Univerzitet u Zenici
Ekonomski fakultet

02.09.2009.

Pismeni ispit iz predmeta Matematika

1. Dokazati metodom matematičke indukcije da za sve prirodne brojeve n važi

$$\frac{1}{6}+\frac{1}{12}+...+\frac{1}{n^2+3n+2}=\frac{n}{2n+4}.$$

2. Koliko ima racionalnih članova u razvoju binoma $(\sqrt[3]{4}+\sqrt[4]{3})^{120}$.

3. Naći sve vrijednosti korijena $\sqrt[4]{z}$, ako je $z=(-1+i)^8$.

4. Riješiti matričnu jednačinu $X^{-1}AB=B^{-1}A^{-1}$, ako su $A=\begin{bmatrix}1&1&6\\2&-3&0\\0&1&1\end{bmatrix}$ i $B=\begin{bmatrix}2&-1&0\\1&-4&1\\1&0&2\end{bmatrix}$.

5. Ispitati i grafički predstaviti funkciju $y=\dfrac{3x}{1+x^3}$.

6. Ispitati i grafički predstaviti funkciju $y=\dfrac{x^2+5x}{x^2+2x+1}$.

7. Ispitati i grafički predstaviti funkciju $y=x^3\,e^{-\frac{x^2}{6}}$.

8. Ispitati i grafički predstaviti funkciju $y=ln\dfrac{x^2}{x+1}$.

9. Izračunati integral $\displaystyle\int\frac{dx}{3cos^2x+4sin^2x}$.

10. Izračunati integral $\displaystyle\int_{6-\sqrt{2}}^{7}\frac{4x+2}{\sqrt{-34+12x-x^2}}\,dx$.

11. Izračunati integral $\displaystyle\int_0^1\sqrt{4-x^2}\,dx$.

12. Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

13. Odrediti ekstremne vrijednosti funkcije $z=8x^3-y^3+6xy+7$.

14. Odrediti ekstremne vrijednosti funkcije $z=\dfrac{xy}{2}+(47-x-y)(\dfrac{x}{3}+\dfrac{y}{4})$.

15. Riješiti diferencijalnu jednačinu $y'=y^4cosx+y\,tgx$.

16. Riješiti diferencijalnu jednačinu $y'+\dfrac{1}{y'}=\dfrac{y}{x}$.

(Ova stranica je ostavljena prazna)

Dokazati metodom matematičke indukcije da za sve prirodne brojeve n važi

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+3n+2}=\frac{n}{2n+4}.$$

Rj. $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{k^2+3k+2}=\frac{k}{2k+4}$, k je pozitivan cijeli br.

BAZA INDUKCIJE

$k=1$: $\frac{1}{6}=\frac{1}{2\cdot 1+4} \Rightarrow \frac{1}{6}=\frac{1}{6}$ jednakost je tačna za $k=1$.

INDUKCISKI KORAK

Pretpostavimo da je jednakost tačna za $k=1,2,\ldots,n$,

tj. $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{k^2+3k+2}=\frac{k}{2k+4}$, $k=1,2,\ldots,n$.

Na osnovu ove pretpostavke dokažimo da je jednakost tačna za $n+1$ tj. da je

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{(n+1)^2+3(n+1)+2}=\frac{n+1}{2(n+1)+4}$$

$(n+1)^2=n^2+2n+1$

$3(n+1)=3n+3$

ili drugačije napisano $\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+5n+6}=\frac{n+1}{2n+6}$

$$\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n^2+3n+2}+\frac{1}{n^2+5n+5}\overset{\text{na osnovu pretpostavke}}{=}\frac{n}{2n+4}+\frac{1}{n^2+5n+6}$$

$n^2+5n+6=0$

$D=25-24=1$

$n_{1,2}=\frac{-5\pm 1}{2}$

$n_1=\frac{-6}{2}=-3 \quad n_2=\frac{-4}{2}=-2$

$$=\frac{n}{2(n+2)}\,{}^{\cdot(n+3)}+\frac{1}{(n+2)(n+3)}\,{}^{\cdot 2}=\frac{n(n+3)+2}{2(n+2)(n+3)}$$

$$=\frac{n^2+3n+2}{2(n+2)(n+3)}=\frac{\cancel{(n+2)}(n+1)}{2\cancel{(n+2)}(n+3)}=\frac{n+1}{2n+6}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

\# Koliko ima racionalnih članova u razvoju binoma $\left(\sqrt[3]{4}+\sqrt[4]{3}\right)^{120}$?

Rj. Koji su racionalni brojevi?

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\left(\sqrt[3]{4}+\sqrt[4]{3}\right)^{120}=\sum_{k=0}^{120}\binom{120}{k}\left(\sqrt[3]{4}\right)^{120-k}\left(\sqrt[4]{3}\right)^k=\sum_{k=0}^{120}\binom{120}{k}4^{\frac{120-k}{3}}\cdot 3^{\frac{k}{4}}=$$

$$=\sum_{k=0}^{120}\binom{120}{k}4^{40-\frac{k}{3}}\cdot 3^{\frac{k}{4}}$$

Da bi član bio racionalan, u posljednjem izrazu, potrebno je da je k djeljiv sa 3 (iz izraza $4^{40-\frac{k}{3}}$) i da je k djeljiv sa 4 (iz izraza $3^{\frac{k}{4}}$).

Kako je potrebno da je k djeljiv sa 3 i sa 4 to treba da je k djeljiv sa 12.

Brojevi djeljivi sa 12 iz intervala $0, 1, 2, \ldots, 120$ su:

0, 12, 24, 36, 48, 60, 72, 84, 96, 108 i 120

Postoji 11 racionalnih članova u razvoju binoma.

(#) Nadi sve vrijednosti korijena $\sqrt[4]{z}$, ako je $z=(-1+i)^8$.

Rj: $\sqrt[4]{z}$, $z=z_1^{\,8}$, $z_1=-1+i$, $|z_1|=\sqrt{2}$

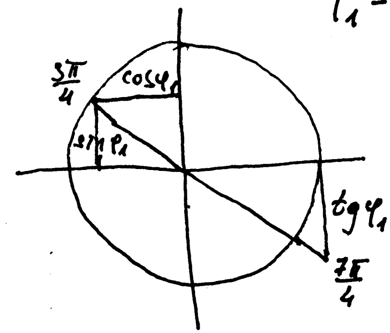
$tg\,45°=1$

$\varphi_1=\frac{3\pi}{4}$

$\cos\varphi_1=\frac{-1}{\sqrt{2}}$

$\sin\varphi_1=\frac{1}{\sqrt{2}}$

$tg\,\varphi_1=\frac{1}{-1}=-1$



$z_1=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$z=z_1^{\,8}=(\sqrt{2})^8\left[\cos 8\cdot\frac{3\pi}{4}+i\sin 8\cdot\frac{3\pi}{4}\right]$

$z=16(\cos 6\pi+i\sin 6\pi)=16(\cos 0+i\sin 0)$

$\sqrt[4]{z}=?$ $\quad z_k=\sqrt[4]{|z|}\left(\cos\frac{0+2k\pi}{4}+i\sin\frac{0+2k\pi}{4}\right)$

$z_0=\sqrt[4]{16}\left(\cos\frac{0}{4}+i\sin\frac{0}{4}\right)=2(1+i\cdot 0)=2$

$z_1=\sqrt[4]{16}\left(\cos\frac{0+2\pi}{4}+i\sin\frac{0+2\pi}{4}\right)=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i\cdot 1)=2i$

$z_2=\sqrt[4]{16}\left(\cos\frac{0+4\pi}{4}+i\sin\frac{0+4\pi}{4}\right)=2(\cos\pi+i\sin\pi)=2(-1+i\cdot 0)=-2$

$z_3=\sqrt[4]{16}\left(\cos\frac{0+6\pi}{4}+i\sin\frac{0+6\pi}{4}\right)=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=2(0+i\cdot(-1))=-2i$

Sve vrijednosti $\sqrt[4]{z}$ su $\{2, 2i, -2, -2i\}$

(#) Riješiti matričnu jednačinu $X^{-1}AB = B^{-1}A^{-1}$,

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Rj.

$$X^{-1}AB = B^{-1}A^{-1}$$

$$X^{-1}AB = (AB)^{-1} \qquad /(AB)^{-1} \text{ sa desne strane}$$

$$X^{-1} = (AB)^{-1}(AB)^{-1}$$

$$X = (AB)\cdot(AB)$$

$$X = (AB)^2$$

$$A\cdot B = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix}$$

$2+1+6$ $4-3+0$ $0+1+1$

$-1-4+0$ $-2+12+0$ $0-4+0$

$0+1+12$ $0-3+0$ $0+1+2$

$$(AB)^2 = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 102 & -147 & 171 \\ 13 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

$81-5+26$ $-45-50-52$ $117+15+39$

$9+10-6$ $-5+100+12$ $13-30-9$

$18-4+6$ $-10-40-12$ $26+12+9$

$$X = \begin{bmatrix} 102 & -147 & 171 \\ 13 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{3x}{1+x^3}$.

