Ekonomski fakultet u Zenici

03.07.2009, Matematika, Pismeni ispit

Nijedan zadatak nije detaljno raspisan.

Za uočene greške pisati na infoarrt@gmail.com

(1.) Izračunati broj $z = \dfrac{\left(\frac{1}{2\sqrt{3}} - \frac{i}{2}\right)^9}{\left(-1 + \frac{i}{\sqrt{3}}\right)^6}$.

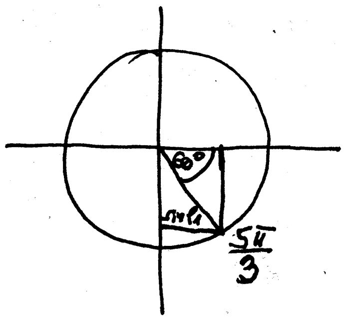
Rj.

$z_1 = \frac{\sqrt{3}}{6} - \frac{1}{2}i$

$|z_1| = \frac{\sqrt{3}}{3}$

$\cos\varphi_1 = \frac{1}{2}$

$\sin\varphi_1 = -\frac{\sqrt{3}}{2}$



$\varphi_1 = \frac{5\pi}{6}$ $\quad z_1 = \frac{\sqrt{3}}{3}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

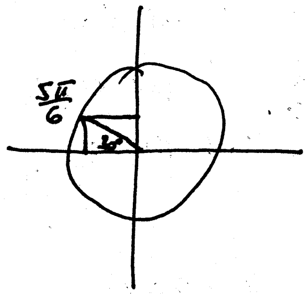
$z_1^9 = \left(\frac{\sqrt{3}}{3}\right)^9\left(\cos 9\cdot\frac{5\pi}{3} + i\sin 9\cdot\frac{5\pi}{3}\right)$

$= \frac{\sqrt{3}}{3^5}\left(\cos\pi + i\sin\pi\right)$

$z_2 = -1 + \frac{1}{\sqrt{3}}i$ $\quad \cos\varphi_2 = -\frac{\sqrt{3}}{2}$

$|z| = \frac{2\sqrt{3}}{3}$ $\quad \sin\varphi_2 = \frac{1}{2}$

$tg\,\varphi = -\frac{\sqrt{3}}{3}$



$z_2 = \frac{2\sqrt{3}}{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$z_2^6 = \left(\frac{2\sqrt{3}}{3}\right)^6\left(\cos 6\cdot\frac{5\pi}{6} + i\sin 6\cdot\frac{5\pi}{6}\right)$

$= \frac{2^6}{3^3}\left(\cos\pi + i\sin\pi\right)$

$z_1^9 = -\frac{\sqrt{3}}{3^5}$, $\quad z_2^6 = -\frac{2^6}{3^3}$, $\quad z = \frac{z_1^9}{z_2^6} = \frac{\sqrt{3}}{576}$ traženi broj

(2.) Diskutovati rang matrice $M = \begin{bmatrix} 10 & \lambda & 1 & -6 \\ -1 & 3 & 2 & \lambda \\ \lambda & 2 & 1 & -1 \end{bmatrix}$ za razne vrijednosti parametra λ.

Rj.

$$M = \begin{bmatrix} 10 & \lambda & 1 & -6 \\ -1 & 3 & 2 & \lambda \\ \lambda & 2 & 1 & -1 \end{bmatrix} \underset{I_k \leftrightarrow III_k}{\sim} \begin{bmatrix} 1 & \lambda & 10 & -6 \\ 2 & 3 & -1 & \lambda \\ 1 & 2 & \lambda & -1 \end{bmatrix} \underset{III_k \leftrightarrow IV_k}{\sim} \begin{bmatrix} 1 & \lambda & -6 & 10 \\ 2 & 3 & \lambda & -1 \\ 1 & 2 & -1 & \lambda \end{bmatrix}$$

$$\underset{III_v - I_v}{\overset{II_v - 2\cdot I_v}{\sim}} \begin{bmatrix} 1 & \lambda & -6 & 10 \\ 0 & 3-2\lambda & \lambda+12 & -21 \\ 0 & 2-\lambda & 5 & \lambda-10 \end{bmatrix} \underset{}{\overset{II_v - III_v}{\sim}} \begin{bmatrix} 1 & \lambda & -6 & 10 \\ 0 & 1-\lambda & \lambda+7 & -\lambda-11 \\ 0 & 2-\lambda & 5 & \lambda-10 \end{bmatrix} \overset{II_v - III_v}{\sim}$$

$$\begin{bmatrix} 1 & \lambda & -6 & 10 \\ 0 & -1 & \lambda+2 & -2\lambda-1 \\ 0 & 2-\lambda & 5 & \lambda-10 \end{bmatrix} \overset{III_v + II_2\cdot(2-\lambda)}{\sim} \begin{bmatrix} 1 & \lambda & -6 & 10 \\ 0 & -1 & \lambda+2 & -2\lambda-1 \\ 0 & 0 & (3-\lambda)(3+\lambda) & 2(\lambda-3)(\lambda+2) \end{bmatrix}$$

