Univerzitet u Zenici
Ekonomski fakultet

14.10.2009.

Pismeni ispit iz predmeta Matematika

1. Dokazati metodom matematičke indukcije da vrijedi za sve $n \in \{2, 3, 4, ...\}$:

$$\frac{1}{log_x 2 \cdot log_x 4} + \frac{1}{log_x 4 \cdot log_x 8} + ... + \frac{1}{log_x 2^{n-1} \cdot log_x 2^n} = (1 - \frac{1}{n})\frac{1}{(log_x 2)^2}.$$

2. Izračunati x ako se zna da u razvoju izraza $(\sqrt{2^x} + \frac{1}{\sqrt{2^{x-1}}})^6$ zbir trećeg i petog člana iznosi 135.

3. Izračunati $\frac{(\sqrt{3}+i)^{22}(1-i)^{15}}{(-1-i)^3}$.

4. Diskutovati rang matrice $A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix}$ u zavisnosti od parametara a i b.

5. Ispitati funkciju i nacrtati joj grafik $y = \frac{(x-1)^4}{x^3}$.

6. Ispitati funkciju i nacrtati joj grafik $y = \frac{(2x-1)(x^2-2x+6)}{4x^2}$.

7. Ispitati funkciju i nacrtati joj grafik $y = (x+1)e^{\frac{1}{2}x^2 - x}$.

8. Ispitati funkciju i nacrtati joj grafik $y = \frac{x}{x-1} ln \frac{x}{x-1}$ (bez analize drugog izvoda).

9. Izračunati integral $I = \int \frac{4x^3+1}{x^4-x}\,dx$.

10. Izračunati integral $I = \int (x^2+x) ln \frac{2x+1}{x-1}\,dx$.

11. Izračunati integral $I = \int \frac{8cosx - sinx}{2cosx + sinx}\,dx$.

12. Izračunati površinu figure koju ograničavaju linije $x = y^2 - 2y - 3$ i $y = 3 - 3x$.

13. Naći stacionarne tačke funkcije $z = xy\,ln(x^2+y^2)$.

14. Naći ekstreme funkcije $z = (2x^2+3y^2)e^{-(x^2+y^2)}$.

15. Riješiti diferencijalnu jednačinu $(x-y-2)dx + (2x-y-5)dy = 0$.

16. Riješiti diferencijalnu jednačinu $2y + y'(2x+y') = 0$.

(Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Dokazati metodom matematičke indukcije da vrijedi za sve $n \in \{2,3,4,\dots\}$:

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\dots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^n}=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}$$

Rj. postavka zadatka

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\dots+\frac{1}{\log_x 2^{k-1}\cdot\log_x 2^k}=\left(1-\frac{1}{k}\right)\frac{1}{(\log_x 2)^2},\quad k=2,3,\dots$$

BAZA INDUKCIJE

$k=2$: $$\frac{1}{\log_x 2\cdot\log_x 4}=\left(1-\frac{1}{2}\right)\cdot\frac{1}{\log_x 2\cdot\log_x 2}=\frac{1}{2}\cdot\frac{1}{\log_x 2\cdot\log_x 2}=\frac{1}{\log_x 2\cdot 2\log_x 2}=\frac{1}{\log_x 2\cdot\log_x 4}$$

Tvrdnja je tačna za $k=2$.

KORAK INDUKCIJE

Pretpostavimo da je jednakost $\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\dots+\frac{1}{\log_x 2^{k-1}\log_x 2^k}=\left(1-\frac{1}{k}\right)\frac{1}{(\log_x 2)^2}$ tačna za svako $k=2,3,\dots,n$.

Na osnovu ove pretpostavke dokažimo da je

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\dots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^n}+\frac{1}{\log_x 2^n\cdot\log_x 2^{n+1}}=\left(1-\frac{1}{n+1}\right)\cdot\frac{1}{(\log_x 2)^2}$$

$$\frac{1}{\log_x 2\cdot\log_x 4}+\frac{1}{\log_x 4\cdot\log_x 8}+\dots+\frac{1}{\log_x 2^{n-1}\cdot\log_x 2^n}+\frac{1}{\log_x 2^n\cdot\log_x 2^{n+1}}$$ na osnovu pretpostavke

$$=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{\log_x 2^n\cdot\log_x 2^{n+1}}=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{n\cdot(n+1)\log_x 2\cdot\log_x 2}$$

$$=\left(1-\frac{1}{n}\right)\frac{1}{(\log_x 2)^2}+\frac{1}{n(n+1)(\log_x 2)^2}=\left(1-\frac{1}{n}+\frac{1}{n(n+1)}\right)\frac{1}{(\log_x 2)^2}$$

$$=\left(1+\frac{-(n+1)+1}{n(n+1)}\right)\frac{1}{(\log_x 2)^2}=\left(1+\frac{-n}{n(n+1)}\right)\cdot\frac{1}{(\log_x 2)^2}=\left(1-\frac{1}{n+1}\right)\cdot\frac{1}{(\log_x 2)^2}$$

što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve brojeve $n\in\{2,3,4,\dots\}$

Izračunati x ako se zna da u razvoju izraza $\left(\sqrt{2^x}+\frac{1}{\sqrt{2^{x-1}}}\right)^6$ zbir trećeg i petog člana iznosi 135.

Rj.

$$\left(\sqrt{2^x}+\frac{1}{\sqrt{2^{x-1}}}\right)^6=\sum_{k=0}^{6}\binom{6}{k}\left(\sqrt{2^x}\right)^{6-k}\cdot\left(\frac{1}{\sqrt{2^{x-1}}}\right)^k=$$

$$=\sum_{k=0}^{6}\binom{6}{k}\left(2^{\frac{x}{2}}\right)^{6-k}\cdot\left(2^{-\frac{x-1}{2}}\right)^k=\sum_{k=0}^{6}\binom{6}{k}2^{3x-\frac{kx}{2}}\cdot2^{-\frac{kx}{2}+\frac{k}{2}}$$

$$=\sum_{k=0}^{6}\binom{6}{k}2^{3x+\frac{k}{2}-kx}$$

$$\binom{6}{2}=\binom{6}{4}=\frac{6\cdot5}{2}=$$
$$=3\cdot5=15$$

treći član $(k=2)$: $\binom{6}{2}2^{3x+1-2x}$

peti član $(k=4)$: $\binom{6}{4}2^{3x+2-4x}$

$$\binom{6}{2}2^{x+1}+\binom{6}{4}2^{-x+2}=15\left(2^{x+1}+2^{-x+2}\right)=135 \quad /:15$$

$$2^{x+1}+2^{-x+2}=9$$

$$2^x\cdot2+2^{-x}\cdot2^2=9$$

$$2^x=t$$

$$2t+4t^{-1}=9 \quad /\cdot t$$

$$2t^2-9t+4=0$$

$$D=81-32$$

$$D=49$$

$$t_{1,2}=\frac{9\pm7}{4}$$

$$t_1=\frac{2}{4}=\frac{1}{2} \qquad t_2=4$$

$$2^x=2^{-1} \qquad 2^x=2^2$$

$$x_1=-1 \qquad x_2=2$$

Vrijednost promjenjive x je -1 ili 2.

Izračunati $\dfrac{(\sqrt{3}+i)^{22}(1-i)^{15}}{(-1-i)^3}$.

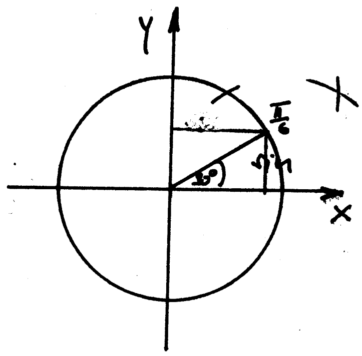
Rj.

$z_1 = \sqrt{3}+i$

$|z_1| = \sqrt{3+1} = \sqrt{4} = 2$

$\cos\theta_1 = \dfrac{a}{|z_1|} = \dfrac{\sqrt{3}}{2}$

$\sin\theta_1 = \dfrac{b}{|z_1|} = \dfrac{1}{2}$



$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$

$\theta_1 = \dfrac{\pi}{6}$

$z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z_1^{22} = 2^{22}\left(\cos\left(22\cdot\frac{\pi}{6}\right) + i\sin\left(22\cdot\frac{\pi}{6}\right)\right)$

$= 2^{22}\left(\cos\frac{11\pi}{3} + i\sin\frac{11\pi}{3}\right)$

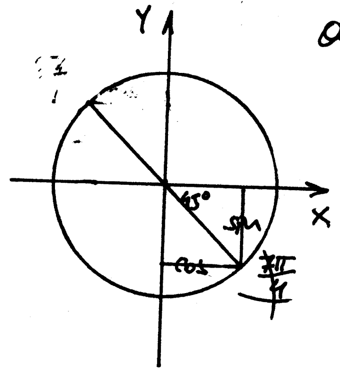
$z_2 = 1-i$

$|z_2| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta_2 = \dfrac{a}{|z_2|} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\sin\theta_2 = \dfrac{b}{|z_2|} = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$tg\,\theta_2 = \dfrac{b}{a} = -1$



$tg\,45° = tg\frac{\pi}{4} = 1$

$\theta_2 = \dfrac{7\pi}{4}$

$z_2 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$z_2^{15} = (\sqrt{2})^{15}\left(\cos 15\cdot\frac{7\pi}{4} + i\sin 15\cdot\frac{7\pi}{4}\right)$

$= 2^7\sqrt{2}\left(\cos\frac{105\pi}{4} + i\sin\frac{105\pi}{4}\right)$

$(1-i)^2 = 1-2i+i^2 = -2i$

$(1-i)^{14} = \left((1-i)^2\right)^7 = (-2i)^7 = -2^7\cdot i^7 = -2^7\cdot i^6\cdot i$

