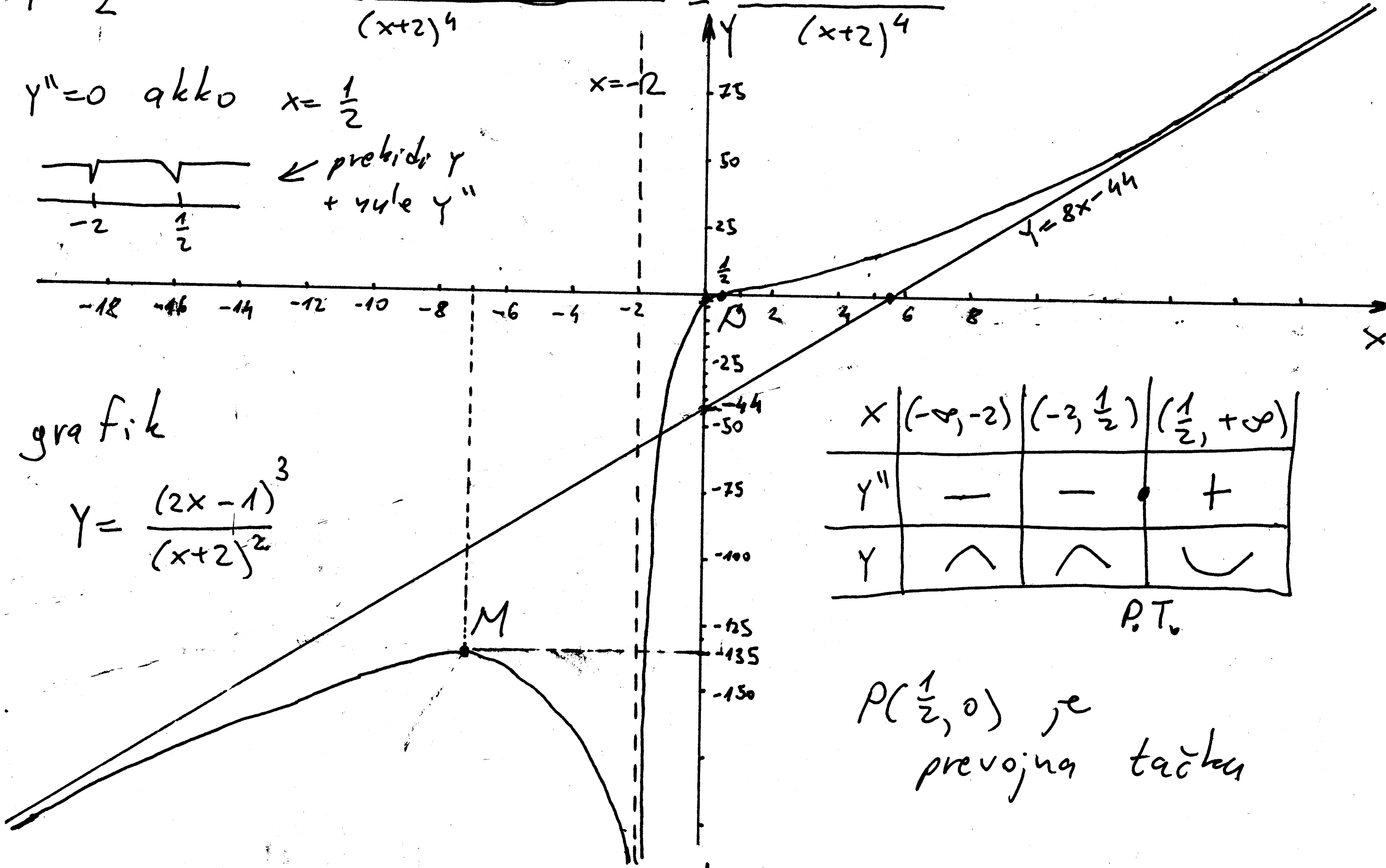
$$Y'' = 2\cdot\frac{(2x-1)(\cancel{6x^2}+39x+54-\cancel{6x^2}-39x+21)}{(x+2)^4} = \frac{150\,(2x-1)}{(x+2)^4}$$

$Y''=0$ akko $x=\frac{1}{2}$

$\leftarrow$ prekidi Y + nule Y''

grafik

$$Y = \frac{(2x-1)^3}{(x+2)^2}$$



X	$(-\infty,-2)$	$(-2,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
Y''	−	−	+
Y	∩	∩	∪
		P.T.	

$P(\frac{1}{2},0)$ je prevojna tačka

Ispitati f-ju i nacrtati joj grafik: $y=(6+x)e^{\frac{1}{x}}$.

Rj. definiciono područje

$x \neq 0$

$\mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

parnost, neparnost, periodičnost

$f(-x)=(6-x)e^{-\frac{1}{x}}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$y=0 \Leftrightarrow (6+x)e^{\frac{1}{x}}=0$

$\Rightarrow 6+x=0$ ili $e^{\frac{1}{x}}=0$

$x=-6$ $\quad e^{\frac{1}{x}} \neq 0 \ \forall x$

$(-6,0)$ je nula f-je

f(0) nije definisana ⇒ f-ja ne siječe Y-osu

$e^{\frac{1}{x}}>0 \ \forall x$

x	$(-\infty,-6)$	$(6,0)$	$(0,+\infty)$
$6+x$	−	+	+
Y	−	+	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$$\lim_{x\to-0} f(x)=\lim_{x\to-0}(6+x)e^{\frac{1}{x}}=(6-0)e^{\frac{1}{-0}}=(6-0)e^{-\infty}=(6-0)\frac{1}{e^{\infty}}=0$$

$$\lim_{x\to+0} f(x)=\lim_{x\to+0}(6+x)e^{\frac{1}{x}}=(6+0)e^{\frac{1}{+0}}=(6+0)e^{\infty}=\infty \Rightarrow x=0 \text{ je V. A.}$$

sa desne strane

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(6+x)e^{\frac{1}{x}}=\infty\cdot e^{0}=\infty\cdot 1=\infty$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}(6+x)e^{\frac{1}{x}}=(-\infty)\cdot e^{0}=-\infty$$

f-ja nema H. A.