Rj. definiciono područje

$1+x^3 \neq 0$

$x^3 \neq -1$

$x \neq -1$

$\mathcal{D}: x \in (-\infty,-1) \cup (-1,+\infty)$

parnost, neparnost, periodičnost

$f(-x)=\frac{3\cdot(-x)}{1+(-x^3)}=-\frac{3x}{1-x^3}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom, znak f-je

$Y=0$

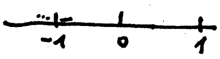
$\frac{3x}{1+x^3}=0$

$x=0$

(0,0) je nula f-je i presjek sa Y-osom

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
$3x$	$-$	$-$	$+$
$1+x^3$	$-$	$+$	$+$
Y	$+$	$-$	$+$

znak f-je



ponašanje na krajevima intervala definisanosti i asimptote

za vrijednost $x=-1$ f-ja ima prekid

$\lim\limits_{x\to-1-0} f(x)=\lim\limits_{x\to-1-0}\frac{3x}{1+x^3}=\frac{3(-1-0)}{1+(-1-0)^3}=\frac{3(-1-0)}{1-1-0}=\frac{-3-0}{-0}=+\infty \Rightarrow x=-1$ je V.A.

$\lim\limits_{x\to-1+0} f(x)=\lim\limits_{x\to-1+0}\frac{3x}{1+x^3}=\frac{3(-1+0)}{1+(-1+0)^3}=\frac{-3+0}{1-1+0}=\frac{-3+0}{+0}=-\infty \Rightarrow x=-1$ je V.A.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\frac{3x \,:x}{1+x^3 \,:x}=\lim\limits_{x\to-\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2}=0 \Rightarrow Y=0$ je H.A.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2}=0 \Rightarrow Y=0$ je H.A. f-ja nema K.A.

rast i opadanje

$Y'=\left(\frac{3x}{1+x^3}\right)'=3\cdot\frac{1\cdot(1+x^3)-x\cdot 3x^2}{(1+x^3)^2}=3\frac{1+x^3-3x^3}{(1+x^3)^2}$

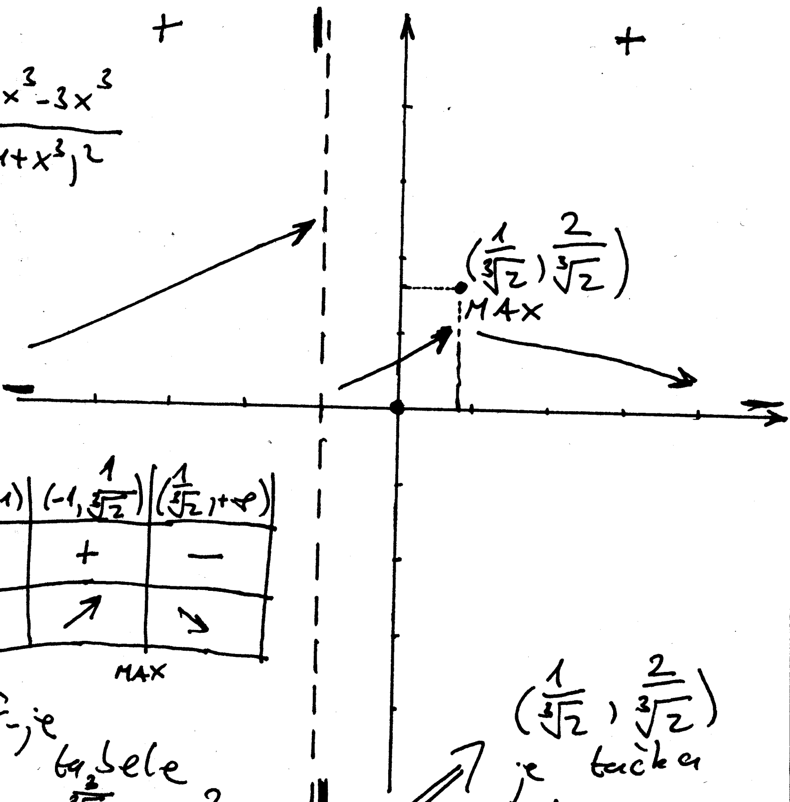
$Y'=3\cdot\frac{1-2x^3}{(1+x^3)^2}$

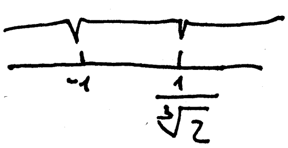
$Y'=0$ akko $1-2x^3=0$

$2x^3=1$

$x^3=\frac{1}{2}$

$x=\frac{1}{\sqrt[3]{2}}\approx 0{,}8$





↖ prekidi Y + nule Y'

x	$(-\infty,-1)$	$(-1,\frac{1}{\sqrt[3]{2}})$	$(\frac{1}{\sqrt[3]{2}},+\infty)$
Y'	$+$	$+$	$-$
Y	↗	↗	↘
			MAX

ekstrem f-je

Na osnovu tabele

$f\left(\frac{1}{\sqrt[3]{2}}\right)=\frac{\frac{3}{\sqrt[3]{2}}}{1+\frac{1}{2}}=\frac{\frac{3}{\sqrt[3]{2}}}{\frac{3}{2}}=\frac{2}{\sqrt[3]{2}}\approx 1{,}6$

$\left(\frac{1}{\sqrt[3]{2}},\frac{2}{\sqrt[3]{2}}\right)$ je tačka maksimuma prevojne tačke i intervali konveksnosti i konkavnosti

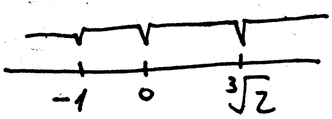
$$y' = 3 \cdot \frac{1-2x^3}{(1+x^3)^2}, \quad y'' = 3 \cdot \frac{-6x^2 \cdot (1+x^3)^{\cancel{2}} - (1-2x^3)\cdot 2\cancel{(1+x^3)} \cdot 3x^2}{(1+x^3)^3 \cdot \cancel{(1+x^3)}} =$$

$$= 3 \cdot \frac{-6x^2 - 6x^5 - 6x^2 + 12x^5}{(1+x^3)^3} = 3 \cdot \frac{6x^5 - 12x^2}{(1+x^3)^3}$$

$$y'' = 18 \cdot \frac{x^5 - 2x^2}{(1+x^3)^3} = \frac{18x^2(x^3-2)}{(1+x^3)^3}$$

$y''=0$ akko $x=0$ ili $x^3-2=0$

$x_1=0$ $\qquad x_2=\sqrt[3]{2}\approx 1{,}3$



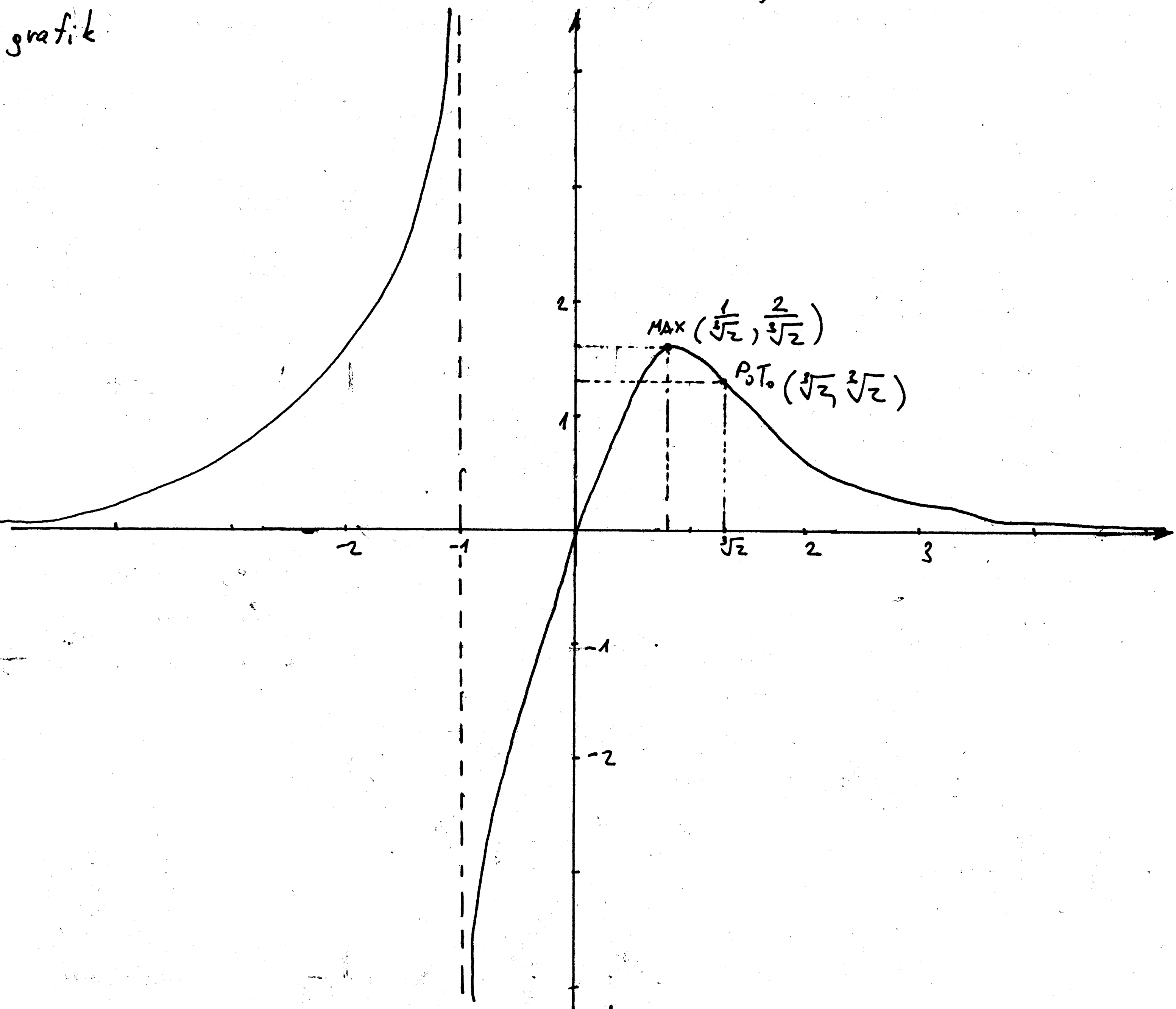
x	$(-\infty,-1)$	$(-1,0)$	$(0,\sqrt[3]{2})$	$(\sqrt[3]{2},+\infty)$
y''	$+$	$-$	$-$	$+$
y	$\cup$	$\cap$	$\cap$	$\cup$

P_0T_0

$$f(\sqrt[3]{2}) = \frac{3\sqrt[3]{2}}{1+2} = \sqrt[3]{2}$$

$(\sqrt[3]{2}, \sqrt[3]{2})$ je prevojna tačka

grafik



(#) Ispitati i grafički predstaviti f-ju $Y=\frac{x^2+5x}{x^2+2x+1}$.

