Univerzitet u Zenici
Ekonomski fakultet
Odsjek: Menadžment preduzeća, Računovodstveni i revizijski menadžment
Zenica, 02.02.2010.

Pismeni ispit iz predmeta Matematika

1. Dokazati matematičkom indukcijom da važi:

$$1 - x + x^2 - x^3 + ... + (-1)^{n-1}x^{n-1} = \frac{1 + (-1)^{n-1}x^n}{1 + x} \ (x \in \mathbb{R},\ n \in \mathbb{N}).$$

2. Naći sve racionalne članove u razvoju binoma $(\sqrt[6]{x} - \sqrt[9]{x})^{42}$.

3. Riješiti jednačinu u skupu kompleksnih brojeva: $(2 + 5i)z^3 - 2i + 5 = 0$.

4. Diskutovati rang matrice $\begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix}$ za razne vrijednosti parametra t.

5. Riješiti matričnu jednačinu $XAB = C$, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $C = [0 \quad 4 \quad 4]$.

6. Riješiti sistem linearnih jednačina:

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \\ -x_2 + x_3 - x_4 &= 1 \ . \end{aligned}$$

7. Ispitati funkciju i nacrtati joj grafik: $y = \frac{x^3 - 3x}{x^2 - 1}$.

8. Grafik kvadratne funkcije $f(x) = ax^2 + bx + c$ prolazi kroz tačke $A(-1, 14)$, $B(2, -4)$ i $C(-2, 24)$. Izračunati konstante a, b, c, pa zatim ispitati funkciju $y = \frac{f(x)}{x-10}$ i nacrtati joj grafik.

9. Ispitati funkciju i nacrtati joj grafik: $y = (x + 3)e^{\frac{1}{x+1}}$.

10. Ispitati funkciju i nacrtati joj grafik: $y = \frac{1}{x}e^{-\frac{1}{x^2}}$.

11. Ispitati funkciju i nacrtati joj grafik: $y = 2xln(e - \frac{2}{x})$ bez analize zbnaka prvog i drugog izvoda.

12. Ispitati funkciju i nacrtati joj grafik: $y = ln\frac{1 + x^3}{1 - x^3}$

13. Izračunati integral $\int \frac{x}{(x^2 - 2x + 2)^2}dx$.

14. Izračunati integral $\int x^3 \sqrt[2]{1+a^2x^2}\, dx$.

15. Izračunati integral $\int x\sqrt{1-x^4}\, dx$.

16. Izračunati integral $\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\, dx$.

17. Izračunati površinu figure koja je određena linijama $y=-2$, $y=x^3+x$, $x+y=3$.

18. Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

19. Naći ekstreme funkcije $z=\frac{2x+2y-1}{\sqrt[2]{x^2+y^2+1}}$.

20. Naći uslovne ekstreme funkcije $z=2x+4y$, ako je $\frac{2}{x}+\frac{4}{y}=3$.

21. Naći ekstreme funkcije $z=\frac{4}{x}+\frac{4}{y}+(x+y)^2$.

22. Riješiti diferencijalnu jednačinu $(x^2y+x^2)dx+(x^4y-y)dy=0$.

23. Riješiti diferencijalnu jednačinu $(5y+7x)dy+(8y+10)dx=0$.

24. Riješiti diferencijalnu jednačinu $(3y^2+3xy+x^2)dx=(x^2+2xy)dy$.

Ovi zadaci su skinuti sa stranice **pf.unze.ba\nabokov**.
Za uočene greške pisati na **infoarrt@gmail.com**.

(#) Dokazati matematičkom indukcijom da važi

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}=\frac{1+(-1)^{n-1}x^n}{1+x} \quad (x\in\mathbb{R}, n\in\mathbb{N}).$$

Rj: BAZA INDUKCIJE

Dokažimo da je jednakost tačna za broj 1

$$1=\frac{1+(-1)^0x^1}{1+x}=\frac{1+x}{1+x}=1$$

Jednakost je tačna za broj 1.

KORAK INDUKCIJE

Pretpostavimo da je jednakost $1-x+x^2-\ldots+(-1)^{k-1}x^{k-1}=\frac{1+(-1)^{k-1}x^k}{1+x}$ tačna za sve brojeve k od 1 do n i na osnovu ove pretpostavke dokažimo da je jednakost tačna za $n+1$ tj. dokažimo $1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+(-1)^nx^n=\frac{1+(-1)^nx^{n+1}}{1+x}$

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+(-1)^nx^n \xlongequal[\text{pretpostavke}]{\text{na osnovu}} \frac{1+(-1)^{n-1}x^n}{1+x}+(-1)^nx^n=$$

$$=\frac{1+(-1)^{n-1}x^n+(-1)^nx^n\cdot(1+x)}{1+x}=\frac{1+\left[(-1)^{n-1}+(-1)^n(1+x)\right]x^n}{1+x}=$$

$$=\frac{1+\left[(-1)^{n-1}(1+(-1)(1+x)\right]x^n}{1+x}=\frac{1+\left[(-1)^{n-1}\cdot(1-1-x\right]x^n}{1+x}=$$

$$=\frac{1+(-1)^{n-1}\cdot(-1)x\cdot x^n}{1+x}=\frac{1+(-1)^nx^{n+1}}{1+x}$$

što je i trebalo dobiti

Jednakost je tačna za $n+1$

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve.