$y=kx+n$ K. A., $\quad k=\lim_{x\to\infty}\frac{f(x)}{x}$, $\quad n=\lim_{x\to\infty}[f(x)-kx]$

$$k=\lim_{x\to\infty}\frac{(6+x)e^{\frac{1}{x}}}{x}=\lim_{x\to\infty}\left(\frac{6}{x}+1\right)e^{\frac{1}{x}}=1\cdot e^{0}=1$$

$$n=\lim_{x\to\infty}\left[(6+x)e^{\frac{1}{x}}-x\right]=\lim_{x\to\infty}\left[6e^{\frac{1}{x}}+xe^{\frac{1}{x}}-x\right]=$$

$$=6\lim_{x\to\infty}e^{\frac{1}{x}}+\lim_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)=$$

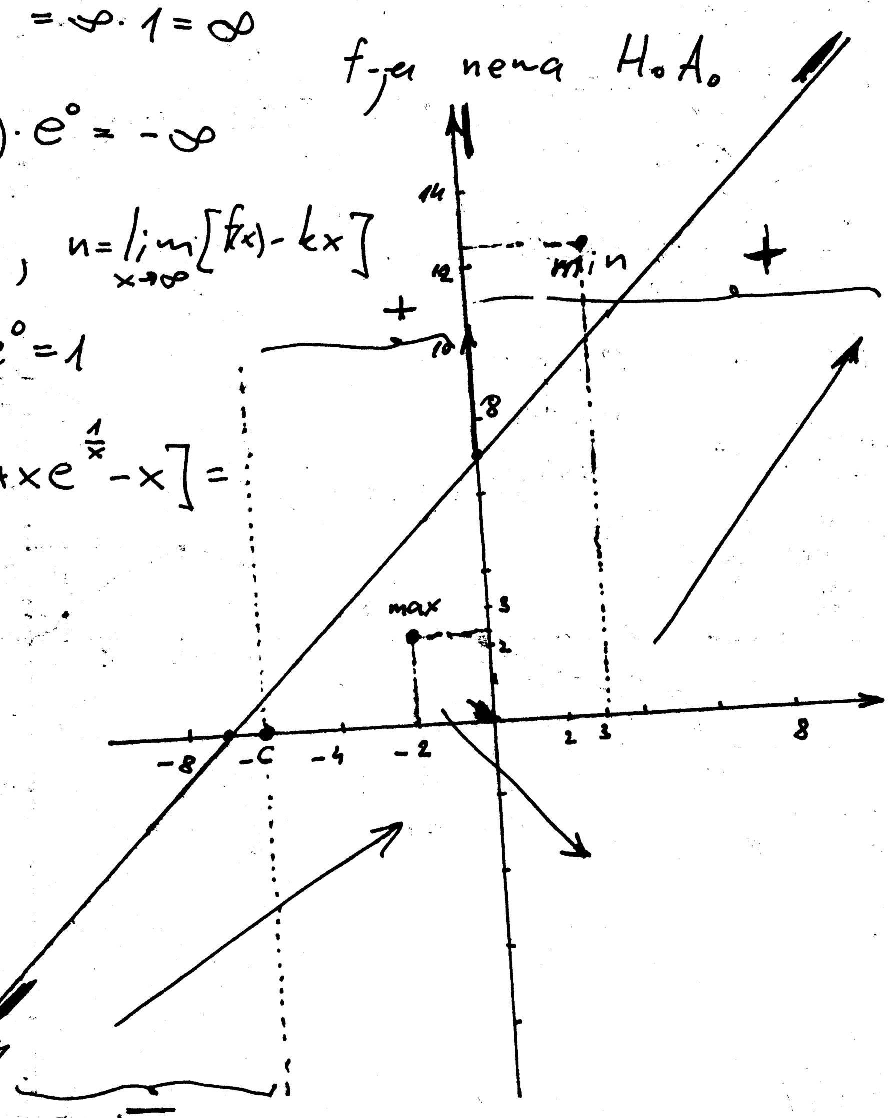
$(\infty\cdot 0)$

$$=6+\lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}\left(\frac{0}{0}\right)\underline{\underline{\text{L.P.}}}$$

$$=6+\lim_{x\to\infty}\frac{e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}=6+1=7$$

$y=x+7$ je K. A.

poslije ovog koraka počinjemo sa skiciranjem grafa



rast i opadanje

$$y' = \left[(6+x)e^{\frac{1}{x}}\right]' = 1\cdot e^{\frac{1}{x}} + (6+x)\,e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)' = e^{\frac{1}{x}} + (6+x)e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}}\left(1-\frac{6}{x^2}-\frac{1}{x}\right)$$

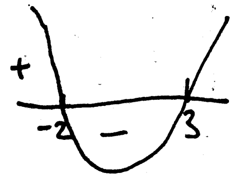
$$\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-2} = -\frac{1}{x^2} \qquad y' = e^{\frac{1}{x}}\cdot\frac{x^2-x-6}{x^2}$$

$x^2-x-6=0$

$D = 1+24 = 25$

$x_{1,2} = \frac{1\pm 5}{2}$

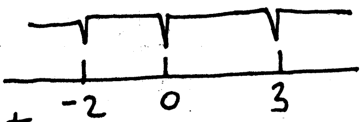
$x_1=-2 \quad x_2=3$



$y'=0$ akko

$x_1=-2 \quad x_2=3$

prelazi y + nule y'



x	$(-\infty,-2)$	$(-2,0)$	$(0,3)$	$(3,+\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗

max $f(-2)\approx 2,4261$ min $f(3)\approx 12,5605$

$$f(-2) = 4e^{\frac{1}{-2}} = \frac{4}{\sqrt{e}} \approx 2,4261$$

$$f(3) = 9e^{\frac{1}{3}} = 9\sqrt[3]{e} \approx 12,5605$$

ekstremi f-je

Na osnovu tabele rasta i opadanja $M_1(-2, \frac{4}{\sqrt{e}})$ je tačka maksimuma a $M_2(3, 9\sqrt[3]{e})$ je tačka minimuma.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\cdot\frac{x^2-x-6}{x^2} + e^{\frac{1}{x}}\cdot\frac{(2x-1)x^2-(x^2-x-6)\cdot 2x}{x^4}$$

$$y'' = e^{\frac{1}{x}}\cdot\frac{1}{x^2}\left[\frac{-x^2+x+6}{x^2} + \frac{\cancel{2x^3}-x^2\cancel{-2x^3}+2x^2+12x}{x^2}\right]$$

$$y'' = e^{\frac{1}{x}}\cdot\frac{1}{x^2}\cdot\frac{13x+6}{x^2} = e^{\frac{1}{x}}\cdot\frac{13x+6}{x^4}$$