$= -2^7\cdot(i^2)^3\cdot i = 2^7\cdot i$

$z_3 = -1-i$

$(-1-i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$

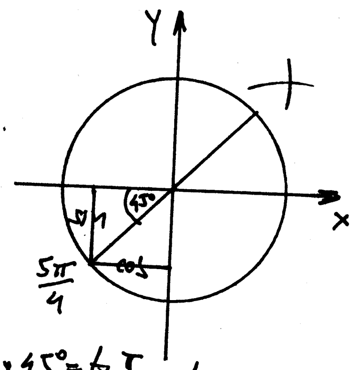
$(-1-i)^3 = (-1-i)^2\cdot(-1-i) = 2i(-1-i) = -2i-2i^2 = 2-2i$

$|z_3| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta_3 = \dfrac{a}{|z_3|} = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$\sin\theta_3 = \dfrac{b}{|z_3|} = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$tg\,\theta_3 = \dfrac{b}{a} = \dfrac{-1}{-1} = 1$



$tg\,45° = tg\frac{\pi}{4} = 1$

$z_3 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_3^3 = (\sqrt{2})^3\left(\cos 3\cdot\frac{5\pi}{4} + i\sin 3\cdot\frac{5\pi}{4}\right)$

$= 2\sqrt{2}\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)$

$$\frac{(1-i)^{15}}{(-1-i)^3} = \frac{2^7\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}\left(\cos\frac{105\pi-15\pi}{4} + i\sin\frac{105-15}{4}\pi\right) = 2^6\left(\cos\frac{90\pi}{4} + i\sin\frac{90\pi}{4}\right)$$

$$z_1^{22}\cdot\frac{z_2^{15}}{z_3^3} = 2^{22}\cdot 2^6\left(\cos\left(\frac{11\pi}{3}+\frac{90\pi}{4}\right) + i\sin\left(\frac{11\pi}{3}+\frac{90\pi}{4}\right)\right) = 2^{28}\left(\cos\frac{314}{12}\pi + i\sin\frac{157}{6}\pi\right)$$

$$z = 2^{28}\left(\cos\left(\frac{\pi}{6}+2\cdot 13\pi\right) + i\sin\left(\frac{\pi}{6}+2\cdot 13\pi\right)\right) = 2^{28}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2^{28}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2^{27}(\sqrt{3}+i)$$

Diskutovati rang matrice u zavisnosti od parametara a i b,

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix}$$

Rj.

$$A = \begin{bmatrix} 2 & 3 & 9 & 6 & 2 \\ 5 & 4 & 12 & 8 & 5 \\ 1 & 2 & 6 & 4 & 1 \\ 4 & 1 & 3 & 2 & a \\ 3 & b & 6 & 4 & 3 \\ 7 & 5 & 15 & 10 & 7 \end{bmatrix} \underset{II_k \leftrightarrow IV_k}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 4 & 2 & 3 & 1 & a \\ 3 & 4 & 6 & b & 3 \\ 7 & 10 & 15 & 5 & 7 \end{bmatrix} \underset{IV_R \leftrightarrow VI_R}{\sim} \begin{bmatrix} 2 & 6 & 9 & 3 & 2 \\ 5 & 8 & 12 & 4 & 5 \\ 1 & 4 & 6 & 2 & 1 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \underset{III_R \leftrightarrow I_R}{\sim}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 5 & 8 & 12 & 4 & 5 \\ 2 & 6 & 9 & 3 & 2 \\ 7 & 10 & 15 & 5 & 7 \\ 3 & 4 & 6 & b & 3 \\ 4 & 2 & 3 & 1 & a \end{bmatrix} \underset{\substack{II_R - I_R \cdot 5 \\ III_R - I_R \cdot 2 \\ IV_R - I_R \cdot 7 \\ V_R - I_R \cdot 3 \\ VI_R - I_R \cdot 4}}{\sim} \begin{bmatrix} 1 & 4 & 6 & 2 & 1 \\ 0 & -12 & -18 & -6 & 0 \\ 0 & -2 & -3 & -1 & 0 \\ 0 & -18 & -27 & -9 & 0 \\ 0 & -8 & -12 & b-6 & 0 \\ 0 & -14 & -21 & -7 & a-4 \end{bmatrix} \underset{II_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \sim$$

$$\underset{II_R \leftrightarrow III_R}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -18 & -6 & -12 \\ 0 & 0 & -27 & -9 & -18 \\ 0 & 0 & -12 & b-6 & -8 \\ 0 & a-4 & -21 & -7 & -14 \end{bmatrix} \underset{\substack{III_R - II_R \cdot 6 \\ IV_R - II_R \cdot 9 \\ V_R - II_R \cdot 4 \\ VI_R - II_R \cdot 7}}{\sim} \begin{bmatrix} 1 & 1 & 6 & 2 & 4 \\ 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-2 & 0 \\ 0 & a-4 & 0 & 0 & 0 \end{bmatrix}$$

Diskusija

1° $a = 4$, $b = 2$ $\quad$ rang $A = 2$

2° $a = 4$, $b \neq 2$ $\quad$ rang $A = 3$

3° $a \neq 4$, $b = 2$ $\quad$ rang $A = 3$

4° $a \neq 4$, $b \neq 2$ $\quad$ rang $A = 4$

#) Ispitati f-ju i nacrtati joj grafik $y=\frac{(2x-1)(x^2-2x+6)}{4x^2}$.

Rj.

definiciono područje

$4x^2 \neq 0$

$x \neq 0$ $\quad D: x \in R \setminus \{0\}$

parnost, neparnost, periodičnost

$f(-x)=\frac{(-2x-1)(x^2+2x+6)}{4x^2}$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $2x-1=0$ ili $x^2-2x+6=0$

$2x=1 \qquad D=4-24<0$

$x=\frac{1}{2} \qquad x^2-2x+6 \neq 0 \quad \forall x \in R$

$(\frac{1}{2},0)$ je nula f-je

$x^2-2x+6>0 \quad \forall x \in R$

$4x^2>0 \quad \forall x \in R$

x	$(-\infty,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},+\infty)$
y	−	−	+

znak f-je

f(0) nije definisano ⇒ f-ja ne siječe Y-osu

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{(2x-1)(x^2-2x+6)}{4x^2}=\lim_{x\to 0-}\frac{2x^3-5x^2+12x-6}{4x^2}=\frac{-6}{4(0-)^2}=-\infty$

$x=0$ je V.A.

$(2x-1)(x^2-2x+6)=2x^3\underline{-4x^2}+12x\underline{-x^2}+2x-6=2x^3-5x^2+12x-6$

$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{2x^3-5x^2+12x-6}{4x^2}=\frac{-6}{0+}=-\infty \Rightarrow x=0$ je V.A.

$\lim_{x\to +\infty} f(x)=\lim_{x\to +\infty}\frac{2x^3-5x^2+12x-6 \; /:x^3}{4x^2 \; /:x^3}=\lim_{x\to +\infty}\frac{2-\frac{5}{x}+\frac{12}{x^2}-\frac{6}{x^3}}{\frac{4}{x}}=\frac{2}{0+}=+\infty$

f-ja nema H.A. (kad $x\to+\infty$)

$\lim_{x\to -\infty} f(x)=\lim_{x\to -\infty}\frac{2-\frac{5}{x}+\frac{12}{x^2}-\frac{6}{x^3}}{\frac{4}{x}}=\frac{2}{0-}=-\infty$

f-ja nema H.A. (kad $x\to-\infty$)

$y=kx+n$ kosa asimptota

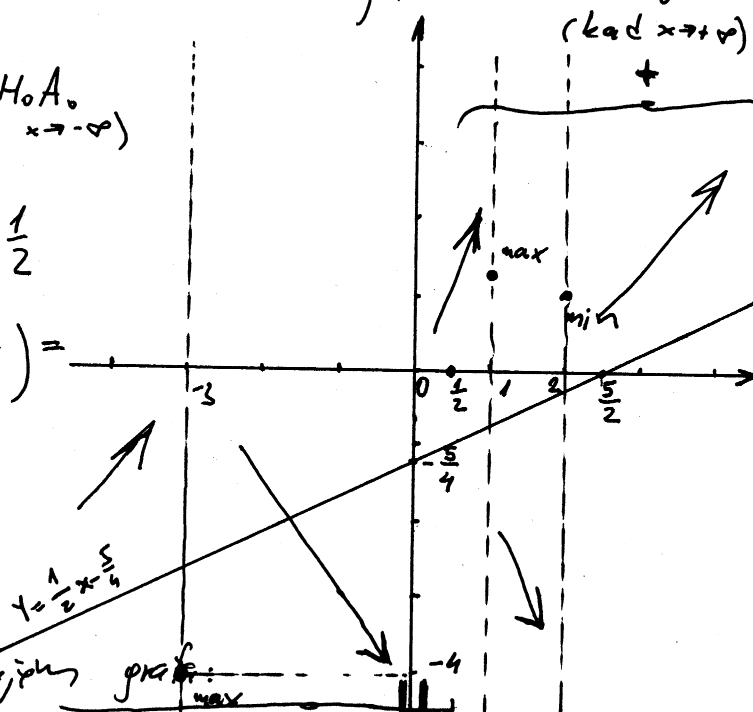
$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{2x^3-5x^2+12x-6 \; /:x^3}{4x^3 \; /:x^3}=\frac{2}{4}=\frac{1}{2}$

$n=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(\frac{2x^3-5x^2+12x-6}{4x^2}-\frac{1}{2}x\right)=$

$=\lim_{x\to\infty}\frac{\cancel{2x^3}-5x^2+12x-6-\cancel{2x^3} \; :x^2}{4x^2 \; :x^2}=-\frac{5}{4}$