(#) Naći sve racionalne članove u razvoju binoma
$(\sqrt[6]{x}-\sqrt[9]{x})^{42}$.

Rj.

$$(\sqrt[6]{x}-\sqrt[9]{x})^{42}=\sum_{k=0}^{42}\binom{42}{k}(\sqrt[6]{x})^{42-k}(\sqrt[9]{x})^{k}=\sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}}\cdot x^{\frac{k}{9}}=$$

$$=\sum_{k=0}^{42}\binom{42}{k}x^{7-\frac{k}{6}+\frac{k}{9}}$$

Da bi član u razvoju našeg binoma bio racionalan potrebno je i dovoljno da je $7-\frac{k}{6}+\frac{k}{9}$ cio broj. tj. da su $\frac{k}{6}$ i $\frac{k}{9}$ cijeli brojevi

$\frac{k}{6}$ je cio broj ako je $k\in\{0,6,12,18,24,30,36,42\}$

$\frac{k}{9}$ je cio broj ako je $k\in\{0,9,18,27,36\}$

Racionalni članovi u razvoju binoma su za vrijednost
$k=0$, $k=18$ i $k=36$.

Prvi, devetnaesti i tridesetsedmi član u razvoju binoma je racionalan.

(#) Riješiti jednačinu u skupu kompleksnih brojeva:

$$(2+5i)z^3-2i+5=0.$$

Rj.

$$(2+5i)z^3-2i+5=0$$

$$(2+5i)z^3=2i-5$$

$$z^3=\frac{(2i-5)\cdot(2-5i)}{(2+5i)\cdot(2-5i)}=\frac{4i-10i^2-10+25i}{4-25i^2}=\frac{29i}{29}$$

$$z^3=i$$

$$z=\sqrt[3]{i}$$

Jednačina $z^n=w$ gdje je w kompleksan broj ima n rješenja koje tražimo u obliku

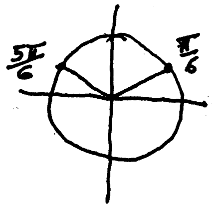
$$z_k=\sqrt[n]{|w|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$$

U našem slučaju $w=i$, $w=a+bi$ $k=0,1,\dots,k-1$

$$|w|=\sqrt{a^2+b^2}=\sqrt{1}=1$$

$$\cos\varphi=\frac{a}{|z|}=0,\quad \sin\varphi=\frac{b}{|z|}=\frac{1}{1}=1 \Rightarrow \varphi=\frac{\pi}{2}$$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$



$$z_0=1\cdot\left(\cos\frac{\frac{\pi}{2}}{3}+i\sin\frac{\pi}{6}\right)=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_1=1\left(\cos\frac{\frac{\pi}{2}+2\pi}{3}+i\sin\frac{\frac{\pi}{2}+2\pi}{3}\right)=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$z_2=1\left(\cos\frac{\frac{\pi}{2}+4\pi}{3}+i\sin\frac{\frac{\pi}{2}+4\pi}{3}\right)=\cos\frac{9\pi}{6}+i\sin\frac{3\pi}{2}=-i$$

Rješenja jednačine u skupu kompleksnih brojeva su $z_0=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z_1=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ i $z_2=-i$.

(#) Diskutovati rang matrice $\begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix}$ za razne vrijednosti parametra t.

Rj.

$$M = \begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix} \overset{III_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 & t \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 0 & 4 & -3 & 0 \end{bmatrix} \overset{I_v \leftrightarrow IV_v}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 2 & -1 & 0 & t \end{bmatrix} \underset{IV_v - I_v}{\overset{II_v - I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix}$$

$$\overset{IV_v + II_v \cdot 2}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & -9 & 11 & t \end{bmatrix} \overset{IV_v - III_v \cdot \frac{3}{2}}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & t \end{bmatrix}$$

$$-9 + 6 \cdot \frac{3}{2} = -9 + 9 = 0$$

$$11 - 7 \cdot \frac{3}{2} = \frac{22}{2} - \frac{21}{2} = \frac{1}{2}$$

Bez obzira na vrijednost parametra t rang matrice M je uvijek 4.

(#) Riješiti matričnu jednačinu $XAB=C$, $A=\begin{bmatrix}1&0&1\\2&1&1\\0&1&1\end{bmatrix}$,

$B=\begin{bmatrix}1&1&-1\\0&1&2\\-1&1&1\end{bmatrix}$, $C=\begin{bmatrix}0&4&4\end{bmatrix}$.

Rj.

$XAB=C$ $/\cdot(AB)^{-1}$ sa desne strane

$X(AB)(AB)^{-1}=C\cdot(AB)^{-1}$

$X=C\cdot(AB)^{-1}$

$$AB=\begin{bmatrix}1&0&1\\2&1&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&-1\\0&1&2\\-1&1&1\end{bmatrix}=\begin{bmatrix}0&2&0\\1&4&1\\-1&2&3\end{bmatrix}$$

$$\det(AB)=\begin{vmatrix}0&2&0\\1&4&1\\-1&2&3\end{vmatrix}=(-2)\begin{vmatrix}1&1\\-1&3\end{vmatrix}=(-2)(3+1)=-8$$