$\lambda=3: \quad M=\begin{bmatrix} 1 & 3 & -6 & 10 \\ 0 & -1 & 5 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{rang}\, M = 2$

$\lambda=-3: \quad M=\begin{bmatrix} 1 & -3 & -6 & 10 \\ 0 & -1 & -1 & 5 \\ 0 & 0 & 0 & -12 \end{bmatrix}, \quad \text{rang}\, M = 3$

$\lambda=-2: \quad M=\begin{bmatrix} 1 & -2 & -6 & 10 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 5 & 0 \end{bmatrix}, \quad \text{rang}\, M = 3$

Zaključak:

$\lambda=3 \quad \text{rang}\, M=2$

$\lambda\neq 3 \quad \text{rang}\, M=3$

rješenje zadatka

(3.) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra:

$$\begin{aligned} x - y - az &= 1 \\ (a+1)y + (a-1)z &= 0 \\ (a+1)x - (a+1)z &= 1. \end{aligned}$$

Rj.

$$D=\begin{vmatrix} 1 & -1 & -a \\ 0 & a+1 & a-1 \\ a+1 & 0 & -(a+1) \end{vmatrix} \underset{=\!=}{\overset{III_k + I_k}{=}} \begin{vmatrix} 1 & -1 & 1-a \\ 0 & a+1 & a-1 \\ a+1 & 0 & 0 \end{vmatrix} = (a+1)\begin{vmatrix} -1 & 1-a \\ a+1 & a-1 \end{vmatrix} \underset{=\!=}{\overset{I_R + II_R}{=}}$$

$$=(a+1)\begin{vmatrix} a & 0 \\ a+1 & a-1 \end{vmatrix} = a(a-1)(a+1), \qquad \begin{aligned} D_x &= -2a \\ D_y &= a(a-1) \\ D_z &= -a(a+1) \end{aligned}$$

$D=0 \iff a=0$ ili $a=1$ ili $a=-1$

Diskusija:

1° $a\neq 0$ i $a\neq 1$ i $a\neq -1$, sistem ima rješenje

$$x=\frac{D_x}{D} = -\frac{2}{(a-1)(a+1)}, \quad y=\frac{D_y}{D}=\frac{1}{a+1}, \quad z=\frac{D_z}{D} = -\frac{1}{a+1}$$

2° $a=1$, $D=0$, $D_x\neq 0$ sistem nema rješenje

3° $a=-1$, $D=0$, $D_x\neq 0$ sistem nema rješenje

4° $a=0$, $D=D_x=D_y=D_z=0$, iz ovoga ne možemo ništa zaključiti

Za $a=0$ sistem postaje:

$$\begin{aligned} x - y &= 1 \\ y - z &= 0 \\ x - z &= 1 \end{aligned} \implies$$ sistem ima ∞ mnogo rješenja $(1+t, t, t)$

4. Trojka vektora $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ i $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ čine bazu vektorskog prostora V_3. Ako je $\vec{a_1}=(1,1,2)$, $\vec{a_2}=(2,3,-1)$, $\vec{a_3}=(-1,0,1)$, $\vec{b_1}=(1,1,2)$, $\vec{b_2}=(2,1,0)$ i $\vec{b_3}=(1,0,-1)$ odrediti koordinate vektora $\vec{c}=3\vec{a_1}+\vec{a_2}-\vec{a_3}$ u odnosu na bazu $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$.

Rj: Neka je $\{\vec{i}, \vec{j}, \vec{k}\}$ jedinična baza vektorskog prostora V_3. Tada je $\vec{a_1}=\vec{i}+\vec{j}+2\vec{k}$, $\vec{a_2}=2\vec{i}+3\vec{j}-\vec{k}$, $\vec{a_3}=-\vec{i}+\vec{j}$
$\vec{b_1}=\vec{i}+\vec{j}+2\vec{k}$, $\vec{b_2}=2\vec{i}+\vec{j}$ i $\vec{b_3}=\vec{i}-\vec{k}$.

$$\vec{c}=3\vec{a_1}+\vec{a_2}-\vec{a_3}=(3,3,6)+(2,3,-1)-(-1,0,1)=(6,6,4)=6\vec{i}+6\vec{j}+4\vec{k}$$

$$\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_3}=\vec{c}$$

$$\begin{aligned} \alpha+2\beta+\gamma&=6\\ \alpha+\beta\quad&=6\\ 2\alpha\quad-\gamma&=4 \end{aligned} \quad\Rightarrow\quad \begin{aligned} \alpha&=-2\\ \beta&=8\\ \gamma&=-8 \end{aligned} \qquad \vec{c}=-2\vec{b_1}+8\vec{b_2}-8\vec{b_3}$$

tražene koordinate vektora $\vec{c}$

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5. Ispitati f-ju i nacrtati njen grafik $y=\dfrac{3x^2-1}{(x^2+1)^3}$.