$y=\frac{1}{2}x-\frac{5}{4}$ je kosa asimptota

Nakon ovog koraka počinjem sa skiciranjem grafa:



rast i opadanje

$$y'=\left(\frac{(2x-1)(x^2-2x+6)}{4x^2}\right)'=\frac{\left[2(x^2-2x+6)+(2x-1)\overset{2(x-1)}{(2x-2)}\right]\cdot \cancel{4}x^2-(2x-1)(x^2-2x+6)\overset{2}{\cancel{8}}x}{16x^4}$$

$$2x^3-4x^2+12x-x^2+2x-6=2x^3-5x^2+14x-6$$

$$=\frac{2(x^2-2x+6+2x^2-2x-x+1)\cdot x-(2x-1)(x^2-2x+6)\cdot 2}{4x^3}=$$

$$=\frac{2(3x^2-5x+7)\cdot x+2(-2x^3+5x^2-14x+6)}{4x^3}=\frac{2(3x^3\cancel{-5x^2}+7x-2x^3\cancel{+5x^2}-14x+6)}{4x^3}$$

$$=\frac{2(x^3-7x+6)}{4x^3}=\frac{x^3-7x+6}{2x^3}=\frac{(x^2+x-6)(x-1)}{2x^3}=\frac{(x+3)(x-2)(x-1)}{2x^3}$$

$g(x)=x^3-7x+6$

$g(1)=1-7+6=0$

$(x^3-7x+6):(x-1)=x^2+x-6$

$-\ \ x^3-x^2$

x^2-7x+6

$-\ \ x^2-x$

$-6x+6$

$-\ \ -6x+6$

$//\ \ //$

prekidi y + nule y'

$y'=0$ akko $x=-3$ ili $x=2$ ili $x=1$

$-3 \quad 0 \quad 1 \quad 2$

$$f(-3)=\frac{(-7)(9+6+6)}{4\cdot 9}=\frac{-7\cdot \overset{7}{\cancel{21}}}{4\cdot \underset{3}{\cancel{9}}}$$

$$f(-3)=-\frac{49}{12}\approx -4{,}083\dot{3}$$

$$f(1)=\frac{5}{4},\quad f(2)=\frac{3\cdot \cancel{6}}{4\cdot \cancel{4}_2}=\frac{9}{8}$$

x	$(-\infty,-3)$	$(-3,0)$	$(0,1)$	$(1,2)$	$(2,+\infty)$
y'	+	−	+	−	+
y	↗	↘	↗	↘	↗

max $\quad$ max $\quad$ min $\quad$ rast i opadanje

ekstremi f-je

Na osnovu tabele rasta i opadanja $\left(-3,-\frac{49}{12}\right)$ i $\left(1,\frac{5}{4}\right)$ su tačke lokalnog maksimuma a $\left(2,\frac{9}{8}\right)$ je tačka lokalnog minimuma.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'=\frac{x^3-7x+6}{2x^3}=\frac{1}{2}+\frac{-7x+6}{2x^3}$$

$$y''=\frac{-7\cdot 2x^{\cancel{3}}-(-7x+6)6x^{\cancel{2}}}{4x^{\cancel{6}4}}=\frac{-14x+42x-36}{4x^4}=\frac{28x-36}{4x^4}$$

$$y''=\frac{7x-9}{x^4}$$

$y''=0$ akko $7x-9=0$

$7x=9$

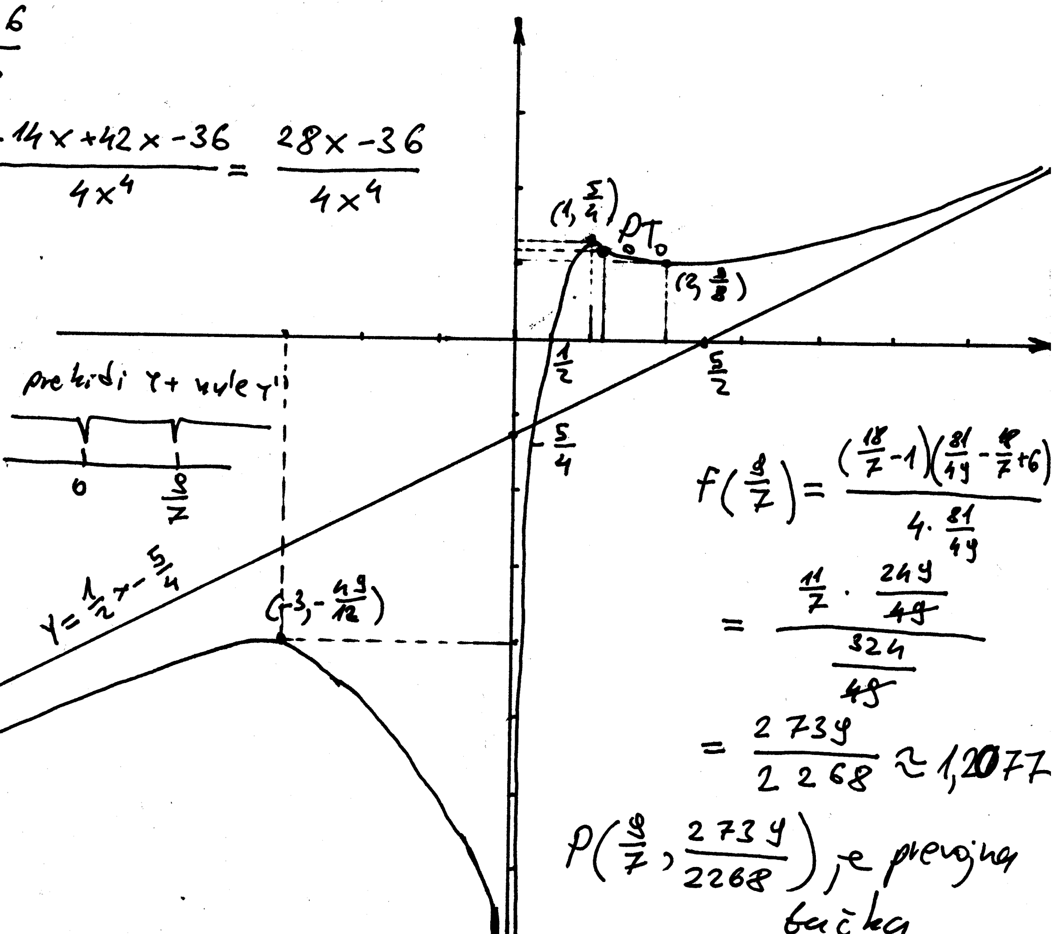
$x=\frac{9}{7}$

prekidi y + nule y''

$0 \quad \frac{9}{7}$

x	$(-\infty,0)$	$(0,\frac{9}{7})$	$(\frac{9}{7},+\infty)$
y''	−	−	+
y	∩	∩	∪

konveksnost i konkavnost $\quad$ PoTo



$$f\left(\frac{9}{7}\right)=\frac{\left(\frac{18}{7}-1\right)\left(\frac{81}{49}-\frac{18}{7}+6\right)}{4\cdot\frac{81}{49}}$$

$$=\frac{\frac{11}{7}\cdot\frac{249}{49}}{\frac{324}{49}}$$

$$=\frac{2739}{2268}\approx 1{,}2077$$

$P\left(\frac{9}{7},\frac{2739}{2268}\right)$ je prevojna tačka

Ispitati f-ju i nacrtati joj grafik $y = \frac{(x-1)^4}{x^3}$.

Rj: definiciono područje
$x^3 \neq 0$
$D: x \in \mathbb{R} \setminus \{0\}$

parnost, neparnost, periodičnost
$f(-x) = \frac{(-x-1)^4}{-x^3}$ f-ja nije ni parna ni neparna
f-ja nije periodična

nule, presjek sa y-osom, znak f-je
$y = 0$ akko $(x-1)^4 = 0$
$x - 1 = 0$
$x = 1$
(1,0) je nula f-je
f-ja ne siječe y-osu

$(x-1)^4 > 0 \quad \forall x \in \mathbb{R}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

x	$(-\infty, 0)$	$(0, +\infty)$
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote
za $x = 0$ f-ja ima prekid

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{(x-1)^4}{x^3} = \frac{((0+)-1)^4}{(0+)^3} = \frac{(-1+0)^4}{0+} = +\infty \Rightarrow x = 0 \text{ je } V_0A_0$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{(x-1)^4}{x^3} = \frac{((0-)-1)^4}{(0-)^3} = \frac{(-1-0)^4}{0-} = -\infty \Rightarrow x = 0 \text{ je } V_0A_0$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to+\infty} \frac{x^4 - 4x^3 + 6x^2 - 4x + 1 \;/:x^3}{x^3 \;/:x^3} = \frac{\infty}{1} = \infty, \quad \lim_{x\to-\infty} f(x) = \frac{\infty}{1} = \infty \Rightarrow$$

$\Rightarrow$ f-ja nema H_0A_0

$y = kx + n$ je oblika K_0A_0

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^4 - 4x^3 + 6x^2 - 4x + 1 \;/:x^4}{x^4 \;/:x^4} = \frac{1}{1} = 1$$

$$n = \lim_{x\to\infty} [f(x) - kx] = \lim_{x\to\infty} \left(\frac{(x-1)^4}{x^3} - x \right) = \lim_{x\to\infty} \frac{\cancel{x^4} - 4x^3 + 6x^2 - 4x + 1 - \cancel{x^4} \;/:x^3}{x^3 \;/:x^3} = \frac{-4}{1} = -4$$