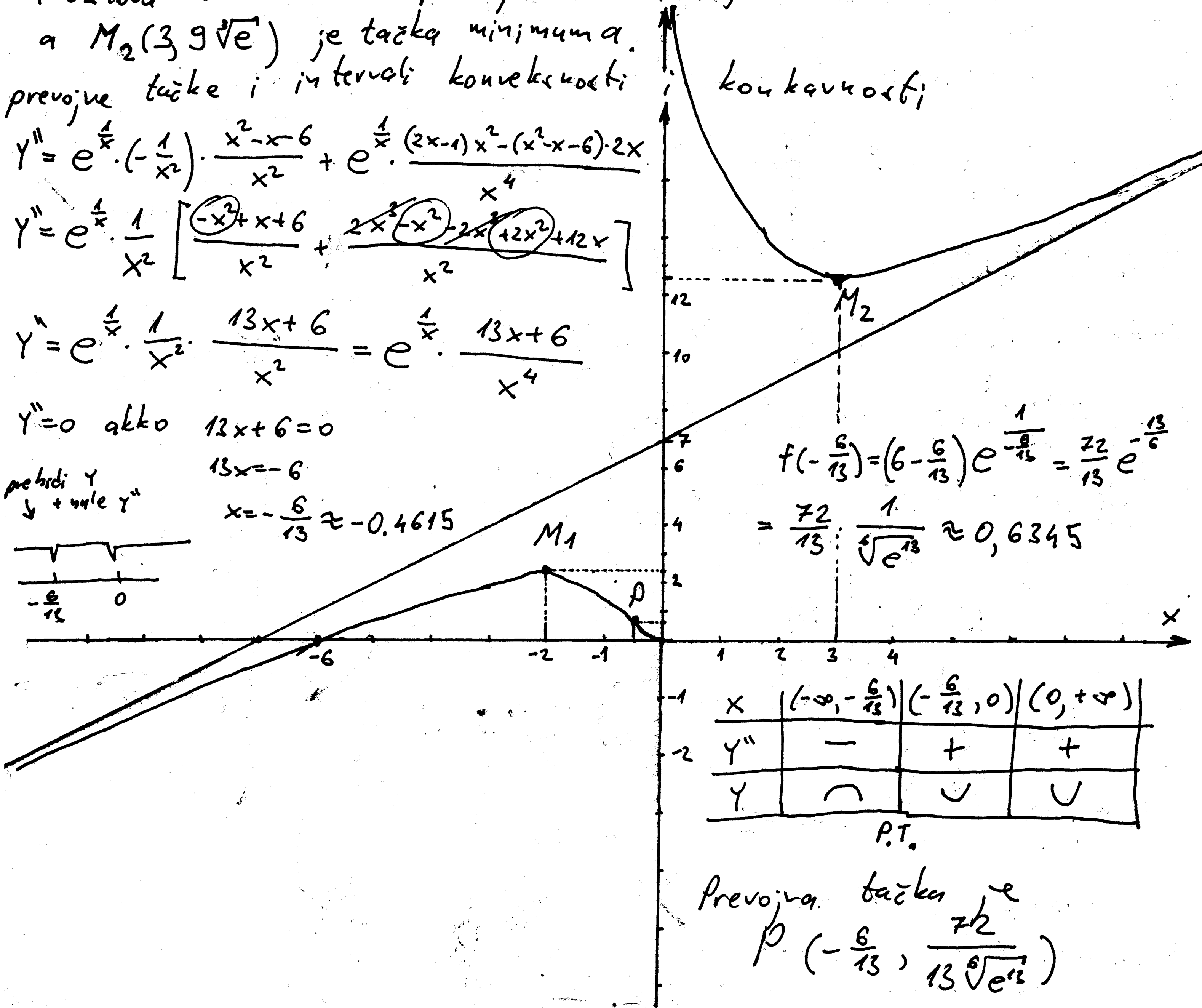
$y''=0$ akko $13x+6=0$

$13x=-6$

$x = -\frac{6}{13} \approx -0.4615$

prelazi y + nule y''

$$f\left(-\frac{6}{13}\right) = \left(6-\frac{6}{13}\right)e^{\frac{1}{-\frac{6}{13}}} = \frac{72}{13}e^{-\frac{13}{6}} = \frac{72}{13}\cdot\frac{1}{\sqrt[6]{e^{13}}} \approx 0,6345$$



x	$(-\infty,-\frac{6}{13})$	$(-\frac{6}{13},0)$	$(0,+\infty)$
y''	−	+	+
y	∩	∪	∪

P.T.

Prevojna tačka je

$$P\left(-\frac{6}{13}, \frac{72}{13\sqrt[6]{e^{13}}}\right)$$

Ispitati f-ju i nacrtati joj grafik $y = \dfrac{\ln x - 1}{x^3}$.

Rj. definiciono područje

$x \neq 0 \quad x > 0$

$\mathcal{D}: x \in (0, +\infty)$

parnost, neparnost, periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y = 0$ akko $\ln x - 1 = 0$

$\ln x = 1$

$x = e$

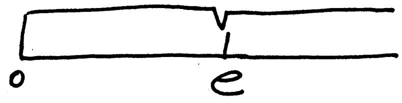
$(e, 0)$ nula f-je

$e \approx 2{,}7183$

$f(0) = ?$

$f(0)$ nije definisano

f-ja ne siječe y-osu



x	$(0, e)$	$(e, +\infty)$
$\ln x - 1$	$-$	$+$
x^3	$+$	$+$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x\to+0} f(x) = \lim_{x\to+0} \frac{\ln x - 1}{x^3} \left(= \frac{-\infty - 1}{+0}\right) = \frac{-\infty}{+0} = -\infty \Rightarrow x = 0 \text{ je V. A.}$$

(sa desne strane)

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x - 1}{x^3} \left(= \frac{\infty}{\infty}\right) \stackrel{L.P.}{=\!=} \lim_{x\to+\infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x\to+\infty} \frac{1}{3x^3} = \frac{1}{\infty} = 0$$

$\Rightarrow y = 0$ je H. A.

f-ja nema K. A.

počinjemo sa skiciranjem grafa

rast i opadanje

$$y' = \left(\frac{\ln x - 1}{x^3}\right)' = \frac{\frac{1}{x}\cdot x^{\cancel{3}\,1} - (\ln x - 1)\cdot 3\cancel{x^2}}{x^{\cancel{6}\,4}}$$

$$y' = \frac{1 - 3\ln x + 3}{x^4} = \frac{4 - 3\ln x}{x^4}$$

$y' = 0$ akko $4 - 3\ln x = 0$

$3\ln x = 4$

$\ln x = \frac{4}{3}$ $\quad x = e^{\frac{4}{3}} = \sqrt[3]{e^4} \approx 3{,}7937$

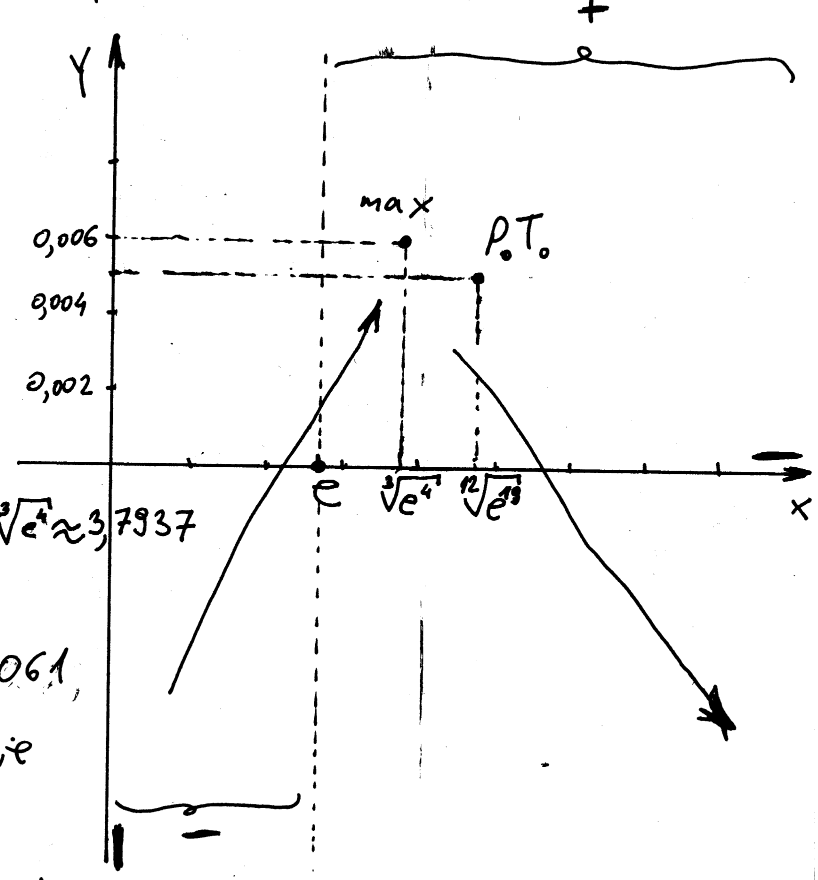
x	$(0, \sqrt[3]{e^4})$	$(\sqrt[3]{e^4}, +\infty)$
y'	$+$	$-$
y	$\nearrow$	$\searrow$