Rj.

$\mathcal{D}: x\in\mathbb{R}$

parna f-ja (simetrična u odnosu na y-osu)

$(0,-1)$ presjek sa y-osom.

$(-\frac{1}{\sqrt{3}},0)$ i $(\frac{1}{\sqrt{3}},0)$ nule f-je

x	$(0,\frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}},+\infty)$
y	$-$	$+$

znak f-je

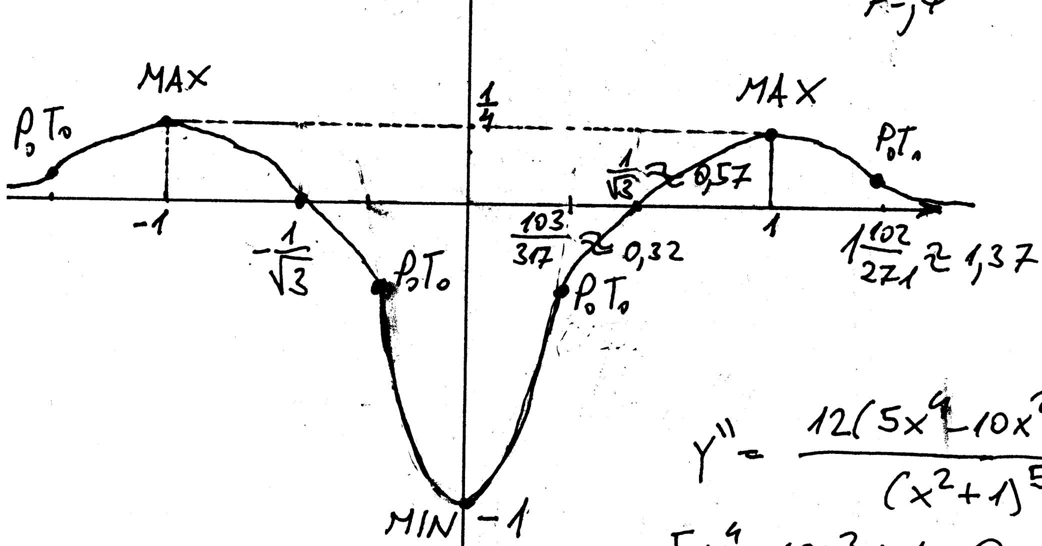
nema tačke prekida ⇓ nema V. A.

$y=0$ je H. A.