$y = x - 4$ je K_0A_0

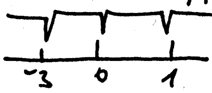
nakon ovog koraka počinjemo sa skiciranjem grafa:

rast i opadanje

$$y' = \left(\frac{(x-1)^4}{x^3} \right)' = \frac{4(x-1)^3 \cancel{x^3}^{\,x} - (x-1)^4 \cdot 3\cancel{x^2}}{x^{\cancel{6}\,4}} = \frac{(x-1)^3 (4x - (x-1)\cdot 3)}{x^4}$$

$$y' = \frac{(x-1)^3 (x+3)}{x^4}$$

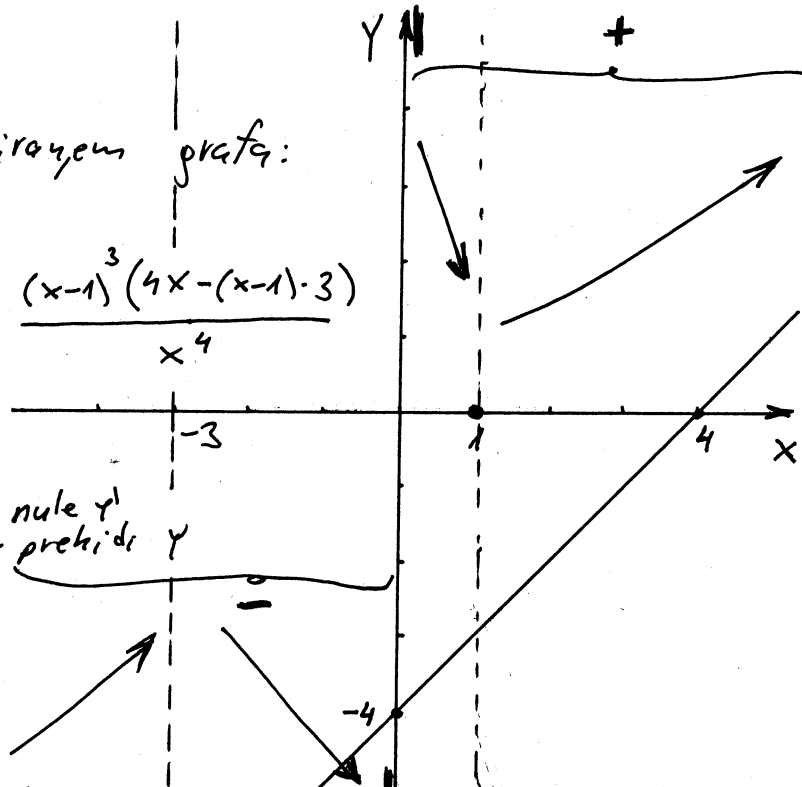
$y' = 0$ akko $x = 1$ ili $x = -3$



← nule y' + prekidi y

x	$(-\infty, -3)$	$(-3, 0)$	$(0, 1)$	$(1, +\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗

max min



$f(-3)=\frac{(-3-1)^4}{(-3)^3}=\frac{16\cdot 16}{-27}=-\frac{256}{27}\approx -9{,}4815$

ekstremi f-je

Iz tabele rasta i opadanja f-ja ima lokalni maksimum u tački $(-3,-\frac{256}{27})$ i lokokalni minimum u tački $(1,0)$.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{(x-1)^3(x+3)}{x^4}\right)'=\frac{(3(x-1)^2(x+3)+(x-1)^3)\cancel{x^4}-(x-1)^3(x+3)\cdot 4\cancel{x^3}}{x^{\cancel{8}5}}$$

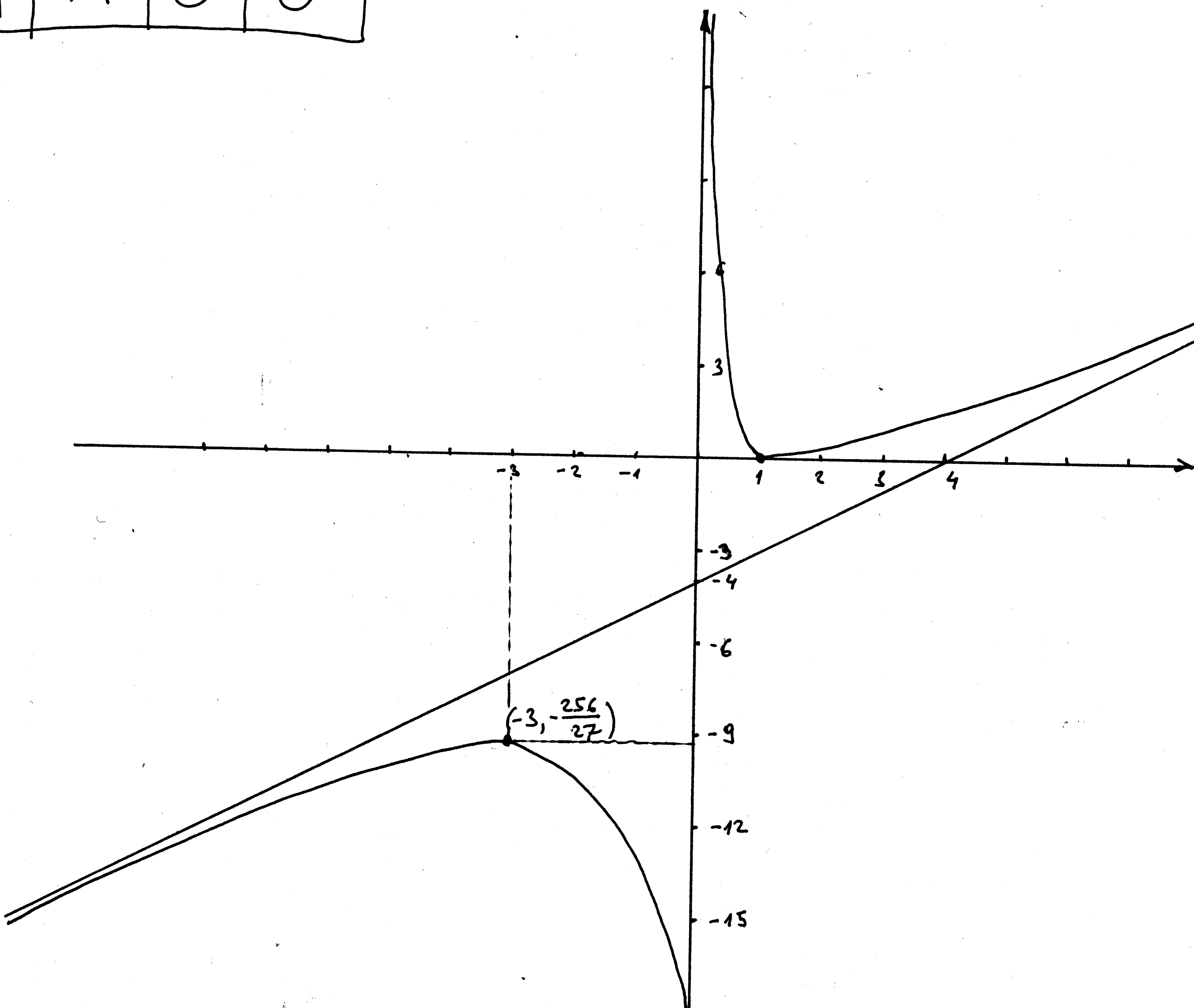
$$y''=\frac{(x-1)^2\left[3x(x+3)+x(x-1)-4\underbrace{(x-1)(x+3)}_{x^2+2x-3}\right]}{x^5}=\frac{(x-1)^2(\cancel{3x^2}+\cancel{9x}+\cancel{x^2}-\cancel{x}-\cancel{4x^2}-\cancel{8x}+12)}{x^5}$$

$$y''=\frac{12(x-1)^2}{x^5}\qquad y''=0 \text{ akko } x=1$$

← prekidi y + nule y''

x	$(-\infty,0)$	$(0,1)$	$(1,+\infty)$
y''	−	+	+
y	∩	∪	∪

f-ja nema prevojnih tački



Ispitati f-ju i nacrtati joj grafik $y=(x+1)\,e^{\frac{1}{2}x^2-x}$.

Rj. definiciono područje

$D: x\in\mathbb{R}$

parnost, neparnost, periodičnost

$f(-x)=(-x+1)\,e^{\frac{1}{2}x^2+x}$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

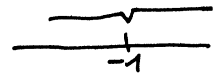
$y=0$ akko $(x+1)=0$

$x=-1$

$(-1,0)$ je nula f-je

$f(0)=(0+1)\,e^0=1$

$(0,1)$ je presjek sa y-osom



$e^{\frac{1}{2}x^2-x}>0 \quad \forall x\in\mathbb{R}$

x	$(-\infty,-1)$	$(-1,\infty)$
$x+1$	−	+
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja je definisana za $\forall x\in\mathbb{R}$ $\Rightarrow$ f-ja nema V. A.

$$\lim_{x\to+\infty} f(x)=\lim_{x\to\infty}(x+1)\,e^{\frac{1}{2}x^2-x}=\infty\cdot\infty=\infty$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}(x+1)\,e^{\frac{1}{2}x^2-x}=(-\infty)\cdot\infty=-\infty$$

$\Rightarrow$ f-ja nema H. A.