max

rast i opadanje

$\frac{1}{3e^4} \approx 0{,}0061$

$$f(e^{\frac{4}{3}}) = \frac{\ln e^{\frac{4}{3}} - 1}{(\sqrt[3]{e^4})^3} = \frac{\frac{4}{3} - 1}{(\sqrt[3]{e^4})^3} = \frac{\frac{1}{3}}{e^4} = \frac{1}{3e^4}$$



ekstremi f-je

na osnovu tabele rasta i opadanja tačka $M\left(\sqrt[3]{e^4}, \frac{1}{3e^4}\right)$ je tačka maksimuma.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{4-3\ln x}{x^4}\right)' = \frac{-3\cdot\frac{1}{x}\cdot x^4 - (4-3\ln x)\cdot 4x^3}{(x^4)^2} = \frac{-3\cancel{x^3} - (4-3\ln x)\cdot 4\cancel{x^3}}{x^5\cdot\cancel{x^3}} = \frac{-3-16+12\ln x}{x^5}$$

$$y'' = \frac{12\ln x - 19}{x^5}$$

$y''=0$ akko $12\ln x - 19 = 0$

$$12\ln x = 19$$

$$\ln x = \frac{19}{12}$$

$$x = e^{\frac{19}{12}} = \sqrt[12]{e^{19}} \approx 4{,}8712$$

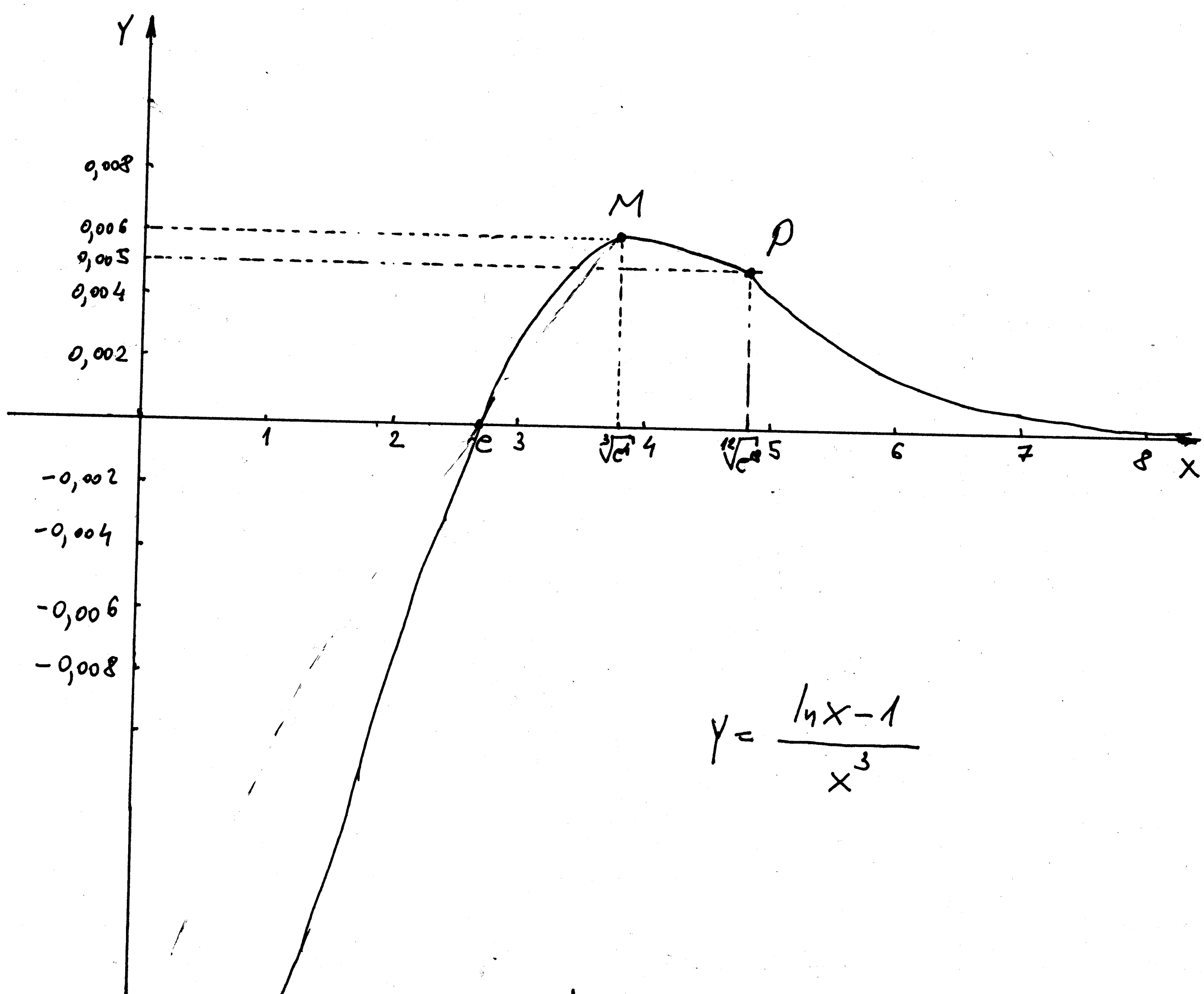
x	$(0, \sqrt[12]{e^{19}})$	$(\sqrt[12]{e^{19}}, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$
	P.T.	

intervali konveksnosti i konkavnosti

$$f\left(e^{\frac{19}{12}}\right) = \frac{\ln e^{\frac{19}{12}} - 1}{\left(e^{\frac{19}{12}}\right)^3} = \frac{\frac{19}{12}-1}{e^{\frac{19}{4}}} = \frac{\frac{7}{12}}{e^{\frac{19}{4}}} = \frac{7}{12\sqrt[4]{e^{19}}} \approx 0{,}005$$

$P\left(\sqrt[12]{e^{19}}, \frac{7}{12\sqrt[4]{e^{19}}}\right)$ je prevojna tačka

grafik



(#) Odrediti parametre a i b tako da f-ja $y=\frac{x}{x^2+ax+b}$ ima ekstrem u tački $T(2,\frac{1}{7})$. Zatim ispitati tako dobijenu f-ju i nacrtati joj grafik.

Rj

$f(2)=\frac{1}{7}$

$$\frac{2}{4+2a+b}=\frac{1}{7}$$

$4+2a+b=14$

$2a+b=10$

Kandidat za ekstreme su stacionarne tačke

$$y'=\frac{x^2+ax+b-x(2x+a)}{(x^2+ax+b)^2}=\frac{x^2+\cancel{ax}+b-2x^2-\cancel{ax}}{(x^2+ax+b)^2}$$

$$y'=\frac{-x^2+b}{(x^2+ax+b)^2}$$

Potreban uslov da f-ja y ima ekstrem u tački $T(2,\frac{1}{7})$ je $y'(2)=0$.