AB označimo sa M, nađimo M^{-1}

$M_{11}=(-1)^2\begin{vmatrix}4&1\\2&3\end{vmatrix}=10$ $\quad M_{21}=(-1)^3\begin{vmatrix}2&0\\2&3\end{vmatrix}=-6$ $\quad M_{31}=(-1)^4\begin{vmatrix}2&0\\4&1\end{vmatrix}=2$

$M_{12}=(-1)^3\begin{vmatrix}1&1\\-1&3\end{vmatrix}=-4$ $\quad M_{22}=(-1)^4\begin{vmatrix}0&0\\-1&3\end{vmatrix}=0$ $\quad M_{32}=(-1)^5\begin{vmatrix}0&0\\1&1\end{vmatrix}=0$

$M_{13}=(-1)^4\begin{vmatrix}1&4\\-1&2\end{vmatrix}=6$ $\quad M_{23}=(-1)^5\begin{vmatrix}0&2\\-1&2\end{vmatrix}=-2$ $\quad M_{33}=(-1)^6\begin{vmatrix}0&2\\1&4\end{vmatrix}=-2$

$$M_{kof}=\begin{bmatrix}10&-4&6\\-6&0&-2\\2&0&-2\end{bmatrix},\quad M_{kof}^{T}=\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}$$

$$(AB)^{-1}=\frac{1}{\det M}\cdot M_{kof}^{T}=\frac{-1}{8}\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}=\begin{bmatrix}-5/4&3/4&-1/4\\1/2&0&0\\-3/4&1/4&1/4\end{bmatrix}$$

$$X=C\cdot(AB^{-1})=\begin{bmatrix}0&4&4\end{bmatrix}\cdot\left(-\frac{1}{8}\right)\begin{bmatrix}10&-6&2\\-4&0&0\\6&-2&-2\end{bmatrix}=\left(-\frac{1}{8}\right)\begin{bmatrix}8&-8&-8\end{bmatrix}$$

$X=\begin{bmatrix}-1&1&1\end{bmatrix}$ rješenje matrične jednačine

(#) Riješiti sistem linearnih jednačina

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Rj: Riješimo sistem Gausovom metodom:

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 & (a) \\ -2x_1 + x_2 - x_3 - 4x_4 &= 0 & (b) \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 &= 2 & (c) \\ -x_2 + x_3 - x_4 &= 1 & (d) \end{aligned}$$

$$\begin{aligned} (a):&\quad 2x_1 - 2x_2 + 2x_3 + 3x_4 = 1 \\ (b)+(a):&\quad -x_2 + x_3 - x_4 = 1 \\ (c)-(a):&\quad -x_2 + x_3 - x_4 = 1 \\ &\quad -x_2 + x_3 - x_4 = 1 \end{aligned}$$

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 3x_4 &= 1 \\ -x_2 + x_3 - x_4 &= 1 \end{aligned}$$

Imamo dvije linearne jednačine sa četri nepoznate $\Rightarrow$

$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $x_3 = s$, $x_4 = t$

$$x_2 = s - t - 1$$

$$2x_1 = 1 + 2x_2 - 2x_3 - 3x_4$$

$$2x_1 = 1 + \underline{2s} \underline{\underline{-2t}} - 2 \underline{-2s} \underline{\underline{-3t}}$$

$$2x_1 = -5t - 1$$

$$x_1 = \frac{-5}{2}t - \frac{1}{2}$$

Rješenje sistema linearnih jednačina je

$$\left(\frac{-5}{2}t - \frac{1}{2}, s - t - 1, s, t\right)$$

(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{x^3-3x}{x^2-1}$.

Rj: definiciono područje

$x^2-1\neq 0$

$x^2\neq 1$

$x\neq -1 \quad x\neq 1$

$\mathcal{D}: x\in(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$

parnost(neparnost), periodičnost

$f(-x)=\frac{(-x)^3-3(-x)}{(-x)^2-1}=$

$=\frac{-x^3+3x}{x^2-1}=-\frac{x^3-3x}{x^2-1}=-f(x)$

f-ja je neparna

f-ja nije periodična

nule, presek grafa sa Y-osom, ~~znak~~

$y=0$ akko $x^3-3x=0$ tj:

$x(x^2-3)=0 \qquad \sqrt{3}\approx 1,7321$

$x(x-\sqrt{3})(x+\sqrt{3})=0$

$(\sqrt{3},0)$ i $(-\sqrt{3},0)$ su nule f-je

$(0,0)$ je nula f-je i presek sa Y-osom

Kako je f-ja neparna dovoljno jer je ispitati za $x>0$

x	$(0,1)$	$(1,\sqrt{3})$	$(\sqrt{3},+\infty)$
$x-1$	$-$	$+$	$+$
$x+1$	$+$	$+$	$+$
x	$+$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$+$
$x+\sqrt{3}$	$+$	$+$	$+$
y	$+$	$-$	$+$

← prekidi od y + nule od y

Znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ f-ja nije definisana

$\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}\frac{x(x^2-3)}{x^2-1}=$

$=\frac{(1-0)((1-0)^2-3)}{(1-0)^2-1}=\frac{(1-0)(1-0-3)}{1-0-1}=+\infty \Rightarrow x=1$ je vertikalna asimptota

$\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}\frac{x(x^2-3)}{x^2-1}=\frac{(1+0)(1+0-3)}{1+0-1}=-\infty \Rightarrow x=1$ je V. A.