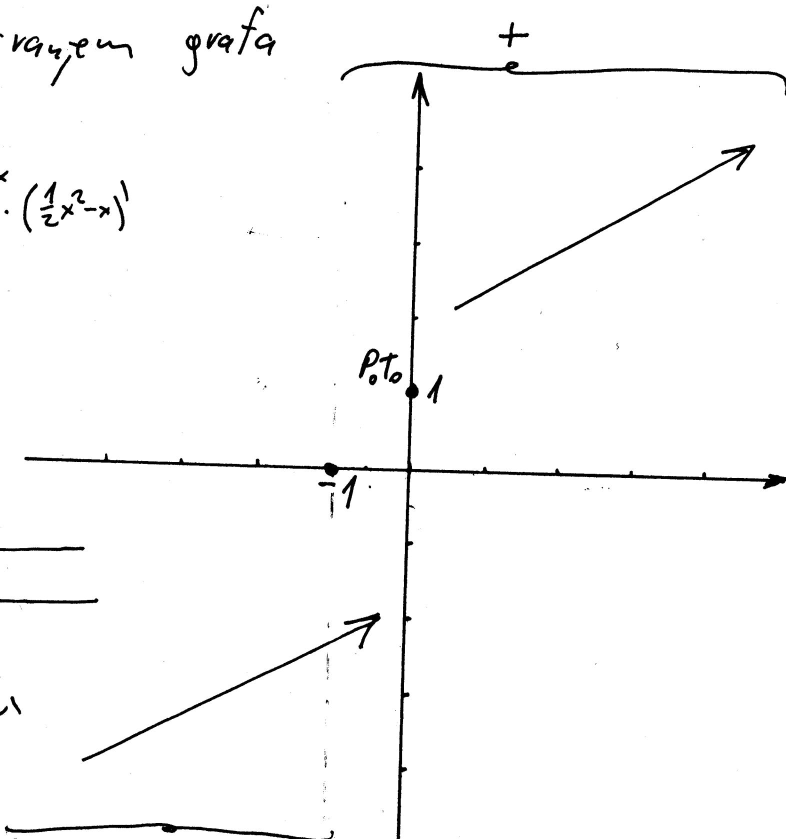
$$e^{\frac{1}{2}x^2-x}=e^{\frac{x^2-x}{2}\,\frac{/:x^2}{/:x^2}}=e^{\frac{1-\frac{1}{x}}{\frac{2}{x^2}}}\quad \infty,\ x\to\infty$$

kosa asimptota je oblika $y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{(x+1)\,e^{\frac{1}{2}x^2-x}}{x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)e^{\frac{1}{2}x^2-x}=1\cdot\infty=\infty$$

f-ja nema K. A.

nakon ovog koraka počinjemo sa skiciranjem grafa



rast i opadanje

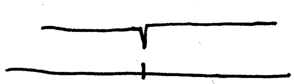
$$y'=\left((x+1)\,e^{\frac{1}{2}x^2-x}\right)'=1\cdot e^{\frac{1}{2}x^2-x}+(x+1)\,e^{\frac{1}{2}x^2-x}\cdot\left(\tfrac{1}{2}x^2-x\right)'$$

$$y'=e^{\frac{1}{2}x^2-x}+(x+1)\,e^{\frac{1}{2}x^2-x}\cdot\left(\tfrac{1}{2}\cdot 2x-1\right)$$

$$y'=e^{\frac{1}{2}x^2-x}\left(1+(x+1)(x-1)\right)$$

$$y'=e^{\frac{1}{2}x^2-x}\left(1+x^2-1\right)=x^2\,e^{\frac{1}{2}x^2-x}$$

$y'=0$ akko $x=0$



prekidi y + nule y'

x	$(-\infty,0)$	$(0,+\infty)$
y'	+	+
y	↗	↗

rast i opadanje

- ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da f-ja nema ekstrema.

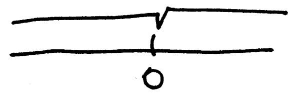
prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(x^2 e^{\frac{1}{2}x^2 - x}\right)' = 2x e^{\frac{1}{2}x^2 - x} + x^2 e^{\frac{1}{2}x^2 - x}(x-1)$$

$$y'' = x e^{\frac{1}{2}x^2 - x}\left(2 + x(x-1)\right) = x e^{\frac{1}{2}x^2 - x}(x^2 - x + 2)$$

prekidi y

+ nule y''



$x^2 - x + 2 > 0 \quad \forall x \in \mathbb{R}$ $\qquad y'' = 0$ akko $x = 0$

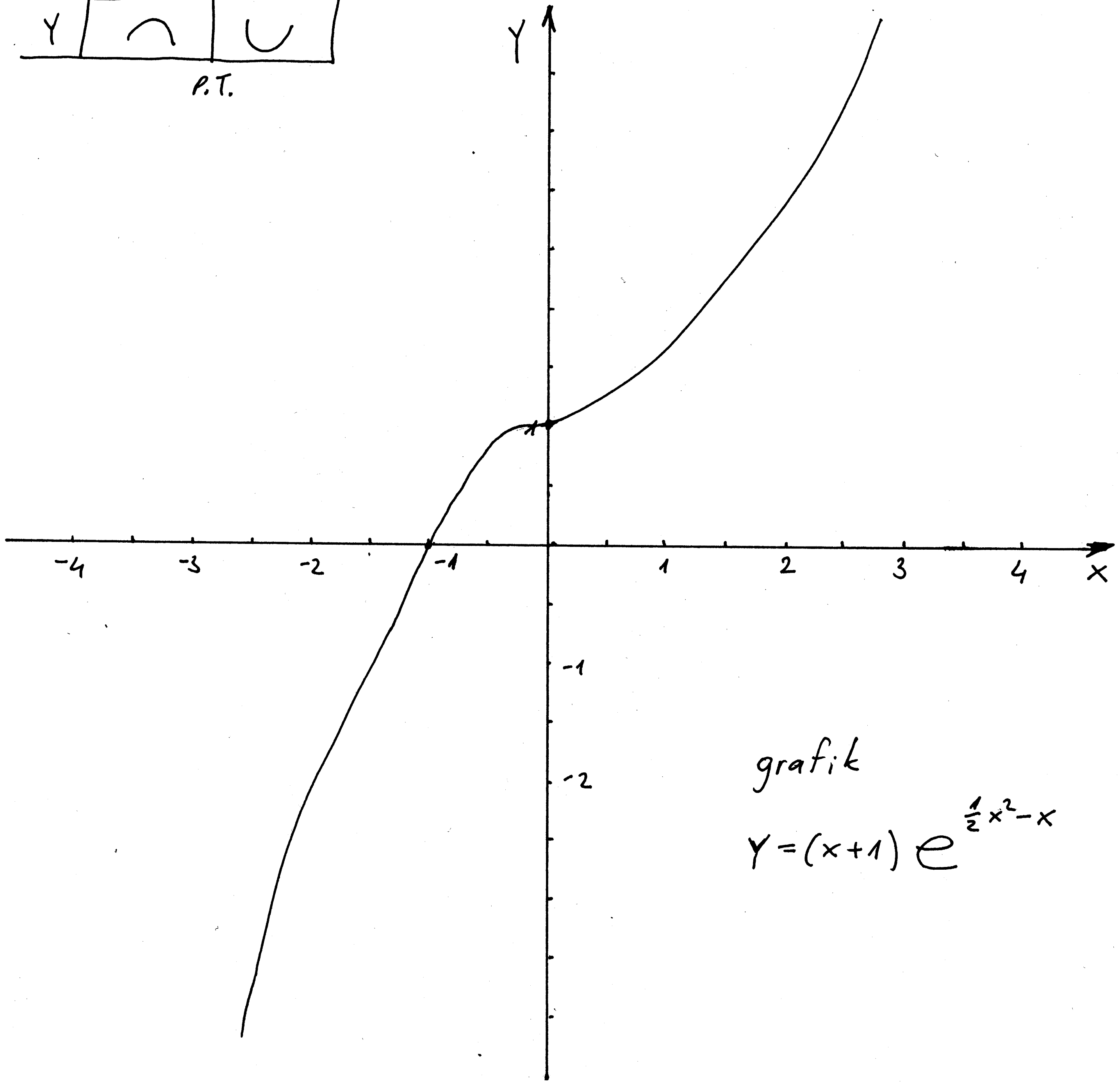
$e^{\frac{1}{2}x^2 - x} > 0 \quad \forall x \in \mathbb{R}$

$$f(0) = (0+1)e^0 = 1$$

(0, 1) je prevojna tačka

x	$(-\infty, 0)$	$(0, +\infty)$
y''	−	+
y	∩	∪

P.T.



grafik

$$y = (x+1)e^{\frac{1}{2}x^2 - x}$$

\# Ispitati f-ju i nacrtati joj grafik $y=\frac{x}{x-1}\ln\frac{x}{x-1}$ (bez analize drugog izvoda).

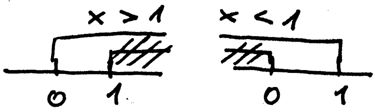
Rj: definiciono područje

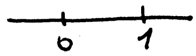
$x-1\neq 0$ $\quad \frac{x}{x-1}>0 \quad$ $D: x\in(-\infty,0)\cup(1,+\infty)$

$x\neq 1$

$x>0 \quad x<0$

$x>1 \quad x<1$





parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $x=0$

za $x=0$ f-ja nije definisana

f-ja nema nulu i ne sječe y-osu

$\ln\frac{x}{x-1}>0 \qquad \frac{x}{x-1}-1>0$

$\ln\frac{x}{x-1}>\ln 1 \qquad \frac{x-x+1}{x-1}>0$

$\frac{x}{x-1}>1 \qquad \frac{1}{x-1}>0$

$x-1>0$

$x>1$

ponašanje na krajevima intervala definisanosti i asimptote

x	$(-\infty,0)$	$(1,+\infty)$
x	$-$	$+$
$x-1$	$-$	$+$
$\ln\frac{x}{x-1}$	$-$	$+$
y	$-$	$+$

znak f-je

$$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\frac{x}{x-1}\ln\frac{x}{x-1}\left(=(0-)\cdot(-\infty)\right)$$

$$=\lim_{x\to 0-}\frac{\ln\frac{x}{x-1}}{\frac{x-1}{x}}\left(=\frac{-\infty}{\infty}\right)\overset{LoPo}{=}\lim_{x\to 0-}\frac{\frac{1}{\frac{x}{x-1}}\left(\frac{x}{x-1}\right)'}{\left(\frac{x-x+1}{x^2}\right)}=\lim_{x\to 0-}\frac{\frac{\cancel{x-1}}{x}\cdot\frac{\cancel{x-1-x}}{(x-1)^{\cancel{2}}}}{\frac{1}{x^2}}=\lim_{x\to 0-}\frac{(-1)x^{\cancel{2}}}{\cancel{x}(x-1)}=0$$

nema V. A. za $x=0$

$$\lim_{x\to 1+0}f(x)=\lim_{x\to 1+0}\frac{x}{x-1}\ln\frac{x}{x-1}=(+\infty)\cdot(+\infty)=+\infty \Rightarrow x=1 \text{ je V. A.}$$