$$y'=\frac{-12x(x^2-1)}{(x^2+1)^4}$$

x	$(0,1)$	$(1,+\infty)$
y'	$+$	$-$
y	↗	↘

MIN $(0,-1)$ MAX $(1,\frac{1}{4})$



$$y''=\frac{12(5x^4-10x^2+1)}{(x^2+1)^5}$$

$5x^4-10x^2+1=0$ akko $x_1=\frac{103}{317}$, $x_2=1\frac{102}{271}$

nule polinoma izračunate na digitronu

6. Ispitati f-ju i nacrtati njen grafik $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

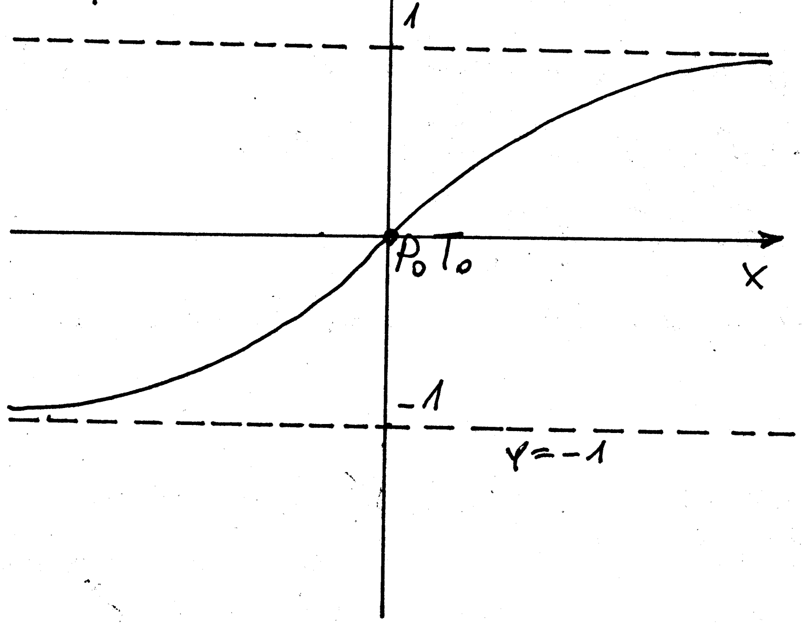
Rj. $D: x\in\mathbb{R}$

(0,0) je nula i presjek sa Y-osom

x	$(0,+\infty)$
y	$+$

f-ja je neparna (simetrična u odnosu na koordinatni početak)

$Y=1$



f-ja je definisana za $\forall x\in\mathbb{R}$
$\Downarrow$
nema V_oA_o

$Y=\pm 1$ je H_oA_o

$$y'=\frac{4\cdot e^{-2x}}{(1+e^{-2x})^2}=\frac{4}{(e^x+e^{-x})}$$

$y'>0\ \forall x\in D$ f-ja raste za $\forall x\in D$

nema ekstrema

$$y''=\frac{-8(e^x-e^{-x})}{(e^x+e^{-x})^3},$$

x	$(0,+\infty)$
y''	$-$
y	$\frown$

P_oT_o

$y''=0$ akko $x=0$

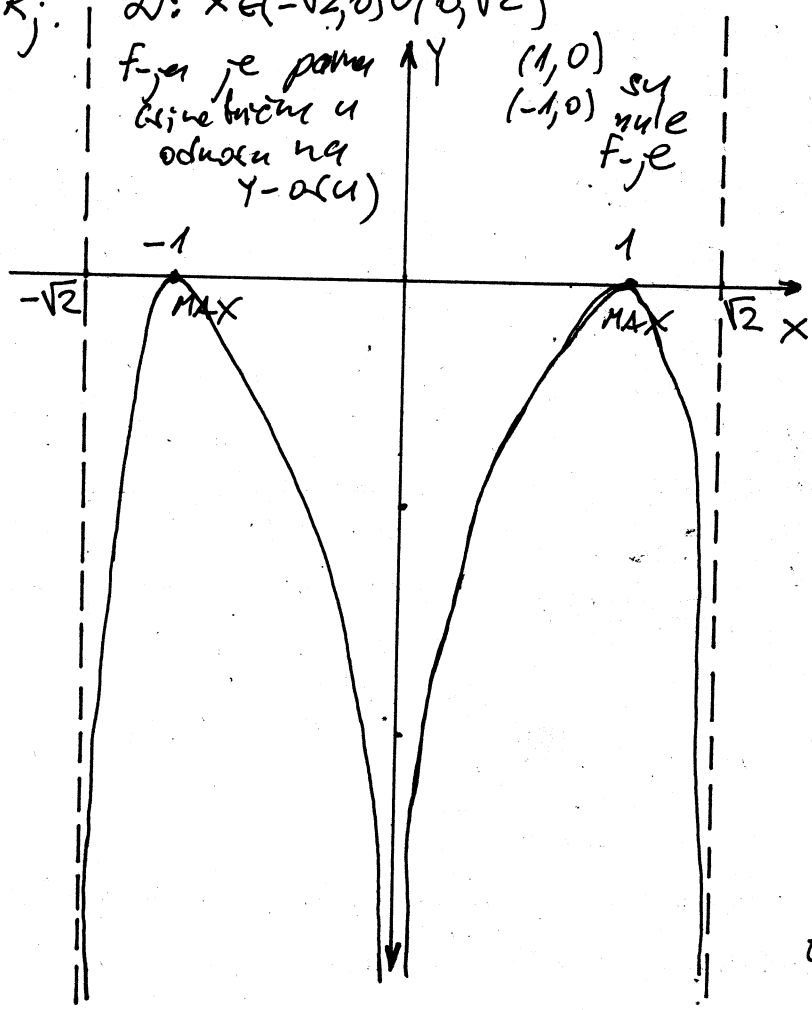
$(0,0)$ je P_oT_o

7. Ispitati f-ju i nacrtati njen grafik $y=\ln(2x^2-x^4)$.

Rj. $D: x\in(-\sqrt{2},0)\cup(0,\sqrt{2})$

f-ja je parna (simetrična u odnosu na Y-osu)