Rj: definiciono područje

$x^2+2x+1\neq 0$

$D=4-4=0$

$(x+1)^2\neq 0$

$x\neq -1$

$D: x\in\mathbb{R}\setminus\{-1\}$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$Y=0$ akko $x^2+5x=0$

$x(x+5)=0$

$x_1=0$ ili $x_2=-5$

$(0,0)$ i $(-5,0)$ su nule f-je

$(0,0)$ je tačka presjeka sa Y-osom.

$Y=\frac{x(x+5)}{(x+1)^2}$ -5 -1 0

x	$(-\infty,-5)$	$(-5,-1)$	$(-1,0)$	$(0,+\infty)$
x	−	−	−	+
x+5	−	+	+	+
Y	+	−	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ f-ja ima prekid

$$\lim_{x\to-1-0} f(x)=\lim_{x\to-1-0}\frac{x(x+5)}{(x+1)^2}=\frac{(-1-0)(-1-0+5)}{(-1-0+1)^2}=\frac{(-1-0)(4-0)}{+0}=-\infty \Rightarrow x=-1 \text{ je } V_oA_o$$

$$\lim_{x\to-1+0} f(x)=\lim_{x\to-1+0}\frac{x(x+5)}{(x+1)^2}=\frac{(-1+0)(-1+0+5)}{(-1+0+1)^2}=\frac{(-1+0)(4+0)}{+0}=-\infty \Rightarrow x=-1 \text{ je } V_oA_o$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^2+5x \;/:x^2}{x^2+2x+1 \;/:x^2}=\lim_{x\to-\infty}\frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}}=1 \Rightarrow Y=1 \text{ je } H_oA_o$$

isto vrijedi i za $\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{1+\frac{5}{x}}{1+\frac{2}{x}+\frac{1}{x^2}}=1 \Rightarrow Y=1$ je H_oA_o

nakon ovog koraka počinjemo skicirati graf

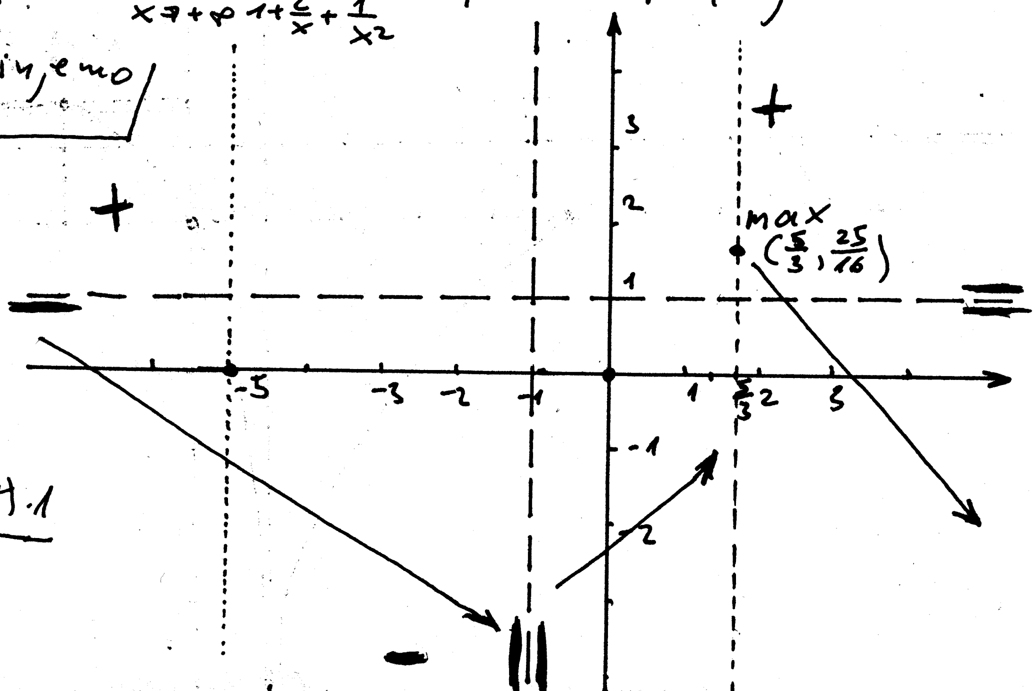
f-ja nema K_oA_o

rast i opadanje

$$Y'=\left(\frac{x^2+5x}{(x+1)^2}\right)'=$$

$$=\frac{(2x+5)(x+1)^{\cancel{2}\,1}-(x^2+5x)\,2\cancel{(x+1)}\cdot 1}{(x+1)^{\cancel{4}\,3}}$$

$$=\frac{\cancel{2x^2}+5x+2x+5-\cancel{2x^2}-10x}{(x+1)^3}$$



$$y' = \frac{-3x+5}{(x+1)^3}$$

$-1 \qquad \frac{5}{3}$ ← nule y' + prekidi y

$y'=0$ akko $-3x+5=0$

$-3x=-5$

$x=\frac{5}{3} \approx 1,6667$

$y'(-2) = \frac{11}{-1} < 0$

x	$(-\infty,-1)$	$(-1,\frac{5}{3})$	$(\frac{5}{3},+\infty)$
y'	−	+	−
y	↘	↗	↘

max — rast i opadanje

ekstremi f-je

na osnovu tabele rasta i opadanja f-ja ima maksimum za $x=\frac{5}{3}$

$$f\left(\frac{5}{3}\right) = \frac{\frac{25}{9}+5\cdot\frac{5}{3}}{\left(\frac{5}{3}+1\right)^2} = \frac{\frac{25+25\cdot 3}{9}}{\left(\frac{8}{3}\right)^2} = \frac{\frac{100}{\cancel{9}}:2}{\frac{64}{\cancel{9}}:2} = \frac{50}{32} = \frac{25}{16} \approx 1,5625$$

$M\left(\frac{5}{3},\frac{25}{16}\right)$ je tačka maksimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{-3x+5}{(x+1)^3}\right)' = \frac{-3(x+1)^{\cancel{3}\,1}-(-3x+5)3\cancel{(x+1)^2}\cdot 1}{(x+1)^4\cdot\cancel{(x+1)^2}} = \frac{-3x-3+9x-15}{(x+1)^4} = \frac{6x-18}{(x+1)^4}$$

$$y'' = 6\cdot\frac{x-3}{(x+1)^4}, \quad y''=0 \text{ akko } x=3$$

$-1 \qquad 3$ ← prekidi y + nule y''

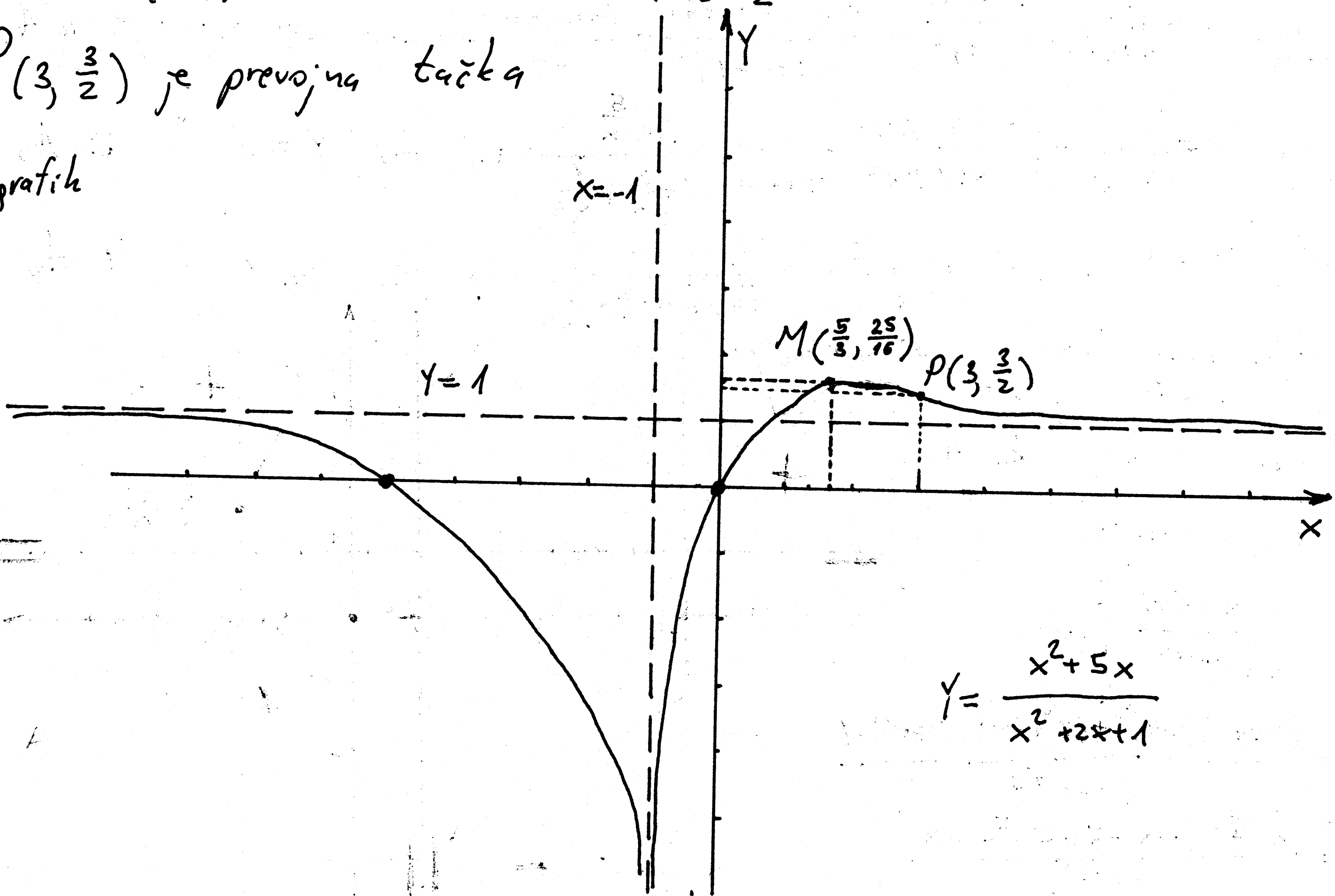
x	$(-\infty,-1)$	$(-1,3)$	$(3,+\infty)$
y''	−	−	+
y	∩	∩	∪