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\frac{x}{x-1}\ln\frac{x}{x-1}=\lim_{x\to+\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0 \Rightarrow y=0 \text{ je H. A.}$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{1-\frac{1}{x}}\ln\frac{1}{1-\frac{1}{x}}=1\cdot\ln 1=1\cdot 0=0 \Rightarrow y=0 \text{ je H. A.}$$

f-ja nema kose asimptote

nakon ovog koraka počinjemo sa skiciranjem grafa

rast i opadanje

$$y'=\left(\frac{x}{x-1}\ln\frac{x}{x-1}\right)'=\frac{x-1-x}{(x-1)^2}\ln\frac{x}{x-1}+\frac{x}{x-1}\cdot\frac{1}{\frac{x}{x-1}}\left(\frac{x}{(x-1)}\right)'$$

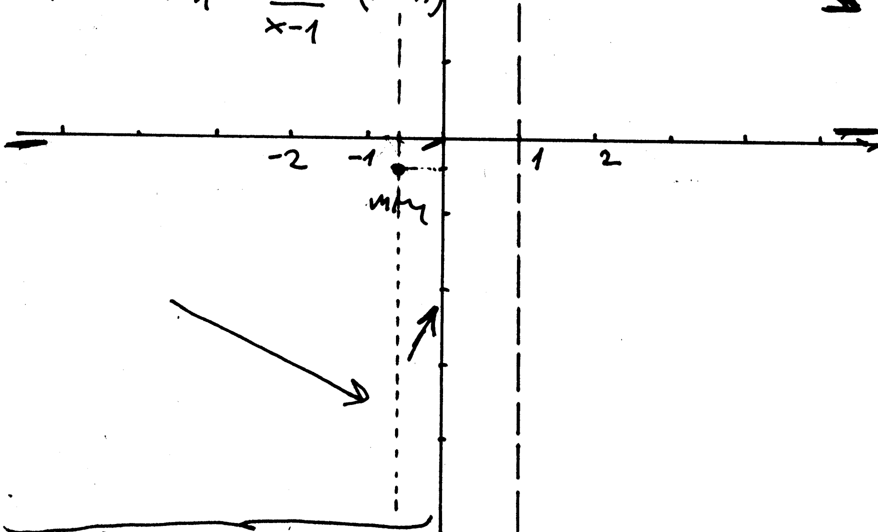
$$y'=\frac{-1}{(x-1)^2}\ln\frac{x}{x-1}+\frac{-1}{(x-1)^2}$$

$$y'=\frac{-1}{(x-1)^2}\left(\ln\frac{x}{x-1}+1\right)$$

$y'=0$ akko $\ln\frac{x}{x-1}+1=0$

$\ln\frac{x}{x-1}=-1$

$\frac{x}{x-1}=e^{-1}$



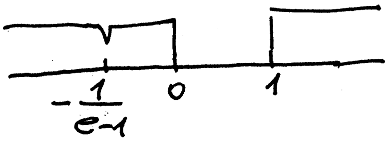
$$\frac{x}{x-1} - \frac{1}{e} = 0 \qquad ex - x + 1 = 0$$

$$\frac{ex-(x-1)}{e(x-1)} = 0 \qquad x(e-1) = -1$$

$$x = -\frac{1}{e-1} \approx -0,5820$$

x	$(-\infty, -\frac{1}{e-1})$	$(-\frac{1}{e-1}, 0)$	$(1, +\infty)$
y'	$-$	$+$	$-$
y	↘	↗	↘

rast i opadanje



← prekidi y + nule y'

$$\ln \frac{-\frac{1}{e^{-1}-1}}{-\frac{1}{e^{-1}-1}-1} = e \qquad \ln\frac{5}{4} \approx 0,22$$

$$\ln \frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1} = e^{-1}$$

$$e > e^{-1}$$

$$e-1 > e^{-1}-1$$

$$\frac{1}{e-1} < \frac{1}{e^{-1}-1} \quad /\cdot(-1)$$

$$-\frac{1}{e-1} > -\frac{1}{e^{-1}-1}$$

$$f\left(-\frac{1}{e-1}\right) = \frac{-\frac{1}{e-1}}{-\frac{1}{e-1}-1} \ln \frac{\frac{-1}{e-1}}{\frac{-1-(e-1)}{e-1}} = \frac{\frac{-1}{e-1}}{\frac{-e}{e-1}} \ln\frac{1}{e} = \frac{1}{e}\cdot(-1) = -\frac{1}{e} \approx -0,3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja tačka minimuma je $\left(-\frac{1}{e-1}, -\frac{1}{e}\right)$.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left[-(x-1)^{-2}\left(\ln\frac{x}{x-1}+1\right)\right]' = 2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right) + \left(-(x-1)^{-2}\right)\frac{x-1}{x}\cdot\frac{-1}{(x-1)^2}$$

$$y'' = 2(x-1)^{-3}\left(\ln\frac{x}{x-1}+1\right) - (x-1)^{-1}\cdot\frac{-1}{x(x-1)^2} = \frac{1}{(x-1)^3}\left[2\left(\ln\frac{x}{x-1}+1\right) + \frac{1}{x}\right]$$

bez analize drugog izvoda
(crtamo graf)

$$2\left(\ln\frac{x}{x-1}+1\right) + \frac{1}{x} = g(x)$$

$g(-2) \approx 0,6891$

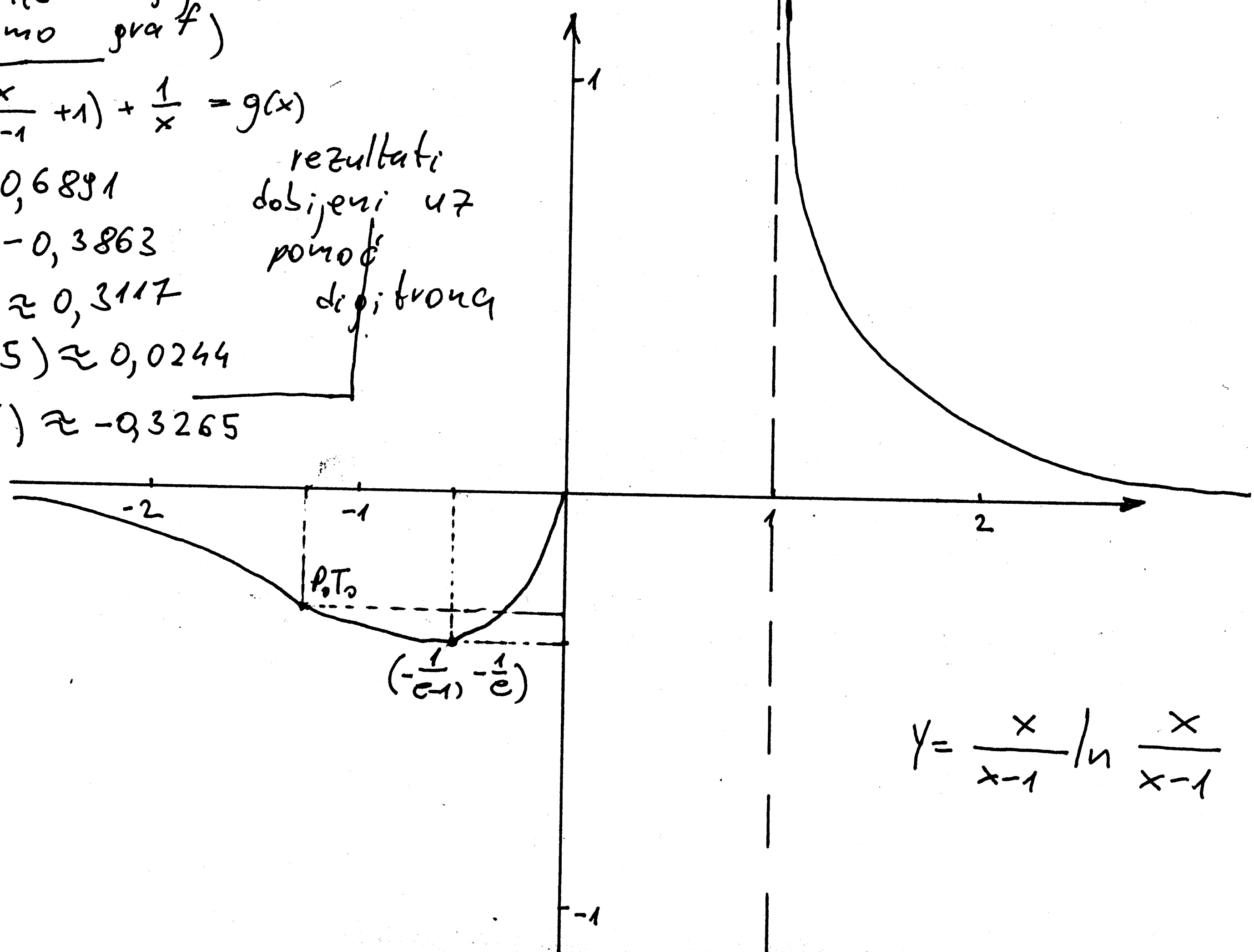
$g(-1) \approx -0,3863$

$g(-1,5) \approx 0,3117$

$g(-1,25) \approx 0,0244$

rezultati dobijeni uz pomoć digitrona

$f(-1,25) \approx -0,3265$



$$y = \frac{x}{x-1}\ln\frac{x}{x-1}$$

(#) Izračunati integral $I = \int \frac{4x^3+1}{x^4-x}\,dx$

Rj:

$$\frac{4x^3+1}{x^4-x} = \frac{4x^3+1}{x(x^3-1)} = A\,\frac{3x^2}{x^3-1} + B\,\frac{1}{x} \quad / x(x^3-1)$$

$$4x^3+1 = 3Ax^3 + B(x^3-1)$$

$$3A+B=4$$
$$-B=1$$

$$B=-1 \qquad 3A-1=4$$
$$3A=5$$
$$A=\frac{5}{3}$$

$$I = \int \frac{4x^3+1}{x^4-x}\,dx = \frac{5}{3}\int \frac{3x^2}{x^3-1}\,dx - \int \frac{dx}{x} =$$

$$= \frac{5}{3}\ln|x^3-1| - \ln|x| + C$$

Izračunati integral $I=\int (x^2+x)\ln\frac{2x+1}{x-1}\,dx$.