$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^3-3x \;/:x^3}{x^2-1\;/:x^3}=\lim_{x\to\pm\infty}\frac{1-\cancel{\frac{3}{x^2}}\to 0}{\cancel{\frac{1}{x}}\to 0-\cancel{\frac{1}{x^3}}\to 0}=\pm\infty \Rightarrow$ f-ja nema H. A.

Tražimo kosu asimptotu u obliku $y=kx+n$

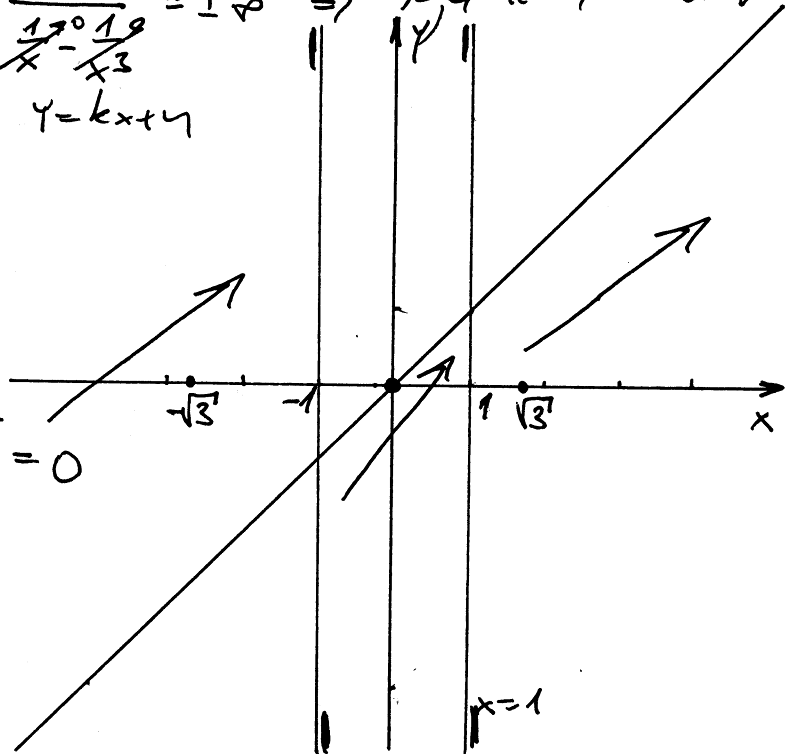
$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^3-3x\;/:x^3}{x^3-x\;/:x^3}=1$

$n=\lim_{x\to\infty}[f(x)-kx]=\lim_{x\to\infty}\left[\frac{x^3-3x}{x^2-1}-x\right]=$

$=\lim_{x\to\infty}\frac{x^3-3x-x^3+x}{x^2-1}=\lim_{x\to\infty}\frac{-2x\;/:x^2}{x^2-1\;/:x^2}=0$

$y=x$ je kosa asimptota

Nakon ovog koraka počinjemo sa skiciranjem grafa f-je



rast i opadanje

$$y' = \left(\frac{x^3-3x}{x^2-1}\right)' = \frac{(3x^2-3)(x^2-1)-(x^3-3x)(2x)}{(x^2-1)^2} = \frac{3x^4\boxed{-3x^2}\boxed{-3x^2}+3-2x^4\boxed{+6x^2}}{(x^2-1)^2}$$

$$y' = \frac{x^4+3}{(x^2-1)^2} \qquad y' \neq 0 \text{ za } \forall x \in D$$

f-ja nema ekstrema

$y' > 0$ za $\forall x \in D$ f-ja ↗ za $\forall x \in D$

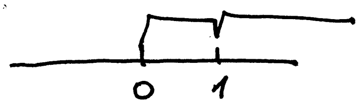
ekstremi f-je

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{x^4+3}{(x^2-1)^2}\right)' = \frac{4x^3\cdot(x^2-1)^{\cancel{2}\,1}-(x^4+3)\,2\cancel{(x^2-1)}\cdot 2x}{(x^2-1)^{\cancel{4}\,3}} = \frac{\cancel{4x^5}-4x^3-\cancel{4x^5}-12x}{(x^2-1)^3}$$