P_0T_0

$$f(3) = \frac{3^2+5\cdot 3}{(3+1)^2} = \frac{9+15}{16} = \frac{24:4}{16:4} = \frac{6:2}{4:2} = \frac{3}{2} = 1,5$$

$P\left(3,\frac{3}{2}\right)$ je prevojna tačka

grafik



Ispitati f-ju i nacrtati joj grafik $y = x^3 e^{-\frac{x^2}{6}}$.

Rj.

definiciono područje

$\mathcal{D}: x \in \mathbb{R}$

parnost, neparnost, periodičnost

$y(-x) = (-x)^3 e^{-\frac{(-x)^2}{6}} = -x^3 e^{-\frac{x^2}{6}}$

f-ja je neparna (simetrična u odnosu na koordinatni početak). Dovoljno ju je ispitati za $x > 0$. F-ja nije periodična

nule, presjek sa y-osom, znak f-je

$x^3 \underbrace{e^{-\frac{x^2}{6}}}_{>0\ \forall x} = 0$ (0,0) je nula f-je i presjek sa y-osom

$x = 0$

x	$(-\infty, 0)$	$(0, +\infty)$
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekid $\Rightarrow$ nema V. A.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 e^{-\frac{x^2}{6}} = \lim_{x\to+\infty} \frac{x^3}{e^{\frac{x^2}{6}}} \left(\frac{+\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to+\infty} \frac{3x^2}{e^{\frac{x^2}{6}} \cdot \frac{1}{6} 2x} =$$

$$= \lim_{x\to+\infty} \frac{9x}{e^{\frac{x^2}{6}}} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to\infty} \frac{9}{e^{\frac{x^2}{6}} \cdot \frac{1}{6} 2x} = \lim_{x\to\infty} \frac{27}{x e^{\frac{x^2}{6}}} = 0$$

$\Rightarrow$ $x = 0$ je H. A., F-ja nema K. A.

rast i opadanje

$$y' = 3x^2 e^{-\frac{x^2}{6}} + x^3 \cdot e^{-\frac{x^2}{6}} \cdot \left(-\frac{1}{6}\right) \cdot 2x$$

$$= 3x^2 e^{-\frac{x^2}{6}} - \frac{1}{3} x^4 e^{-\frac{x^2}{6}}$$

$$= x^2 e^{-\frac{x^2}{6}} \left(3 - \frac{1}{3} x^2\right) = x^2 e^{-\frac{x^2}{6}} \left(\frac{9 - x^2}{3}\right)$$

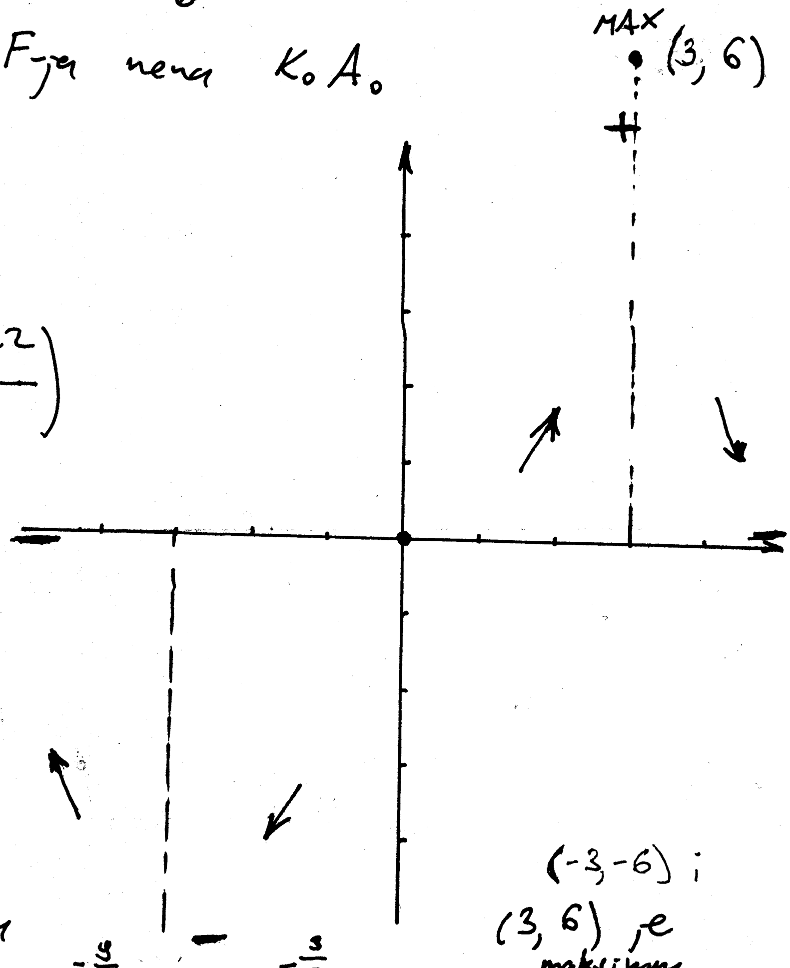
$y' = 0 \Leftrightarrow x_1 = 0,\ x_2 = -3,\ x_3 = 3$

← prekidi y + nule y'

x	$(0, 3)$	$(3, +\infty)$
y'	+	−
y	↗	↘

MAX

rast i opadanje



ekstremi f-je

Iz tabele rasta i opadanja vidimo da f-ja ima ekstrem za $x = 3$ $f(3) = 27 e^{-\frac{9}{6}} = 27 e^{-\frac{3}{2}} \approx 6$

(−3, −6) i (3, 6) je maksimum f-je

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(x^2 e^{-\frac{x^2}{6}}\frac{1}{3}(9-x^2)\right)'=2x\,e^{-\frac{x^2}{6}}\cdot\frac{1}{3}(9-x^2)+x^2e^{-\frac{x^2}{6}}\cdot\left(-\frac{1}{6}\right)2x\cdot\frac{1}{3}(9-x^2)+x^2e^{-\frac{x^2}{6}}\cdot\frac{1}{3}(-2x)=$$

$$=\frac{2}{3}x\,e^{-\frac{x^2}{6}}(9-x^2)-\frac{1}{9}x^3e^{-\frac{x^2}{6}}(9-x^2)-\frac{2}{3}x^3e^{-\frac{x^2}{6}}=x\,e^{-\frac{x^2}{6}}\left(\frac{2}{3}(9-x^2)-\frac{1}{9}x^2(9-x^2)-\right.$$

$$\left.\frac{2}{3}x^2\right)=x\,e^{-\frac{x^2}{6}}\cdot\frac{54-6x^2-9x^2+x^4-6x^2}{9}=x\,e^{-\frac{x^2}{6}}\cdot\frac{x^4-21x^2+54}{9}$$

$y''=0$ akko $x=0$ i $x^4-21x^2+54=0$

$x^2=t$

$t^2-21t+54=0$

$D=441-216=225$

$t_{1,2}=\frac{21\pm15}{2}$

$t_1=\frac{36}{2}=18\quad t_2=\frac{6}{2}=3$

$x^2=18 \qquad x^2=3$

$x=\pm\sqrt{18} \qquad x_3=-\sqrt{3}$

$x_1=3\sqrt{2}\quad x_2=-3\sqrt{2} \qquad x_4=\sqrt{3}\approx1{,}73$

$3\sqrt{2}\approx4{,}24$

f-ja simetrična u odnosu na koordinatni početak pa nas zanimaju samo pozitivne vrijednosti

$0 \quad \sqrt{3} \quad 3\sqrt{2}$ ← prekidi y + nule y''

x	$(0,\sqrt{3})$	$(\sqrt{3},3\sqrt{2})$	$(3\sqrt{2},+\infty)$
y''	+	−	+
y	∪	∩	∪

P.T. P.T. P.T.