$-4+b=0$

$b=4$

$2a+4=10$

$2a=6$

$a=3$

$$y=\frac{x}{x^2+3x+4}$$

definiciono područje

$x^2+3x+4\neq 0$

$D=9-16<0$

$a>0 \quad x^2+3x+4>0 \quad \forall x\in\mathbb{R}$

$\mathcal{D}: x\in\mathbb{R}$

parnost, neparnost, periodičnost

$$f(-x)=\frac{-x}{x^2-3x+4}$$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$f(x)=0$ akko $x=0$

$(0,0)$ je nula f-je i presjek sa y-osom

x	$(-\infty,0)$	$(0,+\infty)$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekida $\Rightarrow$ f-ja nema V.A.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{x^2+3x+4}\,\frac{:x}{:x}=\lim_{x\to\infty}\frac{1}{x+3+\frac{4}{x}}=\frac{1}{\infty}=0$$

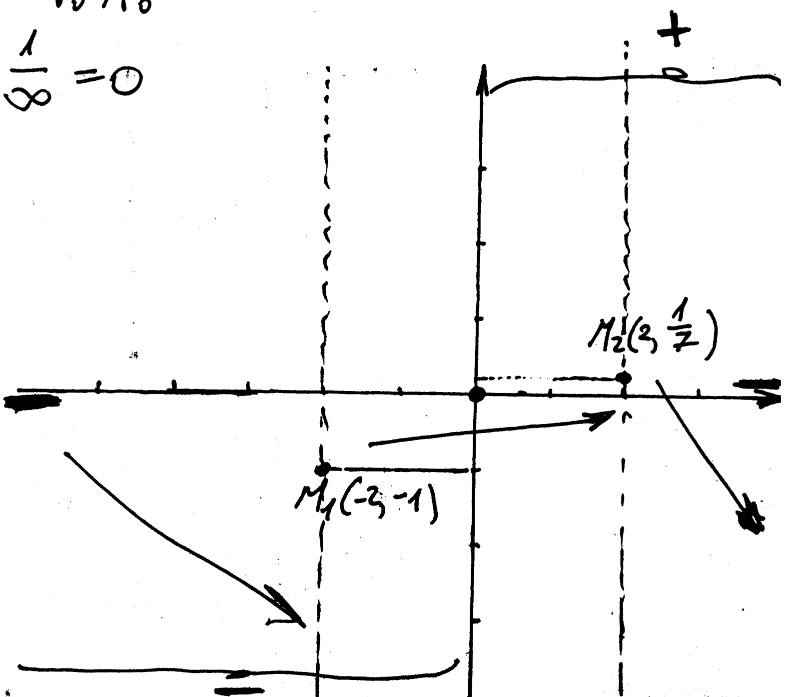
$\Rightarrow y=0$ je H.A.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{1}{x+3+\frac{4}{x}}=\frac{1}{-\infty}=0$$

$\Rightarrow y=0$ je H.A.

F-ja nema K.A.

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$$y' = \frac{-x^2+b}{(x^2+ax+b)^2} \Rightarrow y' = \frac{4-x^2}{(x^2+3x+4)^2}$$

$y'=0$ akko $4-x^2=0$

$x_1=-2,\ x_2=2$

x	$(-\infty,-2)$	$(-2,2)$	$(2,+\infty)$
y'	$-$	$+$	$-$
y	$\searrow$	$\nearrow$	$\searrow$

rast i opadanje

min $f(-2)=-1$ max $f(2)=\frac{1}{7}$

$\frac{-2}{8-6}$

$\frac{2}{8+6}=\frac{2}{14}$

ekstremi f-je

Na osnovu tabele $M_1(-2,-1)$ je tačka min a $M_2(2,\frac{1}{7})$ je max.

prevojne tačke i intervali konv. i konk.

$$y'' = \left(\frac{4-x^2}{(x^2+3x+4)^2}\right)' =$$

$$= \frac{-2x(x^2+3x+4)^{\cancel{2}} - (4-x^2)2\cancel{(x^2+3x+4)}\cdot(2x+3)}{(x^2+3x+4)^{\cancel{4}\,3}} = \frac{-2\left[\underline{x^3}+\cancel{3x^2}+4x+8x+12\underline{-2x^3}-\cancel{3x^2}\right]}{(x^2+3x+4)^3}$$

$$y'' = -2\cdot\frac{-x^3+12x+12}{(x^2+3x+4)^3} = 2\,\frac{x^3-12x-12}{(x^2+3x+4)^3}$$

$y''=0$ akko $x^3-12x-12=0$

$x_1\approx 3{,}88 \quad x_2\approx -1{,}11$

$x_3\approx -2{,}77$

(vrijednosti x_1, x_2 i x_3 su nađene pomoću digitrona koji ima opciju da nađe nule polinoma)

$-2{,}77 \quad -1{,}11 \quad 3{,}88$

x	$(-\infty,-2{,}77)$	$(-2{,}77,-1{,}11)$	$(-1{,}11,\ 3{,}88)$	$(3{,}88,+\infty)$
y''	$-$	$+$	$-$	$+$
y	$\frown$	$\smile$	$\frown$	$\smile$