$$y'' = \frac{-4x(x^2+3)}{(x^2-1)^3} \qquad y''=0 \text{ akko } -4x(x^2+3)=0$$

$$x=0$$



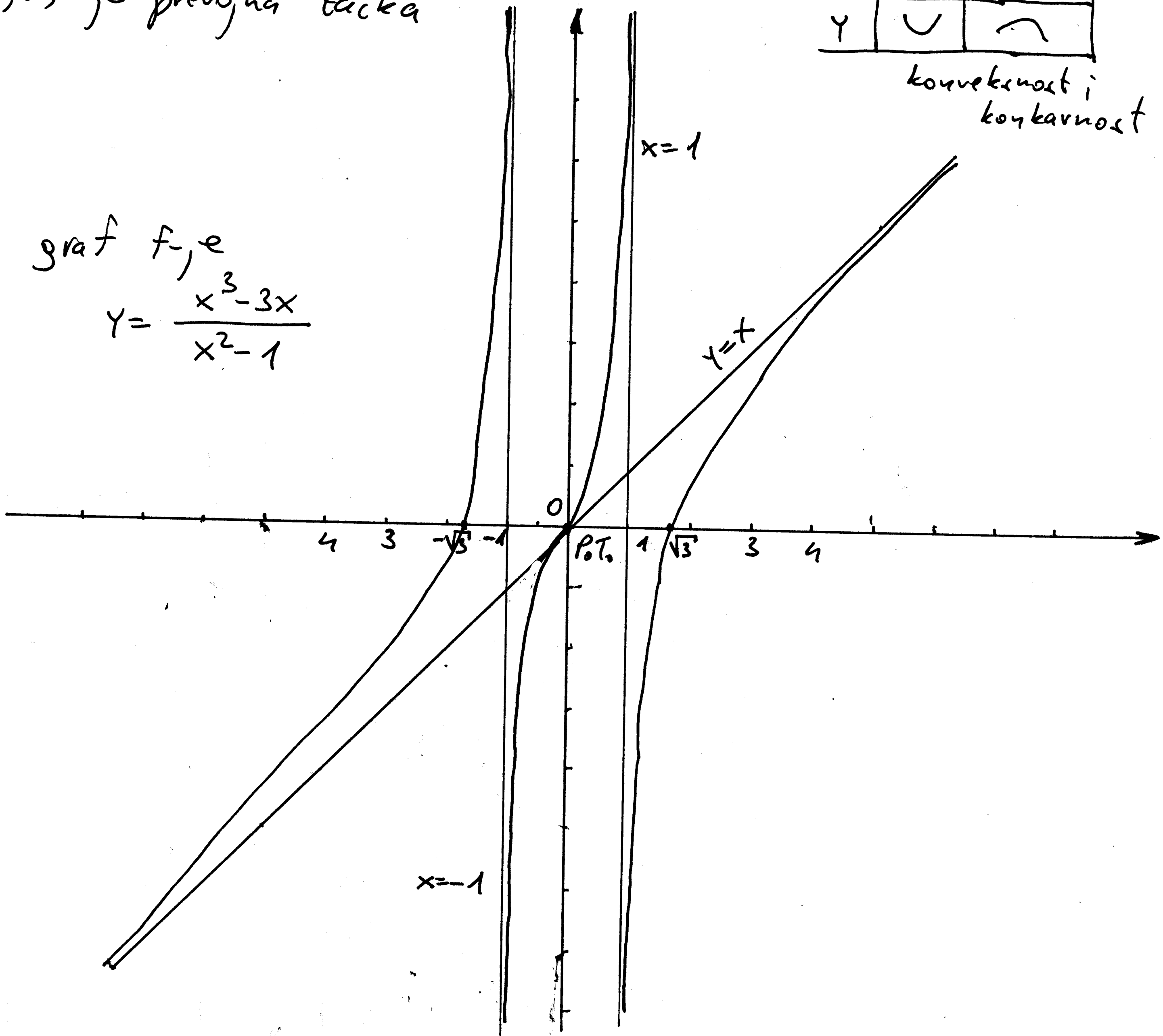
x	(0,1)	(1,+∞)
y''	+	−
y	∪	∩

konveksnost i konkavnost

(0,0) je prevojna tačka

graf f-je

$$y = \frac{x^3-3x}{x^2-1}$$



(#) Grafik kvadratne f-je $f(x)=ax^2+bx+c$ prolazi kroz tačke $A(-1,14)$, $B(2,-4)$ i $C(-2,24)$. Izračunati konstante a, b, c pa zatim ispitati f-ju $Y=\frac{f(x)}{x-10}$ i nacrtati joj grafik.

Rj

$$A(-1,14) \Rightarrow f(-1)=14 \Rightarrow a-b+c=14 \quad (a)$$

$$B(2,-4) \Rightarrow f(2)=-4 \Rightarrow 4a+2b+c=-4 \quad (b)$$

$$C(-2,24) \Rightarrow f(-2)=24 \Rightarrow 4a-2b+c=24 \quad (c)$$

$(a)-(b):\ -3a-3b=18$ $\quad a+b=-6$ $\quad a=1$

$(c)-(b)\ \ -4b=28$ $\quad b=-7$ $\quad c=14-a+b \Rightarrow c=6$

Konstante: $a=1,\ b=-7,\ c=6$

$$Y=\frac{x^2-7x+6}{x-10}$$

definiciono područje

$x\neq 10$

$\mathcal{D}:\ x\in(-\infty,10)\cup(10,+\infty)$

parnost (neparnost)

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna, f-ja nije periodična

nule, presjek sa Y-osom, znak

$Y=0$ akko $x^2-7x+6=0$

$D=49-24=25$

$x_{1,2}=\frac{7\pm5}{2}$

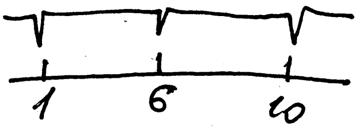
$(x-1)(x-6)=0$

$(1,0)$ i $(6,0)$ su nule f-je

$x=0 \Rightarrow Y=\frac{6}{-10}$

$(0,-\frac{3}{5})$ presjek sa Y-osom

$-\frac{3}{5}\approx -0{,}6$



prekidi Y + nule Y

x	$(-\infty,1)$	$(1,6)$	$(6,10)$	$(10,+\infty)$
x-1	−	+	+	+
x-6	−	−	+	+
x-10	−	−	−	+
Y	−	+	−	+