$y=x^3e^{-\frac{x^2}{6}}$

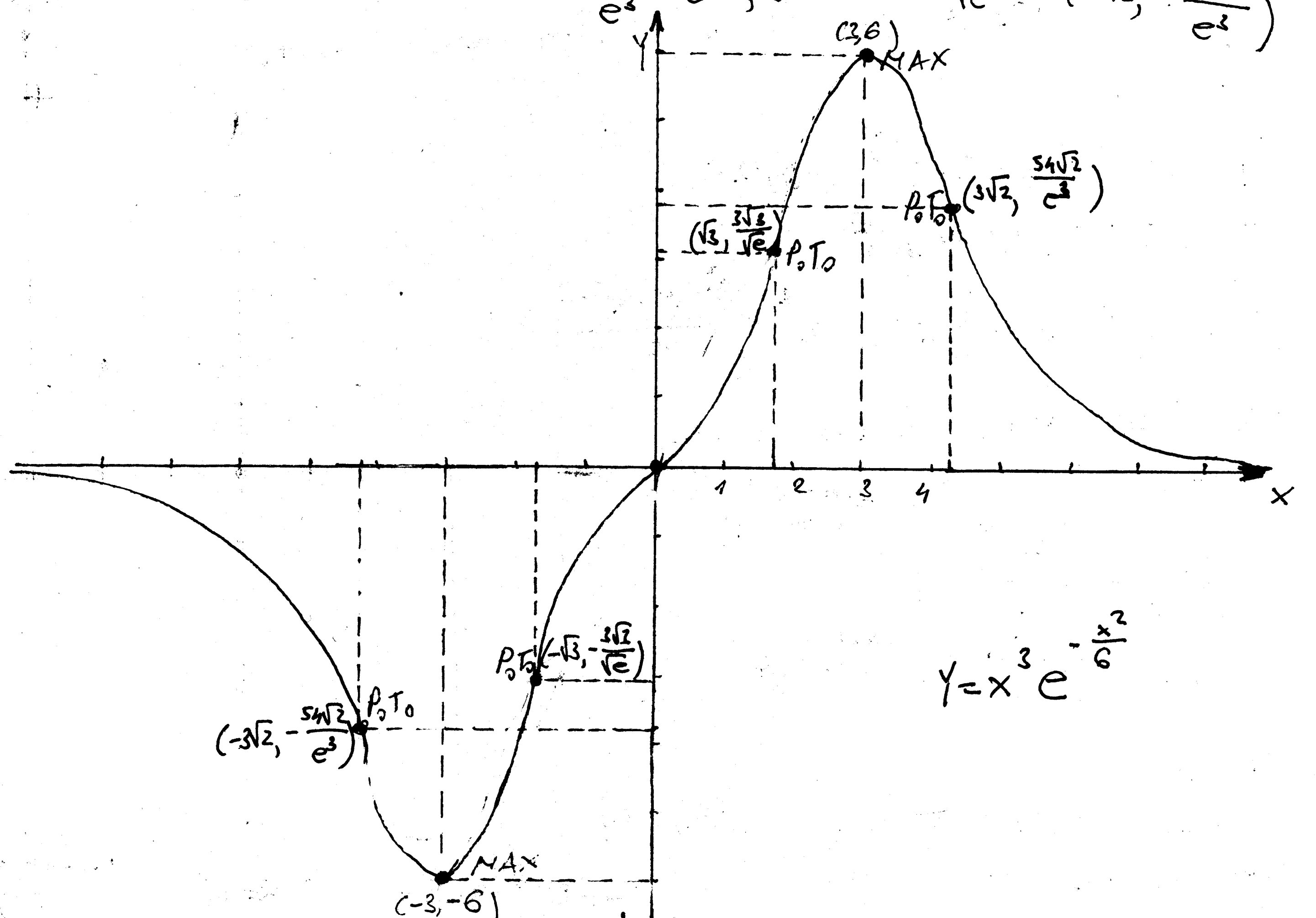
$y(0)=0$

$y(\sqrt{3})=3\sqrt{3}\,e^{-\frac{3}{6}}=\frac{3\sqrt{3}}{\sqrt{e}}\approx3{,}15$

$y(3\sqrt{2})=27\cdot2\sqrt{2}\,e^{-\frac{9\cdot2}{6}}=54\sqrt{2}\,e^{-3}=\frac{54\sqrt{2}}{e^3}\approx3{,}8$

Prevojne tačke su $(0,0)$, $\left(\sqrt{3},\frac{3\sqrt{3}}{\sqrt{e}}\right)$, $\left(3\sqrt{2},\frac{54\sqrt{2}}{e^3}\right)$, $\left(-\sqrt{3},-\frac{3\sqrt{3}}{\sqrt{e}}\right)$ i $\left(-3\sqrt{2},-\frac{54\sqrt{2}}{e^3}\right)$

grafik



(#) Ispitati f-ju i nacrtati joj grafik $y=\ln\frac{x^2}{x+1}$.

Rj. definiciono područje

$x+1\neq 0$; $\frac{x^2}{x+1}>0$

$x\neq -1$; $x\neq 0$ i $x+1>0$, $x>-1$

$D: x\in(-1,0)\cup(0,+\infty)$

parnost, neparnost, periodičnost

D nije simetrično $\Rightarrow$

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$\ln\frac{x^2}{x+1}=0$

$\frac{x^2}{x+1}=e^0$

$\frac{x^2}{x+1}=1$

$\frac{x^2}{x+1}-1=0$

$\frac{x^2-x-1}{x+1}=0$

$x^2-x-1=0$

$D=1+4=5$

$x_{1,2}=\frac{1\pm\sqrt5}{2}$

$x_1\approx 1{,}6$

$x_2\approx -0{,}6$

$\left(\frac{1+\sqrt5}{2},0\right)$ i $\left(\frac{1-\sqrt5}{2},0\right)$ su nule f-je

$y(0)$ nije definisano $\Rightarrow$ f-ja ne siječe y-osu

$y>0$ akko $\ln\frac{x^2}{x+1}>0$

$\ln\frac{x^2}{x+1}>\ln e^0$

$\frac{x^2}{x+1}>1 \iff \frac{x^2-x-1}{x+1}>0$

x	$\left(-1,\frac{1-\sqrt5}{2}\right)$	$\left(\frac{1-\sqrt5}{2},\frac{1+\sqrt5}{2}\right)$	$\left(\frac{1+\sqrt5}{2},+\infty\right)$
x^2-x-1	+	−	+
$x+1$	+	+	+
y	+	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ i $x=0$ f-ja ima prekid

$\lim\limits_{x\to-1+0} f(x)=\lim\limits_{x\to-1+0}\ln\frac{x^2}{x+1}=\ln\frac{(-1+0)^2}{(-1+0)+1}=\ln\frac{1-0}{+0}=$

$=\ln+\infty=+\infty \Rightarrow x=-1$ je V. A.

$\lim\limits_{x\to-0} f(x)=\lim\limits_{x\to-0}\ln\frac{x^2}{x+1}=\ln\lim\limits_{x\to-0}\frac{x^2}{x+1}=\ln\frac{(-0)^2}{-0+1}=\ln\frac{+0}{1-0}=\ln+0=-\infty \Rightarrow x=0$ je V. A.

$\lim\limits_{x\to+0} f(x)=\lim\limits_{x\to+0}\ln\frac{x^2}{x+1}=\ln\lim\limits_{x\to+0}\frac{x^2}{x+1}=\ln\frac{(+0)^2}{+0+1}=\ln\frac{+0}{1+0}=\ln+0=-\infty \Rightarrow x=0$ je V. A.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\ln\frac{x^2}{x+1}=\ln\lim\limits_{x\to+\infty}\frac{x^2:x}{x+1:x}=\ln\lim\limits_{x\to+\infty}\frac{x}{1+\underbrace{\frac{1}{x}}_{\to 0}}=\ln+\infty=+\infty \Rightarrow$ nema H. A.

kosa asimptota je oblika $y=kx+n$

$k=\lim\limits_{x\to\infty}\frac{f(x)}{x}$, $n=\lim\limits_{x\to\infty}(f(x)-kx)$

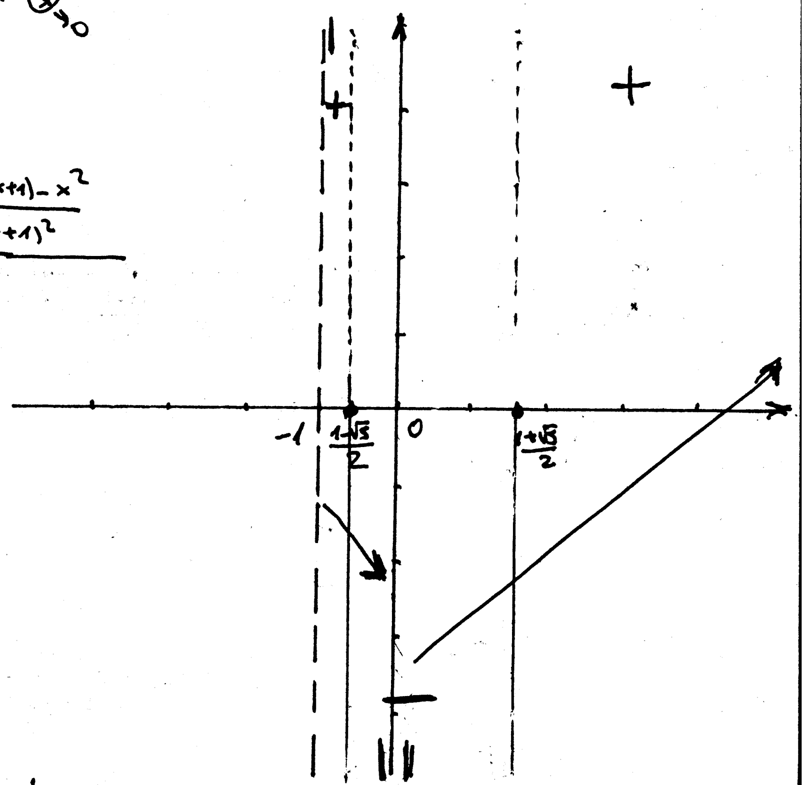
$k=\lim\limits_{x\to\infty}\frac{\ln\frac{x^2}{x+1}}{x}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim\limits_{x\to\infty}\frac{\frac{1}{\frac{x^2}{x+1}}\cdot\frac{2x(x+1)-x^2}{(x+1)^2}}{1}$