Rj.

$$I=\int (x^2+x)\ln\frac{2x+1}{x-1}\,dx = \left|\begin{array}{l} u=\ln\frac{2x+1}{x-1} \\ du=\frac{1}{\frac{2x+1}{x-1}}\cdot\left(\frac{2x+1}{x-1}\right)'dx=\frac{\cancel{x-1}}{2x+1}\cdot\frac{\overset{\cancel{2x}-2-\cancel{2x}-1}{2(x-1)-(2x+1)}}{(x-1)^{\cancel{2}\,1}}dx= \\ =\frac{-3}{(2x+1)(x-1)}dx \qquad dv=(x^2+x)dx \\ \qquad\qquad\qquad\qquad\quad v=\frac{x^3}{3}+\frac{x^2}{2}\end{array}\right| =\left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\ln\frac{2x+1}{x-1}+\int\frac{2x^3+3x^2}{\underset{2}{\cancel{6}}}\cdot\frac{\overset{1}{\cancel{3}}}{(2x+1)(x-1)}dx$$

$$=\left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\ln\frac{2x+1}{x-1}+\frac{1}{2}\int\frac{2x^3+3x^2}{(2x+1)(x-1)}=\frac{2x^3+3x^2}{6}\ln\frac{2x+1}{x-1}+\frac{1}{2}\int(x+2)\,dx+$$

$$+\frac{1}{2}\int\frac{3x+2}{2x^2-x-1}dx$$

$$\frac{2x^3+3x^2}{(2x+1)(x-1)}=\frac{2x^3+3x^2}{2x^2-x-1}=x+2+\frac{3x+2}{(2x+1)(x-1)}$$

$$(2x^3+3x^2):(2x^2-x-1)=x+2+\frac{3x+2}{2x^2-x-1}$$

$$\begin{array}{l} -\ 2x^3-x^2-x \\ \hline \qquad 4x^2+x \\ -\quad 4x^2-2x-2 \\ \hline \qquad\quad 3x+2 \end{array}$$

$$\frac{3x+2}{(2x+1)(x-1)}=\frac{A}{2x+1}+\frac{B}{x-1}\quad /(2x+1)(x-1)$$

$$3x+2=A(x-1)+B(2x+1)$$

$$3x+2=(A+2B)x+(-A+B)$$

$$\begin{array}{l} A+2B=3 \\ +\ -A+B=2 \\ \hline \quad 3B=5 \\ \quad B=\frac{5}{3}\end{array} \qquad \begin{array}{l} A+2B=3 \\ A+\frac{10}{3}=3 \\ A=-\frac{1}{3}\end{array}$$

$$I=\frac{2x^3+3x^2}{6}\ln\frac{2x+1}{x-1}+\frac{1}{2}\cdot\frac{x^2}{2}+\frac{1}{2}\cdot 2x+\frac{1}{2}\cdot\left(-\frac{1}{3}\right)\int\frac{dx}{2x+1}+\frac{1}{2}\cdot\frac{5}{3}\int\frac{dx}{x-1}=$$

$$=\frac{2x^3+3x^2}{6}\ln\frac{2x+1}{x-1}+\frac{x^2}{4}+x-\frac{1}{12}\ln|2x+1|+\frac{5}{6}\ln|x-1|+C$$

Izračunati integral $I=\int \frac{8\cos x-\sin x}{2\cos x+\sin x}\,dx$.

Rj. $(2\cos x+\sin x)' = -2\sin x+\cos x$

$$\frac{8\cos x-\sin x}{2\cos x+\sin x} = A + B\,\frac{-2\sin x+\cos x}{2\cos x+\sin x} \quad /\cdot(2\cos x+\sin x)$$

$$8\cos x-\sin x = A(2\cos x+\sin x)+B(-2\sin x+\cos x)$$

$$8\cos x-\sin x=(2A+B)\cos x+(A-2B)\sin x$$

$$\begin{aligned} 2A+B&=8\\ A-2B&=-1 \quad /\cdot 2\end{aligned}$$

$$\begin{aligned} 2A+B&=8\\ -\;2A-4B&=-2\end{aligned}$$

$$5B=10$$

$$B=2$$

$$2A+B=8$$

$$2A=8-2$$

$$2A=6$$

$$A=3$$

$$I=\int \frac{8\cos x-\sin x}{2\cos x+\sin x}\,dx = \int\left(3+2\,\frac{-2\sin x+\cos x}{2\cos x+\sin x}\right)dx =$$

$$=3\int dx + 2\int \frac{-2\sin x+\cos x}{2\cos x+\sin x}\,dx = \left|\begin{array}{l} 2\cos x+\sin x=t \\ (-2\sin x+\cos x)\,dx=dt\end{array}\right|$$

$$=3x+2\int\frac{dt}{t} = 3x+2\ln|t|+C=$$

$$=3x+2\ln|2\cos x+\sin x|+C$$

(#) Izračunati površinu figure koju ograničavaju linije $x = y^2 - 2y - 3$ i $y = 3 - 3x$

$$y = 3 - 3x$$
$$3x = 3 - y$$
$$x = 1 - \frac{1}{3}y$$

Rj. Nađimo presečnu tačku ovih linija

$$x = y^2 - 2y - 3$$
$$y = 3 - 3x$$

$$x = (3-3x)^2 - 2(3-3x) - 3$$
$$x = 9 - 18x + 9x^2 - 6 + 6x - 3$$
$$9x^2 - 13x = 0$$
$$x(9x - 13) = 0$$
$$x = 0 \quad \text{ili} \quad 9x = 13$$
$$x = \frac{13}{9}$$

$$x = 0 \Rightarrow y = 3$$

$$x = \frac{13}{9} \Rightarrow y = 3 - \cancel{3} \cdot \frac{13}{\cancel{9}_3} = \frac{9}{3} - \frac{13}{3} = -\frac{4}{3}$$

$A(0,3)$ i $B(\frac{13}{9}, -\frac{4}{3})$ su presečne tačke linija

$x = y^2 - 2y - 3$ je kriva oblika parabole C čije je tjeme $T(-\frac{D}{4a}, -\frac{b}{2a})$

$$-\frac{b}{2a} = -\frac{-2}{2} = 1, \quad D = 4 + 12 = 16 \qquad -\frac{D}{4a} = -\frac{16}{4} = -4$$

$$T(1, -4)$$

$$y_{1,2} = \frac{2 \pm 4}{2}$$

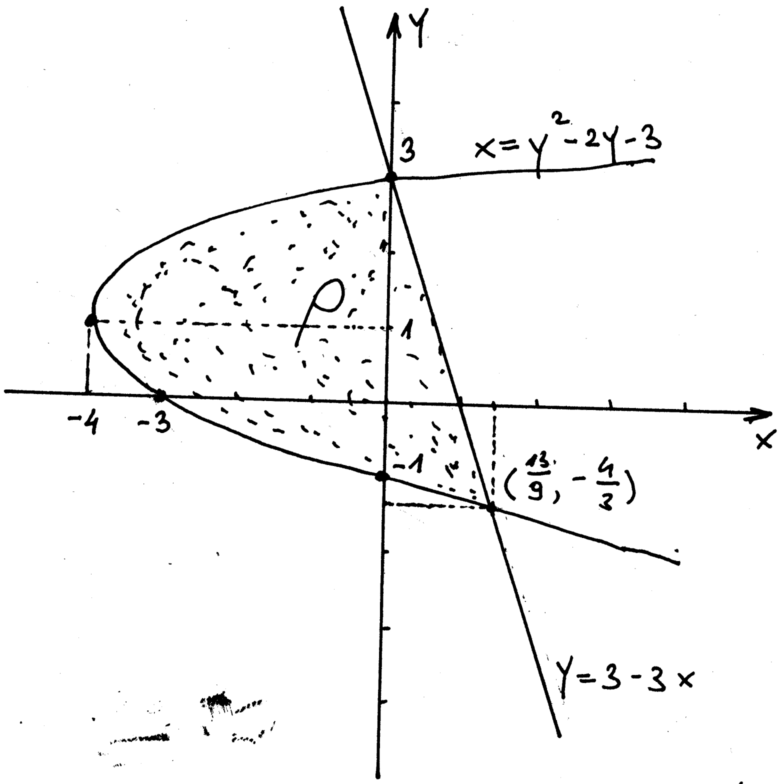
$$y_1 = \frac{-2}{2} = -1 \quad y_2 = \frac{6}{2} = 3$$