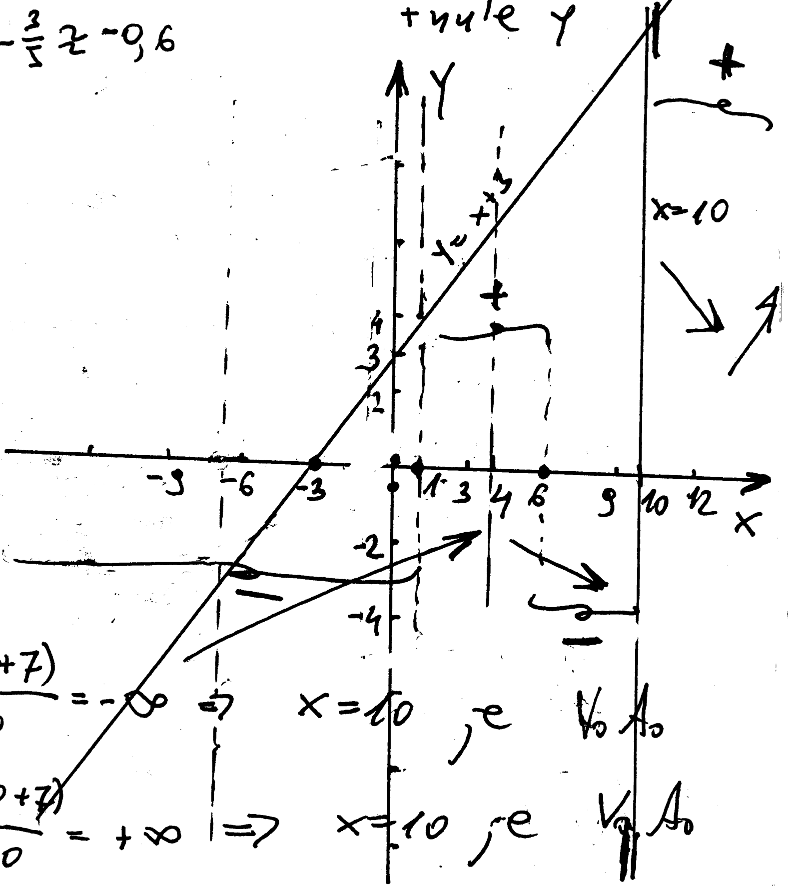
znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=10$ f-ja ima prekid

$$\lim_{x\to10-0} f(x)=\lim_{x\to10-0}\frac{(x-1)(x+7)}{x-10}=\frac{(10-0-1)(10-0+7)}{10-0-10}=-\infty \Rightarrow x=10 \text{ je V.A.}$$

$$\lim_{x\to10+0} f(x)=\lim_{x\to10+0}\frac{(x-1)(x+7)}{x-10}=\frac{(10+0-1)(10+0+7)}{10+0-10}=+\infty \Rightarrow x=10 \text{ je V.A.}$$



$$\left.\begin{array}{l}\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{x^2-7x+6 \quad /:x}{x-10 \quad /:x}=\infty \\ \lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{x^2-7x+6 \quad /:x}{x-10 \quad /:x}=-\infty\end{array}\right\}\Rightarrow \text{f-ja nema HoA}$$

Tražimo kosu asimptotu u obliku $y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2-7x+6 \quad /:x^2}{x^2-10x \quad /:x^2}=1$$

$$n=\lim_{x\to\infty}\left[f(x)-kx\right]=\lim_{x\to\infty}\left(\frac{x^2-7x+6}{x-10}-x\right)=\lim_{x\to\infty}\frac{x^2\underline{-7x}+6-x^2\underline{+10x}}{x-10}=$$

$$n=\lim_{x\to\infty}\frac{3x+6 \quad /:x}{x-10 \quad /:x}=3 \qquad Y=x+3 \text{ je kosa asimptota}$$

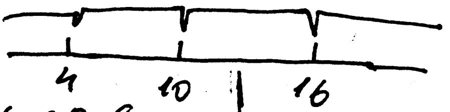
Nakon ovog koraka počinjemo sa skiciranjem grafa.

rast i opadanje

$$y'=\left(\frac{x^2-7x+6}{x-10}\right)'=\frac{(2x-7)(x-10)-(x^2-7x+6)\cdot 1}{(x-10)^2}=\frac{2x^2-20x\cancel{-7x}+70-x^2\cancel{+7x}-6}{(x-10)^2}=$$

$$=\frac{x^2-20x+64}{(x-10)^2}=\frac{(x-4)(x-16)}{(x-10)^2}$$

$y'=0$ akko $x=4$ ili $x=16$ pekidi y +nule y'



x	$(-\infty,4)$	$(4,10)$	$(10,16)$	$(16,+\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗

max min

$$f(4)=\frac{16-28+6}{-6}=\frac{-6}{-6}=1$$

$$f(16)=\frac{220}{6}\approx 36.6667$$

Ekstremi f-je
Na osnovu tabele rasta i opadanja vidimo da je maksimum u tački $(4,1)$ a lokalni minimum u tački $\left(16,\frac{220}{6}\right)$.