$=\lim\limits_{x\to\infty}\frac{\cancel{x+1}}{x^2}\cdot\frac{2x^2+2x-x^2}{(x+1)^{\cancel{2}}}$

$=\lim\limits_{x\to\infty}\frac{x^2+2x \;:x^2}{x^3+x^2 \;:x^2}=\lim\limits_{x\to\infty}\frac{1+\overbrace{\frac{2}{x}}^{\to 0}}{x+1}=0$

$k=0 \Rightarrow$ f-ja nema K. A.

počinjemo sa skiciranjem grafa:



rast i opadanje

$$y' = \frac{1}{\frac{x^2}{x+1}}\left(\frac{x^2}{x+1}\right)' = \frac{x^2+2x}{x^3+x^2} = \frac{x+2}{x(x+1)}$$

$y'=0$ akko $x+2=0$

$x=-2$

$-2 \quad -1 \quad 0$ ← prekidi y + nule y'

x	$(-1,0)$	$(0,+\infty)$
y'	−	+
y	↘	↗

rast i opadanje

ekstremi f-je

$x=-2$ je stacionarna tačka

$f(-2)$ nije definisano $\Rightarrow$

$\Rightarrow$ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y' = \frac{x+2}{x^2+x}, \quad y'' = \frac{x^2+x-(x+2)(2x+1)}{(x^2+x)^2} = \frac{x^2+x-(2x^2+x+4x+2)}{x^2(x+1)^2} =$$

$$= \frac{x^2+x-2x^2-5x-2}{x^2(x+1)^2} = \frac{-x^2-4x-2}{x^2(x+1)^2}$$

$y''=0$ akko $-x^2-4x-2=0$

$D=16-8=8$

$$x_{1,2} = \frac{4\pm\sqrt{8}}{-2}$$

$$x_1 = \frac{4-2\sqrt{2}}{-2} = -2+\sqrt{2} \approx -0{,}59$$

$$x_2 = \frac{4+2\sqrt{8}}{-2} = -2-\sqrt{2} \approx -3{,}4$$

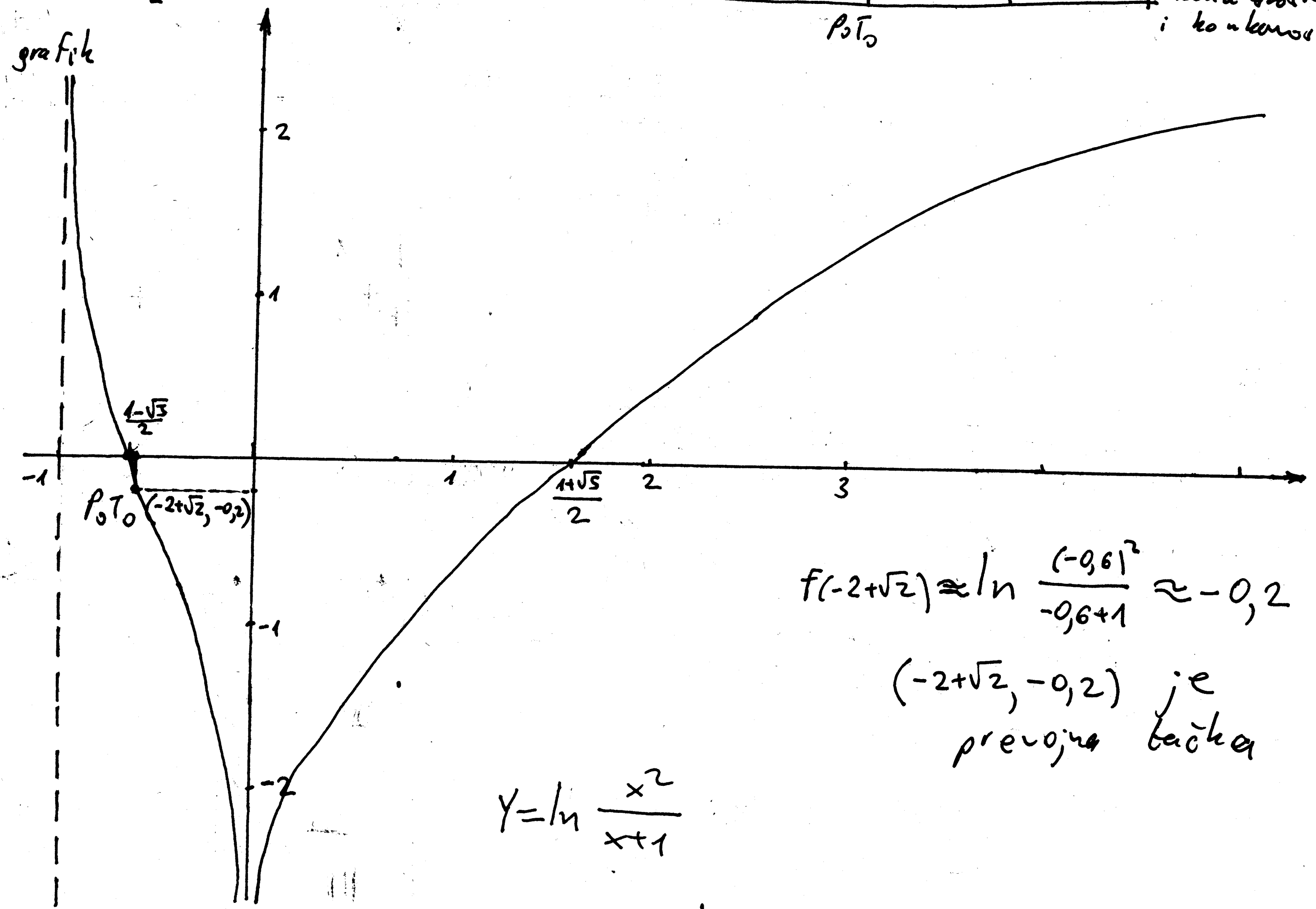
$-2-\sqrt{2} \quad -1 \quad -2+\sqrt{2} \quad 0$ ← prekidi y + nule y''

x	$(-1,-2+\sqrt{2})$	$(-2+\sqrt{2},0)$	$(0,+\infty)$
y''	+	−	−
y	∪	∩	∩

PoTo

intervali konveksnosti i konkavnosti

grafik



$$f(-2+\sqrt{2}) \approx \ln\frac{(-0{,}6)^2}{-0{,}6+1} \approx -0{,}2$$

$(-2+\sqrt{2},-0{,}2)$ je prevojna tačka

Izračunati integral $I=\int \frac{dx}{3\cos^2 x+4\sin^2 x}$.

Rj. $\operatorname{tg} x = t$

$x = \operatorname{arc\,tg} t$

$dx = \frac{dt}{1+t^2}$

$$\sin^2 x = \frac{\sin^2 x \quad :\cos^2 x}{\sin^2 x+\cos^2 x \quad :\cos^2 x} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x+1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x \quad :\cos^2 x}{\sin^2 x+\cos^2 x \quad :\cos^2 x} = \frac{1}{\operatorname{tg}^2 x+1} = \frac{1}{1+t^2}$$

$$I=\int \frac{dx}{3\cos^2 x+4\sin^2 x} = \left| \begin{array}{ll} \operatorname{tg} x = t & \sin^2 x = \frac{t^2}{1+t^2} \\ dx = \frac{dt}{1+t^2} & \cos^2 x = \frac{1}{1+t^2} \end{array} \right| =$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{3}{1+t^2}+\frac{4t^2}{1+t^2}} = \int \frac{\frac{dt}{1+t^2}}{\frac{3+4t^2}{1+t^2}} = \int \frac{dt}{3+4t^2} = \int \frac{dt}{(\sqrt{3})^2+(2t)^2}$$

$$= \left| \begin{array}{l} 2t = \sqrt{3}\,u \\ 2\,dt = \sqrt{3}\,du \\ dt = \frac{\sqrt{3}}{2}\,du \\ u = \frac{2t}{\sqrt{3}} \end{array} \right| = \int \frac{\frac{\sqrt{3}}{2}\,du}{3+3u^2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{3}\int \frac{du}{1+u^2} = \frac{\sqrt{3}}{6}\operatorname{arc\,tg} u + C =$$

$$= \frac{\sqrt{3}}{6}\operatorname{arc\,tg}\frac{2t}{\sqrt{3}} + C = \frac{\sqrt{3}}{6}\operatorname{arc\,tg}\frac{2\operatorname{tg} x}{\sqrt{3}} + C$$

Izračunati integral $I=\int_{6-\sqrt{2}}^{7}\frac{(4x+2)}{\sqrt{-34+12x-x^2}}dx$.