(1,0) (-1,0) su nule f-je



$x=0$ je V_oA_o nema H_oA_o

$x=\sqrt{2}$ je V_oA_o nema K_oA_o

f-ja je negativna za $\forall x\in D$

$$y'=\frac{4(x^2-1)}{x(x^2-2)}$$

x	$(0,1)$	$(1,\sqrt{2})$
y'	$+$	$-$
y	$\nearrow$	$\searrow$

MAX (1,0)

$$y''=(-4)\,\frac{x^4-x^2+2}{x^2(-2+x^2)^2}$$

$y''<0\ \forall x$ f-ja nema prevojnih tački i uvijek je $\frown$

8. Ispitati i nacrtati f-ju $y=\frac{ax+b}{(x-2)^2}$, ako se zna da ima prevojnu tačku $P(-4,-\frac{1}{9})$.

Rj: $y'=-\frac{2a+2b+ax}{(x-2)^3}$

$y''=\frac{8a+6b+2ax}{(x-2)^4}$

$y(-4)=-\frac{1}{9} \Rightarrow \frac{-4a+b}{36}=-\frac{1}{9}$

$y''(-4)=0 \Rightarrow 8a+6b+2ax=0$

$a=1,\ b=0$

$y=\frac{x}{(x-2)^2}$

x	$(-\infty,0)$	$(0,+\infty)$
y	$-$	$+$

znak f-je

(0,0) je nula f-je i prekid sa y-osom

$D: x\in R\setminus\{2\}$

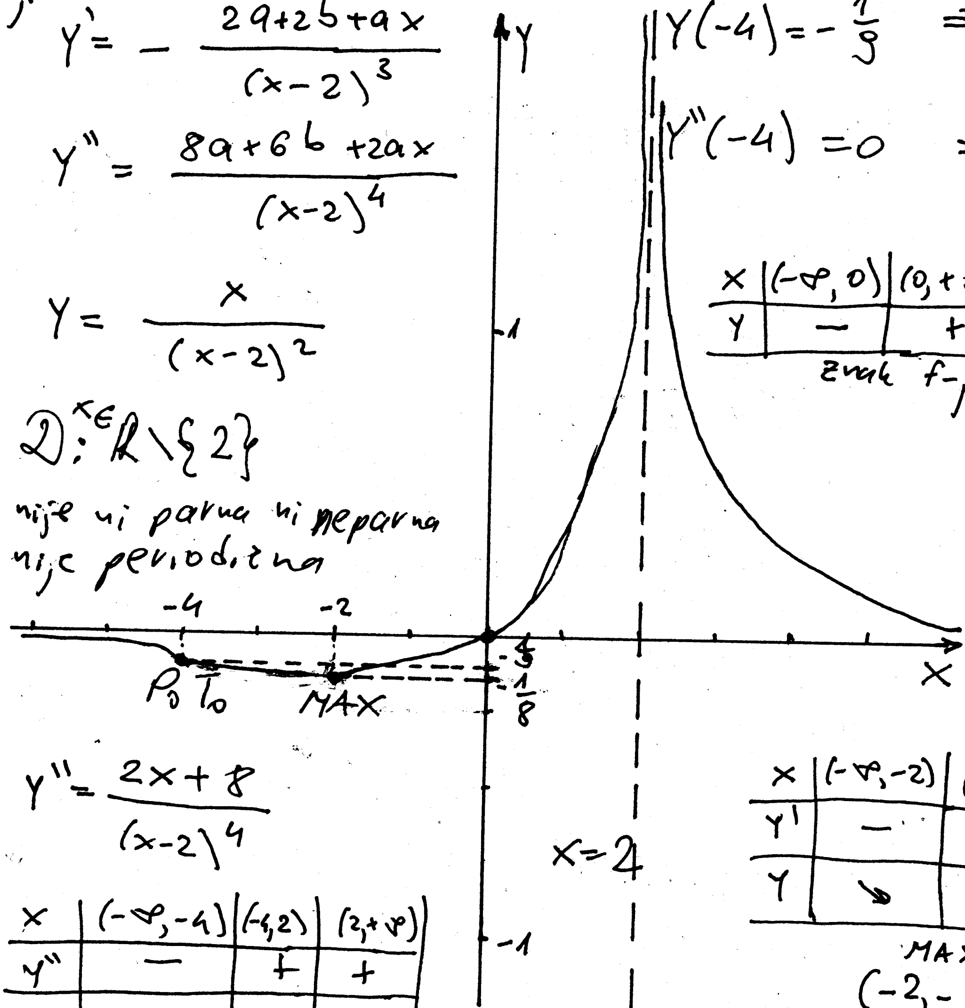
nije ni parna ni neparna

nije periodična

$x=2$ je V.A.

$y=0$ je H.A.

nema K.A.