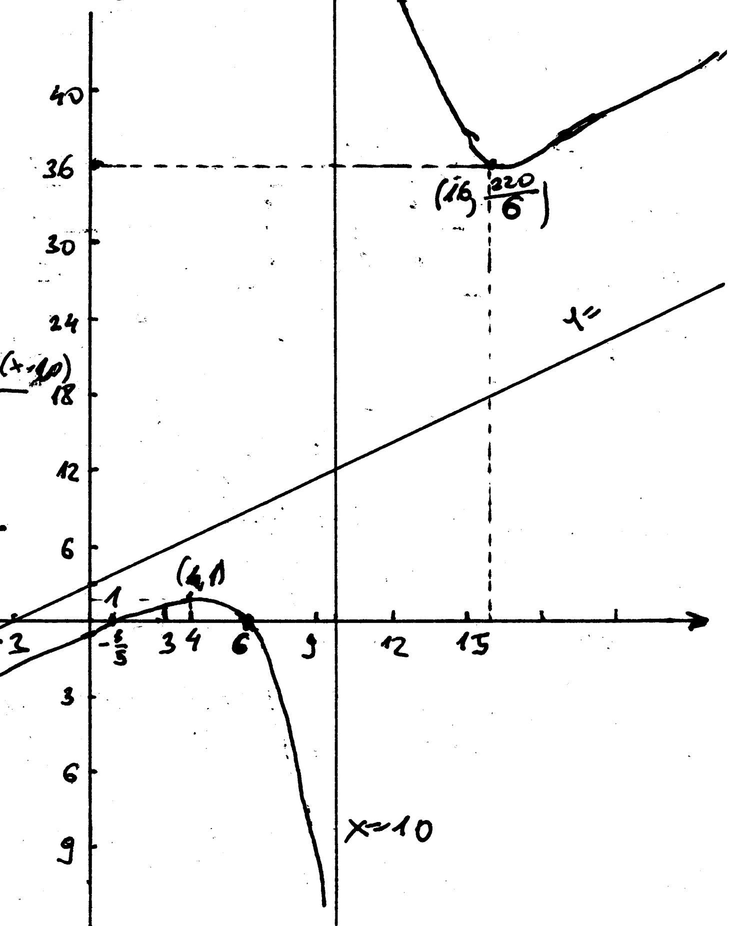
prevojne tačke i intervali konv. i konk.

$$y''=\left(\frac{x^2-20x+64}{(x-10)^2}\right)'=\frac{(2x-20)(x-10)^2-(x^2-20x+64)\cdot 2(x-10)}{(x-10)^4}$$

$$=\frac{\cancel{2x^2}-20x-20x+200\cancel{-2x^2}+40x-128}{(x-10)^3}=\frac{72}{(x-10)^3}$$

f-ja nema prevojnih tačaka

x	$(-\infty,10)$	$(10,+\infty)$
y''	−	+
y	∩	∪



(#) Ispitati f-ju i nacrtati joj grafik $Y=(x+3)\,e^{\frac{1}{x+1}}$.

Rj: definiciono područje

$x+1 \neq 0$
$x \neq -1$

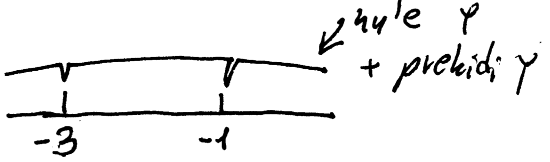
$\mathcal{D}: x \in (-\infty,-1) \cup (-1,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično ⇒ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presek sa Y-osom, znak f-je



$e^{\frac{1}{x+1}} > 0 \quad \forall x \in \mathcal{D}$

$Y=0$ akko $x+3=0$
$x=-3$

$(-3,0)$ je nula f-je

presek sa Y-osom

$x=0$

$Y = 3e \approx 8.1548$

$(0,3e)$ je presek sa Y-osom

x	$(-\infty,-3)$	$(-3,-1)$	$(-1,+\infty)$
$e^{\frac{1}{x+1}}$	+	+	+
$x+3$	−	+	+
Y	−	+	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ f-ja ima prekid

$$\lim_{x\to-1-0} f(x) = \lim_{x\to-1-0}(x+3)\,e^{\frac{1}{x+1}} = (-1-0+3)\,e^{\frac{1}{-1-0+1}} = (2-0)\,e^{-\infty} = 0$$

$$\lim_{x\to-1+0} f(x) = \lim_{x\to-1+0}(x+3)\,e^{\frac{1}{x+1}} = (-1+0+3)\,e^{\frac{1}{-1+0+1}} = (2+0)\,e^{\infty} = \infty \Rightarrow$$

⇒ $x=-1$ je vertikalna asimptota (sa desne strane)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(x+3)\,e^{\frac{1}{x+1}} = \infty\cdot 1 = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x+3)\,e^{\frac{1}{x+1}} = (-\infty)\cdot 1 = -\infty$$

⇒ f-ja nema $H_0 A_0$

tražimo kosu asimptotu u obliku $Y=kx+n$

$$k = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\left(1+\overset{\to 0}{\frac{3}{x}}\right)e^{\overset{\to 0}{\frac{1}{x+1}}} = 1$$

$(\infty-\infty)$

$$n = \lim_{x\to\infty}\left[f(x)-kx\right] = \lim_{x\to\infty}\left[(x+3)\,e^{\frac{1}{x+1}} - x\right] =$$