Rj:

$$I_1=\int\frac{4x+2}{\sqrt{-x^2+12x-34}}dx$$

Metoda Ostrogradskog:

$$\int\frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx=g_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int\frac{4x+2}{\sqrt{-x^2+12x-34}}dx=a\sqrt{-x^2+12x-34}+\lambda\int\frac{dx}{\sqrt{-x^2+12x-34}}\quad\Big/\frac{d}{dx}$$

$$\frac{4x+2}{\sqrt{-x^2+12x-34}}=a\cdot\frac{(-2x+12)}{2\sqrt{-x^2+12x-34}}+\lambda\cdot\frac{1}{\sqrt{-x^2+12x-34}}\quad\Big/\cdot\sqrt{-x^2+12x-34}$$

$$4x+2=a(-x+6)+\lambda$$

$$4x+2=-ax+6a+\lambda$$

$$-a=4\qquad 6a+\lambda=2$$

$$a=-4\qquad -24+\lambda=2$$

$$\lambda=26$$

$$\int\frac{4x+2}{\sqrt{-x^2+12x-34}}dx=-4\sqrt{-x^2+12x-34}+26\int\frac{dx}{\sqrt{-x^2+12x-34}}$$

$$-x^2+12x-34=-(x^2-12x+34)=-(x^2-2\cdot6x+36-36+34)$$

$$=-((x-6)^2-2)=2-(x-6)^2$$

$$\int\frac{dx}{\sqrt{-x^2+12x-34}}=\int\frac{dx}{\sqrt{2-(x-6)^2}}=\left|\begin{array}{l}x-6=\sqrt{2}t\\ dx=\sqrt{2}dt\\ t=\frac{x-6}{\sqrt{2}}\end{array}\right|=\int\frac{\sqrt{2}dt}{\sqrt{2-2t^2}}=\frac{\sqrt{2}}{\sqrt{2}}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\arcsin t+c=\arcsin\left(\frac{x-6}{\sqrt{2}}\right)+C$$

$$\int_{6-\sqrt{2}}^{7}\frac{4x+2}{\sqrt{-34+12x-x^2}}dx=-4\sqrt{-x^2+12x-34}\Big|_{6-\sqrt{2}}^{7}+26\arcsin\frac{x-6}{\sqrt{2}}\Big|_{6\sqrt{2}}^{7}=$$

$$=-4\left(\sqrt{-49+84-34}-\sqrt{-(36-\underline{12\sqrt{2}}+2)+72-\underline{12\sqrt{2}}-34}\right)+26\left(\arcsin\frac{1}{\sqrt{2}}-\arcsin\left(-\frac{\sqrt{2}}{\sqrt{2}}\right)\right)$$

$$=-4(\sqrt{1}-\sqrt{0})+26\left(\frac{\pi}{4}-\left(-\frac{\pi}{2}\right)\right)=-4+26\cdot\frac{3\pi}{4}=-4+\frac{39\pi}{2}$$

(#) Izračunati integral $I=\int_0^1 \sqrt{4-x^2}\,dx$.

Rj: Metoda Ostrogradskog

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = g_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int\sqrt{4-x^2}\,dx = \int\frac{4-x^2}{\sqrt{4-x^2}}dx = (ax+b)\sqrt{4-x^2}+\lambda\int\frac{dx}{\sqrt{4-x^2}} \quad \Big/ \frac{d}{dx}$$

$$\sqrt{4-x^2} = a\sqrt{4-x^2}+(ax+b)\frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2}}+\lambda\cdot\frac{1}{\sqrt{4-x^2}} \quad \Big/ \sqrt{4-x^2}$$

$$4-x^2 = a(4-x^2)-ax^2-bx+\lambda$$

$x^2:\ -a-a=-1$

$-2a=-1$

$a=\frac{1}{2}$

$x:\ -b=0$

$b=0$

$x^0:\ 4a+\lambda=4$

$4\cdot\frac{1}{2}+\lambda=4$

$2+\lambda=4$

$\lambda=2$

$$\int\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2}+2\int\frac{dx}{\sqrt{4-x^2}}$$

$$\int\frac{dx}{\sqrt{4-x^2}} = \left|\begin{matrix}x=2t\\ dx=2dt\\ t=\frac{x}{2}\end{matrix}\right| = \int\frac{2dt}{\sqrt{4-4t^2}} = \frac{2}{\sqrt{4}}\int\frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin\frac{x}{2}+c$$

$$\int\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2}+2\arcsin\frac{x}{2}+c$$

$$\int_0^1\sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2}\Big|_0^1+2\arcsin\frac{x}{2}\Big|_0^1 = \frac{1}{2}\sqrt{3}+2\arcsin\frac{1}{2} -$$

$$-(0+2\arcsin 0) = \frac{\sqrt{3}}{2}+2\cdot\frac{\pi}{6} = \frac{\sqrt{3}}{2}+\frac{\pi}{3}$$ tražena vrijednost

(#) Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj.

$y=1-x^2$ $\qquad$ $1-x^2=0$ $\qquad$ $y=-x^2+1$ $\qquad$ parabola izgleda

$y(0)=1$ $\qquad$ $x^2=1$ $\qquad$ $T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$(0,1)$ je presjek sa y-osom $\qquad$ $x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je $\qquad$ $-\frac{b}{2a}=-\frac{0}{2\cdot(-1)}=0$

$D=0-4(-1)(1)=4$

$y-y_1=y'(x_1)(x-x_1)$ jednačina tangente u tački (x_1,y_1) $\qquad$ $-\frac{D}{4a}=-\frac{4}{4\cdot(-1)}=1$

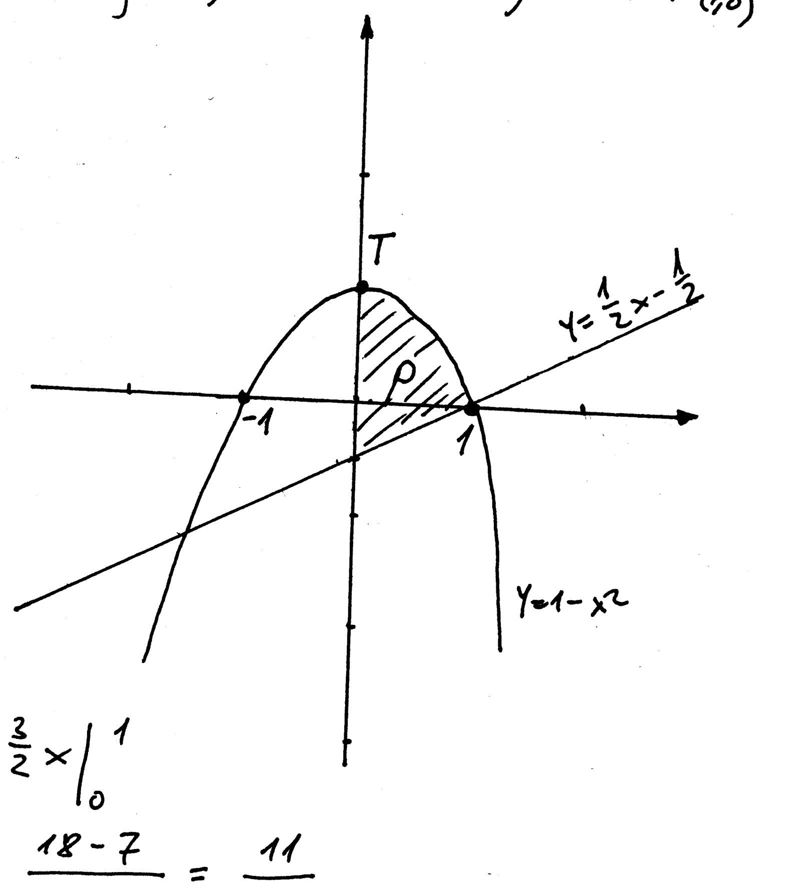
$y-y_1=-\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,y_1) $\qquad$ $T(0,1)$

$y'=-2x$ $\qquad$ presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$ (x_1, y_1)

$y'(1)=-2$

$y-0=-\frac{1}{-2}(x-1)$

$y=\frac{1}{2}x-\frac{1}{2}$ jednačina normale u tački $(1,0)$



$$P=\int_0^1\left[(1-x^2)-\left(\frac{1}{2}x-\frac{1}{2}\right)\right]dx=$$

$$=\int_0^1\left(-x^2-\frac{1}{2}x+\frac{3}{2}\right)dx=-\frac{1}{3}x^3\Big|_0^1-\frac{1}{4}x^2\Big|_0^1+\frac{3}{2}x\Big|_0^1$$

$$=-\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 3}{4\cdot 3}+\frac{3}{2}=\frac{3\cdot 6}{2\cdot 6}-\frac{7}{12}=\frac{18-7}{12}=\frac{11}{12}$$

$P=\frac{11}{12}$ tražena površina

Odrediti ekstremne vrijednosti f-je $z=8x^3-y^3+6xy+7$.