$y'=\frac{-x-2}{(x-2)^3}$

x	$(-\infty,-2)$	$(-2,2)$	$(2,+\infty)$
y'	$-$	$+$	$-$
y	↘	↗	↘

rast i opadanje

MAX $(-2,-\frac{1}{8})$

$y''=\frac{2x+8}{(x-2)^4}$

x	$(-\infty,-4)$	$(-4,2)$	$(2,+\infty)$
y''	$-$	$+$	$+$
y	$\frown$	$\smile$	$\smile$

konveksnost i konkavnost

P.T. $(-4,-\frac{1}{9})$

9. Izračunati integral $I=\int\frac{x^5}{x^4-x^3-x+1}dx$.

Rj. $x^5:(x^4-x^3-x+1)=x+1+\frac{x^3+x^2-1}{x^4-x^3-x+1}$

$I=\int(x+1)dx+\int\frac{x^3+x^2-1}{x^4-x^3-x+1}dx=\frac{x^2}{2}+x+I_1$

$\frac{x^3+x^2-1}{x^4-x^3-x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+x+1} \quad /(x-1)^2(x^2+x+1)$

$\Rightarrow A=\frac{4}{3},\ B=\frac{1}{3},\ C=-\frac{1}{3},\ D=0$

$\int\frac{dx}{x^2+x+1}=\left|\begin{array}{c}x^2+x+1=x^2+2x\cdot\frac{1}{2}+(\frac{1}{2})^2+1)\\=(x+\frac{1}{2})^2+\frac{3}{4}\end{array}\right|=\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}=\left|\begin{array}{c}x+\frac{1}{2}=\frac{\sqrt{3}}{2}t\\dx=\frac{\sqrt{3}}{2}dt\end{array}\right|=\dots$

$-\frac{1}{3}\cdot\frac{1}{x-1}+\frac{4}{3}\ln|x-1|-\frac{1}{6}|x^2+x+1|+\frac{\sqrt{3}}{9}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+c$

10. Izračunati integral $I=\int \frac{2-\cos x}{\sin x-1}\,dx$.

Rj. $I=\int \frac{2}{\sin x-1}\,dx-\int \frac{\cos x}{\sin x-1}\,dx$

$$\int \frac{dx}{\sin x-1}=\left|\begin{matrix} tg\frac{x}{2}=t \\ \frac{x}{2}=arctg\, t \\ dx=\frac{2dt}{1+t^2}\end{matrix}\quad \sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=\ldots=\frac{2t}{t^2+1}\right|=\int\frac{\frac{2}{1+t^2}}{\frac{2t}{t^2+1}-1}\,dt$$

$$=2\int\frac{dt}{-t^2+2t-1}=\frac{2}{t-1}+C=\frac{2}{tg\frac{x}{2}-1}+C$$

$$\int\frac{\cos x}{\sin x-1}\,dx=\left|\begin{matrix}\sin x-1=t\\ \cos x\,dx=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|t|+C=\ln|\sin x-1|+C$$

$$I=\frac{4}{tg\frac{x}{2}-1}-\ln|\sin x-1|+C$$

11. Izračunati površinu figure koju čine prave $y-x=3$, $y-x=6$, $y+2x=-12$, $y+2x=-6$.

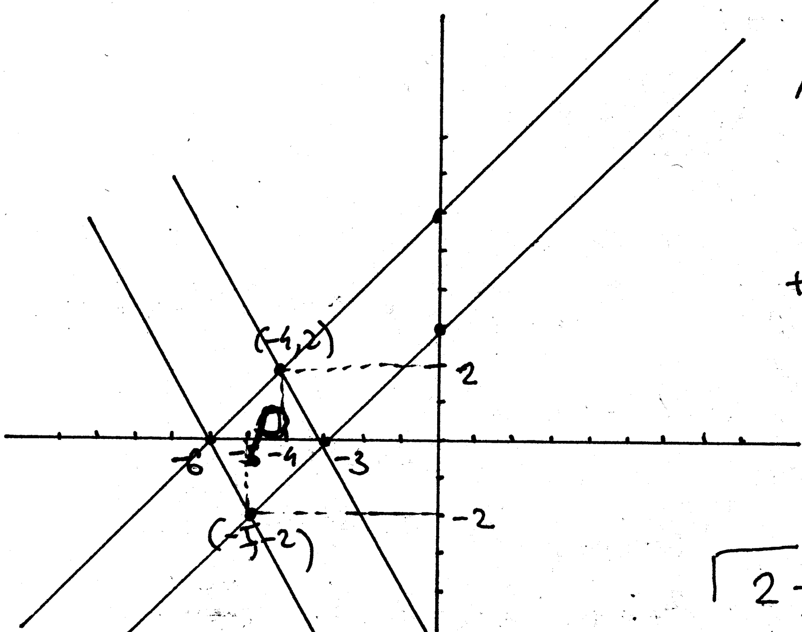
Rj. Tačka presjeka pravih $y-x=3$ i $y=2x=-12$ je $A(-5,-2)$

Tačka presjeka pravih $y-x=3$ i $y+2x=-6$ je $B(-3,0)$

Tačka presjeka pravih $y-x=6$ i $y+2x=-12$ je $C(-6,0)$

Tačka presjeka pravih $y-x=6$ i $y+2x=-6$ je $D(-4,2)$.