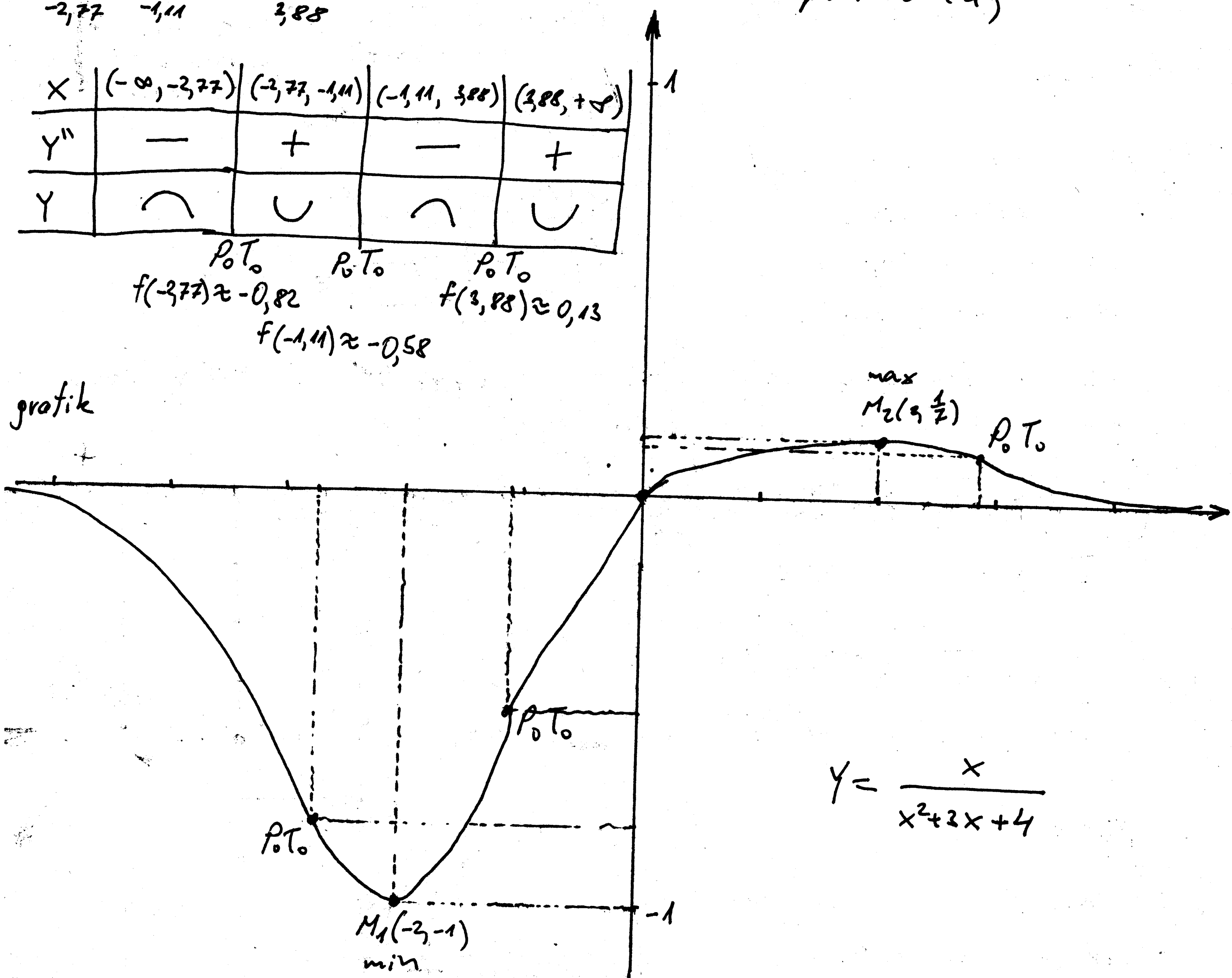
P_0T_0 $\quad P_0T_0 \quad P_0T_0$

$f(-2{,}77)\approx -0{,}82$

$f(3{,}88)\approx 0{,}13$

$f(-1{,}11)\approx -0{,}58$

grafik



Izračunati integral $I=\int \frac{x}{9x^2+24x+17}dx$.

Rj.

$$\int \frac{x}{9x^2+24x+17}dx = \frac{1}{18}\int \frac{18x+24}{9x^2+24x+17}dx - \frac{24}{18}\int \frac{dx}{9x^2+24x+17}$$

$$x = \frac{1}{18}\cdot 18x + \frac{24}{18} - \frac{24}{18}$$

$$\int \frac{18x+24}{9x^2+24x+17}dx = \left| \begin{matrix} 9x^2+24x+17=t \\ (18x+24)\,dx=dt \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + c = \ln|9x^2+24x+17| + c$$

$$(3x)^2 + 2\cdot 3x\cdot 4 + 4^2 + 1 = (3x+4)^2+1$$

$$\int \frac{dx}{9x^2+24x+17} = \int \frac{dx}{(3x+4)^2+1} = \left| \begin{matrix} 3x+4=s \\ 3\,dx=ds \\ dx=\frac{1}{3}ds \end{matrix} \right| = \frac{1}{3}\int \frac{ds}{s^2+1} = \frac{1}{3}\operatorname{arctg} s + c$$

$$= \frac{1}{3}\operatorname{arctg}(3x+4)+c$$

$$I = \frac{1}{18}\ln|9x^2+24x+17| - \frac{4}{3}\cdot\frac{1}{3}\operatorname{arctg}(3x+4)+c$$

$$= \frac{1}{18}\ln|9x^2+24x+17| - \frac{4}{9}\operatorname{arctg}(3x+4)+c$$

(#) Izračunati integral $I=\int x^3 e^{\frac{x}{2}}dx$.

Rj.

$$\int e^{\frac{x}{2}}dx = \left| \begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2\,dt \end{array} \right| = 2\int e^t dt = 2e^t + c = 2e^{\frac{x}{2}} + c$$

$$\int x^3 e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x^3 & dv=e^{\frac{x}{2}}dx \\ du=3x^2dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2x^3e^{\frac{x}{2}} - 6\int x^2 e^{\frac{x}{2}}dx$$

$$\int x^2 e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x^2 & dv=e^{\frac{x}{2}}dx \\ du=2x\,dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2x^2e^{\frac{x}{2}} - 4\int x e^{\frac{x}{2}}dx$$

$$\int x e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x & dv=e^{\frac{x}{2}}dx \\ du=dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2xe^{\frac{x}{2}} - 2\int e^{\frac{x}{2}}dx = 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c$$

$$I = 2x^3e^{\frac{x}{2}} - 6\left[2x^2e^{\frac{x}{2}} - 4\left(2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}}\right)\right] + c =$$

$$= 2x^3e^{\frac{x}{2}} - 6\left(2x^2e^{\frac{x}{2}} - 8xe^{\frac{x}{2}} + 16e^{\frac{x}{2}}\right) + c$$

$$= 2x^3e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$$

$$I = 2e^{\frac{x}{2}}\left(x^3 - 6x^2 + 24x - 48\right) + c$$

(#) Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.

$$\int \arcsin\frac{x}{2}\,dx = \left|\begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2dt \end{array}\right| = 2\int \arcsin t\,dt = \left|\begin{array}{ll} u=\arcsin t & dv=dt \\ du=\frac{dt}{\sqrt{1-t^2}} & v=t \end{array}\right| =$$

$$= 2\left(t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \stackrel{(**)}{=}$$

$$\int \frac{-2t\,dt}{\sqrt{1-t^2}} = \left|\begin{array}{l} 1-t^2=s \\ -2t\,dt=ds \end{array}\right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}}ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{s}+C =$$

$$= 2\sqrt{1-t^2}+C$$

$$\stackrel{(**)}{=} 2t\arcsin t + 2\sqrt{1-t^2}+C = x\arcsin\frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}} + C$$

$$\int_0^1 \arcsin\frac{x}{2}\,dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}} - 2\right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6} + \sqrt{3} - 2$$

(#) Izračunati površinu koju pravi kriva $y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nul-tačkama krive.