$M_1(0,-1)$ i $M_2(0,3)$ su presjek parabole sa y-osom

$x = y^2 - 2y - 3$

$y = 0 \Rightarrow x = -3$

$(-3, 0)$ je presjek krive sa x-osom



$$P = \int_{-\frac{4}{3}}^{3} \left[\left(1 - \frac{1}{3}y\right) - \left(y^2 - 2y - 3\right)\right] dy =$$

$$= \int_{-\frac{4}{3}}^{3} \left(-y^2 + \frac{5}{3}y + 4\right) dy =$$

$$= -\frac{1}{3}y^3 \Big|_{-\frac{4}{3}}^{3} + \frac{5}{3} \cdot \frac{1}{2} y^2 \Big|_{-\frac{4}{3}}^{3} + 4y \Big|_{-\frac{4}{3}}^{3} =$$

$$= -\frac{1}{3}\left(27 + \frac{64}{27}\right) + \frac{5}{6}\left(9 - \frac{16}{9}\right) + 4\left(3 + \frac{4}{3}\right)$$

$$= -\frac{1}{3} \cdot \frac{793}{27} + \frac{5}{6} \cdot \frac{65}{9} + 4 \cdot \frac{13}{3} =$$

$$= -\frac{793}{81} + \frac{325}{54} + \frac{52}{3} = \frac{-793 \cdot 2 + 325 \cdot 3 + 52 \cdot 54}{162} = \frac{-1586 + 975 + 2808}{162}$$

$$P = \frac{2197}{162} = 13\frac{91}{162}$$ tražena površina

(#) Naći stacionarne tačke f-je $z = xy\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = y\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2x = y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = x\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2y = x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2}$$

$$y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2} = 0$$

$$x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2} = 0$$

$$y\left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right) = 0$$

$$x\left(\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right) = 0$$

$y=0$ ili $\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$

$x=0$ ili $\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$

$y=0$ i $x=0$ $\quad M_1(0,0)$

za M_1 f-ja nije definisana

ili

$y=0$ i $\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$

$$\ln x^2 = 0$$

$$x^2 = 1$$

$$x_1=-1,\quad x_2=1$$

$$M_2(-1,0),\quad M_3(1,0)$$

ili

$\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$ i $x=0$

$\ln y^2 = 0$ $\quad M_4(0,-1)$

$y_1=-1 \quad y_2=1$ $\quad M_5(0,1)$

ili

$$\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0 \quad (1)$$

$$\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0 \quad (2)$$

(1)-(2): $\frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2} = 0$

$$2x^2-2y^2=0$$

$$2(x^2-y^2)=0$$

$$2(x-y)(x+y)=0$$

$$x=y \text{ ili } x=-y$$

za $x=y$: $\ln(2x^2)+1=0$

$$\ln(2x^2) = -1$$

$$e^{-1} = 2x^2$$

$$x^2 = \frac{1}{2e}$$

$$x_{1,2} = \pm\frac{1}{\sqrt{2e}}$$

$M_6\left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)$, $M_7\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

za $y=-x$: $\ln(2x^2)+1=0$

$$x_{1,2} = \pm\frac{1}{\sqrt{2e}}$$

$M_8\left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

$M_9\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8$ i M_9.

(#) Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x}=4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x)=(4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y}=6y\,e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y)=(6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \quad \forall (x,y\in\mathbb{R})$

$x=0$ i $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$

$y^2=1$

$y_{1,2}=\pm 1$

$M_2(0,-1)$

$M_3(0,1)$

ili

$2-2x^2-3y^2=0$ i $y=0$

$2x^2=2$

$x^2=1$

$x_{1,2}=\pm 1$

$M_4(-1,0)$

$M_5(1,0)$

ili

$$\begin{array}{r} 2-2x^2-3y^2=0\\ -\quad 3-2x^2-3y^2=0\\ \hline -1=0 \end{array}$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2}=(4-12x^2-6y^2)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x)=(8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=(-12xy)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y)=(-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(6-4x^2-18y^2)e^{-x^2-y^2}+(6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y)=(-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

(#) Riješiti diferencijalnu jednačinu $2y+y'(2x+y')=0$.

Rj.

$y=-xy'-\frac{1}{2}(y')^2$ ovo je Lagranžova diferenc. jedn. $(y=xf(y')+g(y'))$ uvodimo smjene $y'=p$, $x=uv$

$y'=p$

$y=-xp-\frac{1}{2}p^2 \quad \Big/ \frac{d}{dx}$

$y'=-p-xp'-\frac{1}{2}\cdot 2pp'$

$p=-p-xp'-pp'$

$2p=(-x-p)p' \quad /:p'$

$\frac{2p}{p'}=-x-p\,, \quad p'=\frac{dp}{dx}$

$\frac{1}{p'}=\frac{dx}{dp}=x'$

$x'=-\frac{1}{2p}x-\frac{1}{2}$

$x'+\frac{1}{2p}x=-\frac{1}{2}$ ovo je linearna dif. jedn. $(y'+f(x)y=g(x))$

uvodimo smjenu

$x=uv,\quad x'=u'v+uv'$

$u'v+uv'+\frac{1}{2p}uv=-\frac{1}{2}$

$u'v+u\underbrace{\left(v'+\frac{1}{2p}v\right)}_{=0}=-\frac{1}{2}$

$v'+\frac{1}{2p}v=0$

$\frac{dv}{dp}=-\frac{1}{2p}v$

$\frac{dv}{v}=-\frac{1}{2}\cdot\frac{dp}{p} \quad \Big/\int$

$\ln|v|=-\frac{1}{2}\ln|p|$

$v=p^{-\frac{1}{2}}=\frac{1}{\sqrt{p}}$

$u'v=-\frac{1}{2}$

$u'\cdot\frac{1}{\sqrt{p}}=-\frac{1}{2}$

$\frac{du}{dp}=-\frac{1}{2}\sqrt{p}$

$du=-\frac{1}{2}p^{\frac{1}{2}}dp \quad \Big/\int$

$u=-\frac{1}{2}\cdot\frac{p^{\frac{3}{2}}}{\frac{3}{2}}+c$

$u=-\frac{1}{2}\cdot\frac{2}{3}\sqrt{p^3}+c$

$u=-\frac{1}{3}p\sqrt{p}+c$

$x=uv=\left(-\frac{1}{3}p\sqrt{p}+c\right)\cdot\frac{1}{\sqrt{p}}$

$x=-\frac{p}{3}+\frac{c}{\sqrt{p}}$

$y=-xp-\frac{1}{2}p^2$

$y=\left(\frac{p}{3}-\frac{c}{\sqrt{p}}\right)p-\frac{1}{2}p^2$

$y=-\frac{1}{6}p^2-\frac{p}{\sqrt{p}}c$

$$\left.\begin{aligned} x&=-\frac{p}{3}+\frac{c}{\sqrt{p}}\\ y&=-\frac{p^2}{6}-\frac{p}{\sqrt{p}}c\end{aligned}\right\}$$ opće rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu

$$(x-y-2)\,dx+(2x-y-5)\,dy=0$$

Rj.

$$(2x-y-5)\,dy=-(x-y-2)\,dx \quad \Big/\cdot\frac{1}{dx}\cdot\frac{1}{2x-y-5}$$

$$\frac{dy}{dx}=\frac{-x+y+2}{2x-y-5}$$

$$y'=\frac{-x+y+2}{2x-y-5}$$

diferencijalna jednačina koja se svodi na homogenu $y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

$$a_1b_2-a_2b_1=1-2=-1\neq 0$$

uvodimo smjenu $x=u+\alpha$, $y=v+\beta$

$$\begin{aligned}-\alpha+\beta+2&=0\\ +\;2\alpha-\beta-5&=0\\ \hline \alpha-3&=0\\ \alpha&=3\end{aligned}$$

$$\begin{aligned}-\alpha+\beta+2&=0\\ -3+\beta+2&=0\\ \beta&=1\end{aligned}$$

$x=u+3$, $y=v+1$, $y'=v'$ $\Rightarrow$ $u=x-3$, $v=y-1$

$$v'=\frac{-u-3+v+1+2}{2u+6-v-1-5}$$

$$v'=\frac{-u+v}{2u-v}\quad \begin{matrix}/:u\\ /:u\end{matrix}$$

$$v'=\frac{-1+\frac{v}{u}}{2-\frac{v}{u}}$$

ovo je homogena diferenc. jednačina

uvodimo smjenu $z=\frac{v}{u}$

$v=z\cdot u \quad /'_u$

$v'=z'\cdot u+z$

$$z'\cdot u+z=\frac{-1+z}{2-z}$$

$$z'u=\frac{-1+z}{2-z}-z$$

$$z'u=\frac{-1+z-2z+z^2}{2-z}$$

$$z'u=\frac{z^2-z-1}{2-z}\;,\quad z'=\frac{dz}{du}$$

$$\frac{2-z}{z^2-z-1}\,dz=\frac{du}{u}\quad /\int$$

$$2-z=(-1)(z-2)=\left(-\frac{1}{2}\right)(2z-4)=\left(-\frac{1}{2}\right)(2z-1-3)$$

$$\int\frac{2-z}{z^2-z-1}\,dz=-\frac{1}{2}\int\frac{2z-1}{z^2-z-1}\,dz+\frac{3}{2}\int\frac{dz}{z^2-z-1}\overset{(*)}{=}-\frac{1}{2}\ln(z^2-z-1)+\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|+C_1$$

$$\frac{3}{2}\int\frac{dz}{z^2-z-1}=\left|\begin{aligned}&z^2-z-1=z^2-2\cdot\frac{1}{2}\cdot z+\frac{1}{4}-\frac{1}{4}-1=\\&=\left(z-\frac{1}{2}\right)^2-\frac{5}{4}\end{aligned}\right|$$

$$=\frac{3}{2}\int\frac{dz}{\left(z-\frac{1}{2}\right)^2-\frac{5}{4}}=\left|\begin{aligned}&z-\frac{1}{2}=\frac{\sqrt5}{2}t\\&dz=\frac{\sqrt5}{2}\,dt\\&t=\frac{2z-1}{\sqrt5}\end{aligned}\right|=\frac{3}{2}\cdot\frac{\sqrt5}{2}\cdot\frac{4}{5}\int\frac{dt}{t^2-1}$$

$$=\frac{3\sqrt5}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|\quad \dots(*)$$

$$\ln u=-\frac{1}{2}\ln(z^2-z-1)+\frac{3\sqrt5}{10}\ln\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|+\ln C_2\quad /\cdot 10$$

$$u^{10}=\frac{C}{(z^2-z-1)^5}\left|\frac{2z-1-\sqrt5}{2z-1+\sqrt5}\right|^{3\sqrt5}$$

opšte rješenje difer. jedn.

$z=\frac{v}{u}$, $v=y-1$, $u=x-3$