$y-x=3$ i $y-x=6$ su paralelne prave (kao i $y+2x=-12$ i $y+2x=-6$)



$$P=\int_{-6}^{-4}(x+6)\,dx+\int_{-4}^{-3}(-2x-6)\,dx+\left|\int_{-6}^{-5}(-2x-12)\,dx\right|+\left|\int_{-5}^{-3}(x+3)\,dx\right|$$

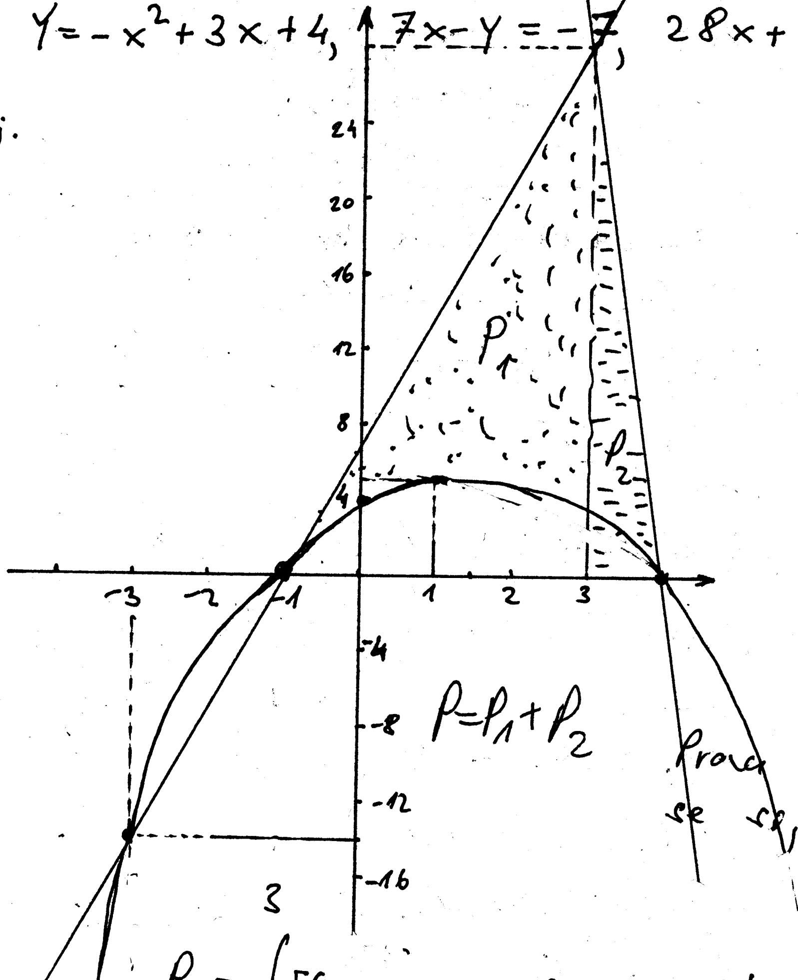
$$P=6$$

$2+1-(-1)-(-2)=\underline{6}$

12. Izračunati površinu figure koju čine linije

$y=-x^2+3x+4,\quad 7x-y=-7,\quad 28x+y=112.$

Rj.



$y=-x^2+3x+4$

$T\left(\frac{3}{2},\frac{25}{4}\right)$,

$(0,4)$ presjek sa y-osom

$(-1,0)$ i $(4,0)$ nule f-je

Prave $7x-y=-7$ i $28x+y=112$ se sijeku u tački $A(3,28)$

Prava $7x-y=-7$ i kriva $y=-x^2+3x+4$ se sijeku u tačkama $B(-1,0)$ i $C(-3,-14)$

Prava $28x+y=112$ i kriva $y=-x^2+3x+4$ se sijeku u tačkama $D(4,0)$ i $E(27,-644)$

$P=P_1+P_2$

$$P_1=\int_{-1}^{3}\left[(7x+7)-(-x^2+3x+4)\right]dx=\frac{112}{3}$$

$$P_2=\int_{3}^{4}\left[(-28x+112)-(-x^2+3x+4)\right]dx=\frac{71}{6}$$

$$P=\frac{295}{6}$$

13. Naći ekstreme f-je $z=\frac{x^5}{5}+\frac{x^3+2y^3}{3}-2(y+3x)+1.$

Rj.