R: $y=x^2+x-6$

$D=1+24=25$

$y=(x-2)(x+3)$

$x_1=2 \quad x_2=-3$

$(2,0)$ i $(-3,0)$ su nule f-je

$f(0)=-6$

$(0,-6)$ je tačka presjeka f-je sa y-osom

$y'=2x+1$

$(-3,0),\ y'(-3)=-5$

$y-0=-5(x+3)$

$y=-5x-15$ jednačina tangente na krivu y u tački $(-3,0)$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$a>0$ f-ja je $\cup$ oblika

$-\frac{b}{2a}=-\frac{1}{2}, \quad -\frac{D}{4a}=-\frac{25}{4}=-6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$y-y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1,y_1) i koeficijentom k

u slučaju tangente $k=y'(x_1)$

$(2,0),\ y'(2)=5$

$y-0=5(x-2)$

$y=5x-10$ jednačina tangente na krivu y u tački $(2,0)$

presjek pravih:

$y=-5x-15 \quad (1)$

$y=5x-10 \quad (2)$

$(1)+(2):\ 2y=-25$

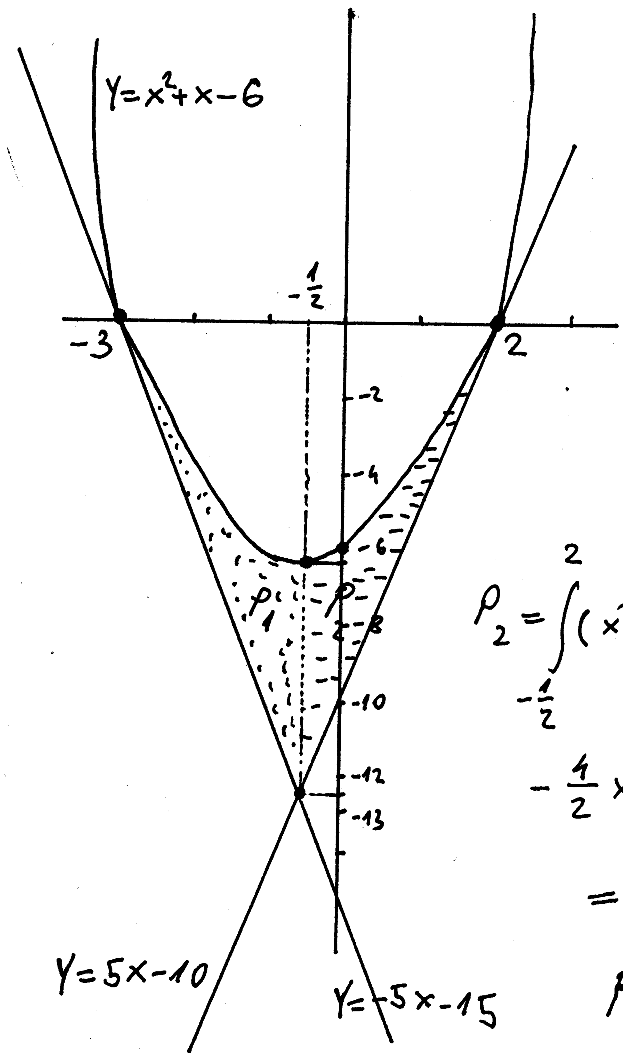
$y=-\frac{25}{2}=-12\frac{1}{2}$

$(1)-(2):\ -10x-5=0$

$-10x=5$

$x=-\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P=P_1+P_2$

$P_1=\int_{-3}^{-\frac{1}{2}}\left(x^2+x-6-(-5x-15)\right)dx=\int_{-3}^{-\frac{1}{2}}(x^2+6x+9)dx=$

$=\frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}}+\frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}}+9x\Big|_{-3}^{-\frac{1}{2}}=\frac{1}{3}\left(-\frac{1}{8}+27\right)$

$+3\left(\frac{1}{4}-9\right)+9\left(-\frac{1}{2}+3\right)=\frac{1}{3}\cdot\frac{215}{8}+3\cdot\frac{-35}{4}$

$+9\cdot\frac{5}{2}=\frac{215}{24}-\frac{630}{24}+\frac{540}{24}=\frac{125}{24}$

$P_2=\int_{-\frac{1}{2}}^{2}\left(x^2+x-6-(5x-10)\right)dx=\int_{-\frac{1}{2}}^{2}(x^2-4x+4)dx=\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2}-$

$-\frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2}+4x\Big|_{-\frac{1}{2}}^{2}=\frac{1}{3}\left(8+\frac{1}{8}\right)-2\left(4-\frac{1}{4}\right)+4\left(2+\frac{1}{2}\right)=$

$=\frac{1}{3}\cdot\frac{65}{8}-2\cdot\frac{15}{4}+4\cdot\frac{5}{2}=\frac{65}{24}-\frac{180}{24}+\frac{240}{24}=\frac{125}{24}$

$P=P_1+P_2=\frac{125}{24}+\frac{125}{24}=\frac{125}{12}$ tražena površina

(#) Nađi sve ekstreme f-je $z=y^2e^{-x^2-y^2}$, $y\neq 0$.

Rj.

$$\frac{\partial z}{\partial x}=y^2e^{-x^2-y^2}\cdot(-x^2-y^2)'_x=-2xy^2e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y}=2ye^{-x^2-y^2}+y^2e^{-x^2-y^2}\cdot(-x^2-y^2)'_y=2ye^{-x^2-y^2}+(-2y)y^2e^{-x^2-y^2}$$

$$=2ye^{-x^2-y^2}(1-y^2)$$

$$-2xy^2e^{-x^2-y^2}=0$$

$$2ye^{-x^2-y^2}(1-y^2)=0$$

Kako je $y\neq 0$ i $e^{-x^2-y^2}\neq 0$ $\forall x, y$

to $x=0$ i ($y=1$ ili $y=-1$)

Stacionarne tačke su $M_1(0,1)$ i $M_2(0,-1)$.

$$\frac{\partial^2 z}{\partial x^2}=-2y^2e^{-x^2-y^2}+(-2xy^2)e^{-x^2-y^2}\cdot(-2x)=-2y^2e^{-x^2-y^2}(1-2x^2)$$

$$\frac{\partial^2 z}{\partial x\partial y}=-4xye^{-x^2-y^2}+(-2xy^2)e^{-x^2-y^2}\cdot(-2y)=-4xye^{-x^2-y^2}+4xy^3e^{-x^2-y^2}$$

$$=4xye^{-x^2-y^2}(y^2-1)$$

$$\frac{\partial^2 z}{\partial y^2}=2e^{-x^2-y^2}(1-y^2)+2ye^{-x^2-y^2}\cdot(-2y)(1-y^2)+2ye^{-x^2-y^2}\cdot(-2y)$$

$$=2e^{-x^2-y^2}(1-y^2)-4y^2e^{-x^2-y^2}(1-y^2)-4y^2e^{-x^2-y^2}$$

$$=2e^{-x^2-y^2}(\underbrace{1-y^2-2y^2(1-y^2)-2y^2}_{1-y^2-2y^2+2y^4-2y^2})=2e^{-x^2-y^2}(2y^4-5y^2+1)$$

$D=AC-B^2$

$M_1(0,1)$, $A=-2e^{-1}=-\frac{2}{e}$, $B=0$, $C=2e^{-1}(2-5+1)=-\frac{4}{e}$

$D=\left(-\frac{2}{e}\right)\left(-\frac{4}{e}\right)=\frac{8}{e^2}>0$ f-ja ima ekstrem, $A<0$ f-ja ima MAX

$z_{max}(0,1)=1\cdot e^{-1}=\frac{1}{e}$

$M_2(0,-1)$, $A=-\frac{2}{e}$, $B=0$, $C=-\frac{4}{e}$, $D=\frac{8}{e^2}>0$ f-ja ima ekstrem

$A<0 \Rightarrow$ f-ja ima MAX

$z_{max}(0,-1)=1\cdot e^{-1}=\frac{1}{e}$

Ekstremi f-je z su u tačkama M_1 i M_2 i iznose $\frac{1}{e}$.

(#) Naći uslovne ekstreme f-je $z = 2x^4 + 8y^4 + 24$ ako je $8x + 4y = 1$.