Rj. $z=8x^3-y^3+6xy+7$

$$\frac{\partial z}{\partial x}=24x^2+6y$$

$$\frac{\partial z}{\partial y}=-3y^2+6x$$

$$24x^2+6y=0 \quad /:6$$
$$6x-3y^2=0 \quad /:3$$

$$4x^2+y=0$$
$$2x-y^2=0$$

$$y=-4x^2$$
$$2x-y^2=0$$

$$2x-(-4x^2)^2=0$$
$$2x-16x^4=0$$
$$2x(1-8x^3)=0$$
$$2x(1-2x)(1+2x+4x^2)=0$$
$$x_1=0, \quad x_2=\frac{1}{2}$$

$$y=-4x^2$$
$$x_1=0 \Rightarrow y_1=0$$
$$x_2=\frac{1}{2} \Rightarrow y_2=-4\cdot\frac{1}{4}=-1$$

Stacionarne tačke su $(0,0)$ i $(\frac{1}{2},-1)$

$$\frac{\partial^2 z}{\partial x^2}=48x$$

$$\frac{\partial^2 z}{\partial x\partial y}=6$$

$$\frac{\partial^2 z}{\partial y^2}=-6y$$

$$D=AC-B^2$$

$M_1(0,0)$

$A=0$

$B=6$, $D=0-36=-36<0$

$C=0$

f-ja nema ekstrem u tački M_1.

$M_2(\frac{1}{2},-1)$

$A=24$, $B=6$, $C=6$, $D=84-36=48>0$ f-ja z ima ekstrem u tački $M_2(\frac{1}{2},-1)$

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(\tfrac{1}{2},-1)=8\cdot\frac{1}{8}-(-1)+6\cdot\frac{1}{2}\cdot(-1)+7=$$
$$=1+1-3+7=6$$

$z_{min}=6$ u tački $M_2(\frac{1}{2},-1)$

(#) Odrediti ekstremne vrijednosti f-je

$$z=\frac{xy}{2}+(47-x-y)\left(\frac{x}{3}+\frac{y}{4}\right).$$

Rj.

$$\frac{\partial z}{\partial x}=\frac{1}{2}y+(-1)\left(\frac{x}{3}+\frac{y}{4}\right)+(47-x-y)\cdot\frac{1}{3}=\underline{\underline{\frac{1}{2}y}}\underline{-\frac{1}{3}x}\underline{\underline{-\frac{1}{4}y}}+\frac{47}{3}\underline{-\frac{1}{3}x}\underline{\underline{-\frac{1}{3}y}}$$

$$=-\frac{2}{3}x+\frac{6-3-4}{12}y+\frac{47}{3}=-\frac{2}{3}x-\frac{1}{12}y+\frac{47}{3}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2}x+(-1)\left(\frac{x}{3}+\frac{y}{4}\right)+(47-x-y)\cdot\frac{1}{4}=\underline{\frac{1}{2}x}\underline{-\frac{1}{3}x}\underline{\underline{-\frac{1}{4}y}}+\frac{47}{4}\underline{-\frac{1}{4}x}\underline{\underline{-\frac{1}{4}y}}$$

$$=-\frac{1}{12}x-\frac{1}{2}y+\frac{47}{4}$$

$$-\frac{2}{3}x-\frac{1}{12}y+\frac{47}{3}=0\quad /\cdot 12$$

$$-\frac{1}{12}x-\frac{1}{2}y+\frac{47}{4}=0\quad /\cdot 12$$

$$-8x-y+188=0$$

$$-x-6y+141=0$$

$$-8x-y+188=0$$

$$x=-6y+141$$

$$-8(-6y+141)-y+188=0$$

$$48y-1128-y+188=0$$

$$47y=940$$

$$y=20$$

$$x=-6y+141=-120+141=21$$

Stacionarna tačka je $M(21,20)$.

$$\frac{\partial^2 z}{\partial x^2}=-\frac{2}{3}$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{12}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{1}{2}$$

$$D=AC-B^2$$

$M(21,20)$

$$A=-\frac{2}{3},\quad B=-\frac{1}{12},\quad C=-\frac{1}{2}$$

$$D=\frac{2}{6}-\frac{1}{144}=\frac{1}{3}-\frac{1}{144}>0$$

f-ja z ima ekstrem

$A<0$ f-ja ima maksimum

$$z_{max}(21,20)=21\cdot10+(47-41)(7+5)=210+6\cdot12=210+72=282$$

$z_{max}(21,20)=282$ traženi ekstrem f-je

(#) Riješiti diferencijalnu jednačinu $y' = y^4\cos x + y\,tg\,x$,

Rj: $y' - y\,tg\,x = \cos x\; y^4$ ovo je Bernulijeva diferencijalna jednačina, uvodimo smjenu $y = uv$, $y' = u'v + uv'$.

$u'v + uv' - uv\,tg\,x = u^4v^4\cos x$

$u'v + u\underbrace{(v' - v\,tg\,x)}_{=0} = u^4v^4\cos x$

$$\int tg\,x\,dx = \left| \begin{array}{l} tg\,x = t \\ x = arctg\,t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{t}{1+t^2}\,dt =$$

$$= \left| \begin{array}{l} 1+t^2 = s \\ 2t\,dt = ds \\ t\,dt = \frac{ds}{2} \end{array} \right| = \frac{1}{2}\int \frac{ds}{s} = \frac{1}{2}\ln|s| + C = \frac{1}{2}\ln|1+t^2| + C$$

$$= \frac{1}{2}\ln\left|1 + \frac{\sin^2 x}{\cos^2 x}\right| + C = \frac{1}{2}\ln\left|\frac{1}{\cos^2 x}\right| + C = \ln\frac{1}{\cos x} + C$$

$v' - v\,tg\,x = 0$

$v' = v\,tg\,x$

$\frac{dv}{dx} = v\,tg\,x$

$\frac{dv}{v} = tg\,x\,dx \quad / \int$

$\ln v = \ln\frac{1}{\cos x}$

$v = \frac{1}{\cos x}$

$u'v = u^4v^4\cos x \quad /:v$

$u' = u^4v^3\cos x$

$u' = u^4 \cdot \frac{1}{\cos^2 x}$

$\frac{u'}{u^4} = \frac{1}{\cos^2 x}$, $u' = \frac{du}{dx}$

$\frac{du}{u^4} = \frac{dx}{\cos^2 x} \quad / \int$

$\int \frac{du}{u^4} = \int \frac{dx}{\cos^2 x}$

$\int u^{-4}\,du = \int \frac{dx}{\cos^2 x}$

$\frac{u^{-3}}{-3} = tg\,x + C_1$

$\frac{1}{u^3} = -3\,tg\,x + C$

$$\frac{1}{v} = \cos x$$

$$\frac{1}{u^3} = -3\cdot\frac{\sin x}{\cos x} + C$$

$$\frac{1}{v^3} = \cos^3 x$$

$$\frac{1}{u^3} = -3\,\frac{\sin x}{\cos x} + C$$

$$\frac{1}{y^3} = \frac{1}{u^3v^3} = -3\,\frac{\sin x}{\cos x}\cdot\cos^3 x + C\cdot\cos^3 x$$

$$y^{-3} = -3\sin x\cos^2 x + C\cos^3 x$$

rješenje diferencijalne jednačine

Riješiti diferencijalnu jednačinu $y' + \frac{1}{y'} = \frac{y}{x}$.

Rj.
$$y' + \frac{1}{y'} = \frac{y}{x} \quad /\cdot x$$

$$y = xy' + \frac{x}{y'}$$ uvodimo smjenu $y' = p$

$$y = xp + \frac{x}{p} \quad / \frac{d}{dx}$$

$$y' = p + xp' + \frac{p - xp'}{p^2}$$ (kako je $y' = p$ imamo)

$$p = p + xp' + \frac{1}{p^2}(p - xp')$$

$$xp' + \frac{1}{p} - \frac{xp'}{p^2} = 0$$

$$\left(x - \frac{x}{p^2}\right)p' = -\frac{1}{p} \quad /\cdot p$$

$$\left(px - \frac{x}{p}\right)p' = -1 \quad /\cdot \frac{1}{p'}$$

$$-\frac{1}{p'} = px - \frac{1}{p}x$$

$$-\frac{1}{p'} = \left(p - \frac{1}{p}\right)x$$

znamo da je $\frac{1}{p'} = \frac{1}{\frac{dp}{dx}} = \frac{dx}{dp} = x'$

pa imamo

$$-x' = \left(p - \frac{1}{p}\right)x \quad /\cdot(-1)$$

$$x' = \left(\frac{1}{p} - p\right)x$$ ovo je diferencijalna jednačina sa razdvojenim promjenjivim

$$x' = \left(\frac{1}{p} - p\right)x$$

$$\frac{dx}{dp} = \left(\frac{1}{p} - p\right)x$$

$$\frac{dx}{x} = \left(\frac{1}{p} - p\right)dp \quad /\int$$

$$\int\frac{dx}{x} = \int\left(\frac{1}{p} - p\right)dp$$

$$\ln|x| = \ln|p| - \frac{p^2}{2} + C_1$$

$$\ln|x| = \ln|p| + \ln e^{-\frac{p^2}{2}} + \ln C$$

$$x = pCe^{-\frac{p^2}{2}}$$

$$y = xp + \frac{x}{p} = Cp\,e^{-\frac{p^2}{2}}\cdot p + \frac{pCe^{-\frac{p^2}{2}}}{p}$$

$$y = Cp^2e^{-\frac{p^2}{2}} + Ce^{-\frac{p^2}{2}}$$

$$y = Ce^{-\frac{p^2}{2}}(p^2+1)$$

$$\left.\begin{array}{l} x = pCe^{-\frac{p^2}{2}} \\ y = Ce^{-\frac{p^2}{2}}(p^2+1) \end{array}\right\}$$ opšte rješenje