$\frac{\partial z}{\partial x}=x^4+x^2-6$

$\frac{\partial z}{\partial y}=2y^2-2$

$x^4+x^2-6=0$

$2y^2-2=0$

$\Rightarrow$ stacionarne tačke su $M_1(-\sqrt{2},-1)$, $M_2(-\sqrt{2},1)$, $M_3(\sqrt{2},-1)$ i $M_4(\sqrt{2},1)$

$\frac{\partial^2 z}{\partial x^2}=4x^3+2x$

$\frac{\partial^2 z}{\partial x\partial y}=0$

$\frac{\partial^2 z}{\partial y^2}=4y$

$D=AC-B^2$

za $M_1(-\sqrt{2},-1)$, $A=-10\sqrt{2}$, $B=0$, $C=-4$

$D=40\sqrt{2}$ f-ja ima maksimum, $z_{max}=8,7444$

za $M_2(-\sqrt{2},1)$, $D<0$ f-ja nema ekstrema

za $M_3(\sqrt{2},-1)$, $D<0$ f-ja nema ekstrema

za $M_4(\sqrt{2},1)$, $D>0$ f-ja ima minimum

$z_{min}=-6,7444$

14. Nađi uslovne ekstreme f-je $z=(x-y)^4+1$ ako je $x^2+y^2=18$.

Rj. $F(x,y,\lambda)=(x-y)^4+1+\lambda(x^2+y^2-18)$

$$\frac{\partial F}{\partial x}=4(x-y)^3+2\lambda x$$

$$\frac{\partial F}{\partial y}=4(x-y)^3\cdot(-1)+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-18$$

$$\begin{aligned} 4(x-y)^3+2\lambda x&=0\\ -4(x-y)^3+2\lambda y&=0\\ x^2+y^2&=18 \end{aligned}$$

Stacionarne tačke su $M_1(3,-3)$ i $M_2(-3,3)$ za $\lambda=-144$ i $M_3(3,3)$ i $M_4(-3,-3)$ za $\lambda=0$.

$$\frac{\partial^2 F}{\partial x^2}=12(x-y)^2+2\lambda$$

$$\frac{\partial^2 F}{\partial x\partial y}=-12(x-y)^2$$

$$\frac{\partial^2 F}{\partial y^2}=12(x-y)^2+2\lambda$$

$D=AC-B^2$

$M_1(3,-3)$, $\lambda=-144 \Rightarrow D<0$ nema ekstrema

$M_2(-3,3)$, $\lambda=-144 \Rightarrow D<0$ f-ja nema ekstre.

$M_3(3,3)$, $\lambda=0 \Rightarrow D=0$ potrebno je ispitati f-ju u okolini tačke $(3,3)$

$$\Delta z=z(3+\varepsilon,\ 3+\omega)-z(3,3)$$

$$z=(x-y)^4+1,\quad z(3,3)=1$$

$$\Delta z=(3+\varepsilon-3-\omega)^4+1-1=(\varepsilon-\omega)^4>0\ \ \forall \varepsilon \text{ i } \omega$$

Priraštaj Δz je pozitivan u okolini tačke $(3,3)$ pa f-ja z u tački $M_3(3,3)$ ima minimum $z_{min}=1$.

$M_4(-3,-3)$, $\lambda=0$, $D=0 \Rightarrow$ ispitujemo f-ju u okolini tačke, $\Delta z=(\varepsilon-\omega)>0\ \forall \varepsilon$ i ω

$z_{min}=1$

15. Riješiti diferencijalnu jednačinu $\dfrac{y-xy'}{x+yy'}=2$.

Rj. $y=\dfrac{\frac{y}{x}-2}{1+2\frac{y}{x}}$ ovo je homogena dif. jednačina, uvodimo smjenu $u=\dfrac{y}{x}$

$$\frac{2u+1}{u^2+1}\,du=(-2)\frac{dx}{x} \Rightarrow \operatorname{arctg}\frac{y}{x}=\ln\frac{C}{x^2+y^2}$$

16. Riješiti diferencijalnu jednačinu $xy'-y+\ln x=0$ uz početni uslov $y(1)=3$.

Rj. $y'-\dfrac{1}{x}y=-\dfrac{\ln x}{x}$ ovo je linearna dif. jedn., uvodimo smjenu $y=uv$

$$u'v+\underbrace{u\left(v'-\frac{1}{x}v\right)}_{=0}=-\frac{\ln x}{x}$$

$$v=x,\quad u=-\frac{1}{x}\ln x-\frac{1}{x}+C$$

$y=\ln x+1+Cx$ opće rješenje dif. jedn.

$y=\ln x+2x+1$ partikularno rješenje diff. jedn.