Rj.

$$F(x, y, \lambda) = 2x^4 + 8y^4 + 24 + \lambda(8x + 4y - 1)$$

$$\frac{\partial F}{\partial x} = 8x^3 + 8\lambda$$

$$\frac{\partial F}{\partial y} = 32y^3 + 4\lambda$$

$$\frac{\partial F}{\partial \lambda} = 8x + 4y - 1$$

$$8x^3 + 8\lambda = 0 \quad /:8$$
$$32y^3 + 4\lambda = 0 \quad /:4$$

$$x^3 + \lambda = 0$$
$$- \quad 8y^3 + \lambda = 0$$

$$x^3 - 8y^3 = 0$$
$$x^3 = 8y^3$$
$$x = 2y$$

$$8x + 4y - 1 = 0$$
$$8 \cdot 2y + 4y - 1 = 0$$
$$20y = 1$$
$$y = \frac{1}{20}$$
$$x = 2 \cdot \frac{1}{20} = \frac{1}{10}$$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$ je stacionarna tačka

$$\frac{\partial^2 F}{\partial x^2} = 24x^2$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y^2} = 96y^2$$

$$D = AC - B^2$$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$

$$A = 24 \cdot \frac{1}{100} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$$B = 0$$

$$C = \overset{24}{\cancel{\overset{48}{\cancel{96}}}} \cdot \frac{1}{\underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}}} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$D = \left(\frac{6}{25}\right)^2 > 0$ f-ja ima ekstrem

$A > 0$ f-ja ima minimum

$$z_{min}\left(\frac{1}{10}, \frac{1}{20}\right) = 2 \cdot \frac{1}{10^4} + 8 \cdot \frac{1}{20^4} + 24 = \frac{2}{10\,000} + \frac{\overset{1}{\cancel{\overset{2}{\cancel{\overset{4}{\cancel{8}}}}}}}{\underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}} \cdot 20} + 24$$

$$= \frac{2}{10\,000} + \frac{1}{20\,000} + \frac{24\,000}{1\,000} = \frac{4 + 1 + 480\,000}{20\,000} =$$

$$= \frac{480\,005 \;:5}{20\,000 \;:5} = \frac{96\,001}{4\,000}$$

$z_{min} = \frac{96\,001}{4\,000}$ je minimum f-je u tački $M\left(\frac{1}{10}, \frac{1}{20}\right)$

(#) Riješiti diferencijalnu jednačinu $y' = 2^{2x+y}$.

Rj.

$$y' = 2^{2x} \cdot 2^{y}$$ diferencijalna jednačina sa razdvojenim promjenjivim

$$\frac{dy}{dx} = 2^{2x} \cdot 2^{y}$$

$$\frac{dy}{2^{y}} = 4^{x} dx$$

$$\left[\int a^x dx = \frac{a^x}{\ln a} + C, \quad 0 < a \neq 1 \right]$$

$$2^{-y} dy = 4^{x} dx \quad / \int$$

$$\int 2^{-y} dy = \int 4^{x} dx$$

$$\left[\ln 4 = \ln 2^2 = 2 \ln 2 \right]$$

$$-\frac{2^{-y}}{\ln 2} = \frac{4^{x}}{\ln 4} + C_1$$

$$\left[\int 2^{-y} dy = \left| \begin{array}{l} -y = t \\ -dy = dt \\ dy = -dt \end{array} \right| = -\int 2^{t} dt = -\frac{2^{t}}{\ln 2} + C = -\frac{2^{-y}}{\ln 2} + C \right]$$

$$-\frac{2^{-y}}{\ln 2} = \frac{4^{x}}{2\ln 2} + C_1 \quad / \cdot \ln 2 \cdot 2$$

$$-2 \cdot 2^{-y} = 4^{x} + C$$ opšte rješenje dif. jednačine

$$2^{-y} = \frac{4^{x} + C}{-2}$$

ili

$$4^{x} = C - 2 \cdot 2^{-y}$$

$$-y = \log_2 \frac{4^{x} + C}{-2}$$

ili

$$x = \log_4 (C - 2^{1-y})$$ opšte rješenje

$$y = \log_2 \frac{-2}{4^{x} + C}$$ opšte rješenje

(#) Riješiti diferencijalnu jednačinu $y'\cos x - y\sin x = x^3 e^{x^2}$ uz početni uslov $y(0)=1$.

Rj. $y'\cos x - y\sin x = x^3 e^{x^2} \quad /:\cos x$

$$y' - y\,tg x = \frac{x^3 e^{x^2}}{\cos x}$$

ovo je linearna diferencijalna jednačina

uvodimo smjenu

$y = uv$

$y' = u'v + uv'$

$$u'v + uv' - uv\,tg x = \frac{x^2 e^{x^2}}{\cos x}$$

$$u'v + \underbrace{u(v' - v\,tg x)}_{=0} = \frac{x^3 e^{x^2}}{\cos x}$$

$$\int tg x\,dx = \left| \begin{array}{l} tg x = t \\ x = arc\,tg\,t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{t}{1+t^2}dt = \left| \begin{array}{l} 1+t^2 = s \\ 2t\,dt = ds \\ t\,dt = \frac{ds}{2} \end{array} \right| = \frac{1}{2}\int \frac{ds}{s} =$$

$$= \frac{1}{2}\ln|s| = \frac{1}{2}\ln|1+t^2| =$$

$$= \frac{1}{2}\ln|1+tg^2 x| = \frac{1}{2}\ln\left|1+\frac{\sin^2 x}{\cos^2 x}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right| = \ln\left|\frac{1}{\cos x}\right|$$

$v' - v\,tg x = 0$

$$\frac{dv}{dx} = v\,tg x$$

$$\frac{dv}{v} = tg x\,dx$$

$$\int \frac{dv}{v} = \int tg x\,dx$$

$$\ln v = \ln\left|\frac{1}{\cos x}\right|$$

$$v = \frac{1}{\cos x}$$

$$u'v = \frac{x^3 e^{x^2}}{\cos x}$$

$$u' \cdot \frac{1}{\cos x} = \frac{x^3 e^{x^2}}{\cos x} \quad /\cdot \cos x$$

$$\frac{du}{dx} = x^3 e^{x^2}$$

$$du = x^3 e^{x^2} dx$$

$$I = \int x^3 e^{x^2} dx = \left| \begin{array}{ll} u = x^2 & dv = x e^{x^2} dx \\ du = 2x & v = \int x e^{x^2} dx = \left| \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int e^t dt = \frac{1}{2}e^{x^2} \end{array} \right| =$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\cdot 2\int x\,e^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

$du = x^3 e^{x^2} dx \quad //\int$

$$u = \frac{1}{2}e^{x^2}(x^2-1) + c_1$$

$$y = uv = \frac{e^{x^2}(x^2-1)+c}{2\cos x}$$

opće rješenje diferencijalne jednačine

$y(0) = 1$

$$y(0) = \frac{e^0(0-1)+c}{2\cos 0} = \frac{-1+c}{2} = 1$$

$-1 + c = 2$

$c = 3$

$$y = \frac{e^{x^2}(x^2-1)+3}{2\cos x}$$

partikularno rješenje diferencijalne jednačine