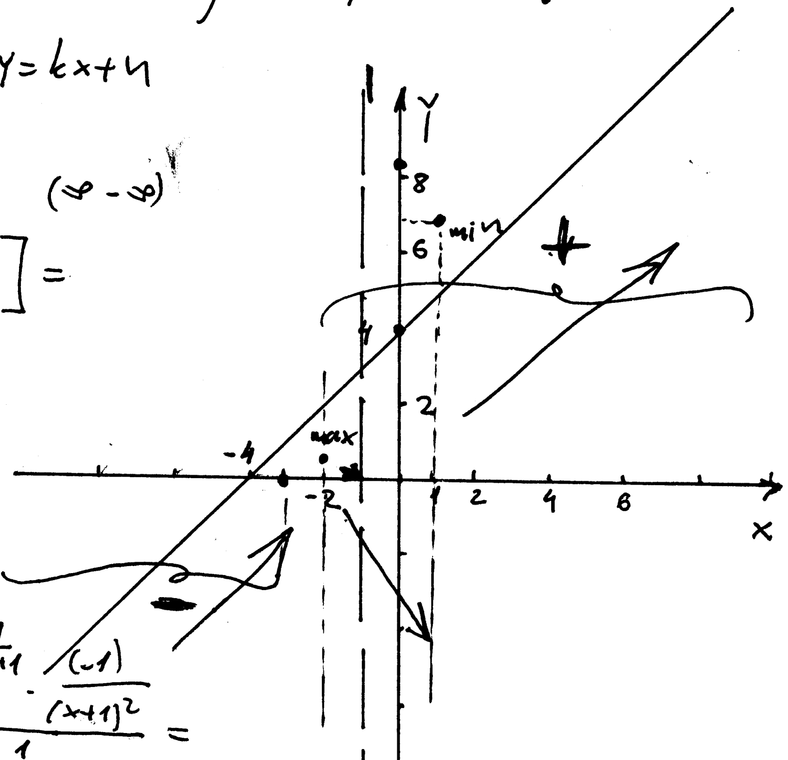
$$= \lim_{x\to\infty}\left[x\,e^{\frac{1}{x+1}} + 3e^{\frac{1}{x+1}} - x\right] =$$

$$= \lim_{x\to\infty} 3e^{\frac{1}{x+1}} + \lim_{x\to\infty}\left[x\,e^{\frac{1}{x+1}} - x\right] =$$

$$= 3 + \lim_{x\to\infty} x\cdot\left(e^{\frac{1}{x+1}}-1\right) =$$

$$= 3 + \lim_{x\to\infty}\frac{e^{\frac{1}{x+1}}-1}{\frac{1}{x}}\left(\frac{0}{0}\right) \overset{L.P.}{=} 3 + \lim_{x\to\infty}\frac{e^{\frac{1}{x+1}}\cdot\frac{(-1)}{(x+1)^2}}{-\frac{1}{x^2}} =$$

$$= 3 + \lim_{x\to\infty}\frac{x^2 \;:x^2}{x^2+2x+1 \;:x^2}\,e^{\frac{1}{x+1}} \xrightarrow{} 1 = 3+1 = 4$$



$Y = x+4$ je kosa asimptota

Počinjemo sa skiciranjem grafa

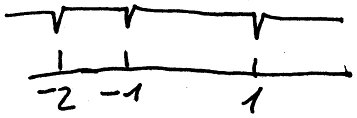
rast i opadanje

$$y' = \left[(x+3)\, e^{\frac{1}{x+1}}\right]' = e^{\frac{1}{x+1}} + (x+3)\, e^{\frac{1}{x+1}} \cdot \frac{-1}{(x+1)^2} =$$

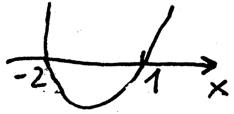
$$= e^{\frac{1}{x+1}}\left[1 - \frac{x+3}{(x+1)^2}\right] = \frac{x^2+\underline{2x}+1\,\underline{\underline{-x}}-3}{(x+1)^2}\, e^{\frac{1}{x+1}} = \frac{x^2+x-2}{(x+1)^2}\, e^{\frac{1}{x+1}}$$

$y'=0$ akko $x^2+x-2=0$

$(x-1)(x+2)=0$



↙ nule y' + prekidi y



x	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,+\infty)$
y'	+	−	−	+
y	↗	↘	↘	↗
		max		min

rast i opadanje

$f(-2) = e^{-1} = \frac{1}{e} \approx 0,3679$

$f(1) = 4e^{1/2} = 4\sqrt{e} \approx 6,5949$

ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da f-ja ima maksimum u tački $(-2, \frac{1}{e})$, a minimum u tački $(1, 4\sqrt{e})$.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \frac{(2x+1)\overset{x^2+2x+1}{(x+1)^2} - (x^2+x-2)2(x+1)}{(x+1)^4}\, e^{\frac{1}{x+1}} + \frac{x^2+x-2}{(x+1)^2}\, e^{\frac{1}{x+1}} \cdot \frac{-1}{(x+1)^2}$$

$$= \frac{2x^3\boxed{+4x^2}+2x\boxed{+x^2}+2x+1-2(x^3\boxed{+x^2}-2x+\boxed{x^2}+x-2)}{(x+1)^4}\, e^{\frac{1}{x+1}} + \frac{-x^2-x+2}{(x+1)^4}\, e^{\frac{1}{x+1}} =$$

$$= \frac{x^2+6x+5-x^2-x+2}{(x+1)^4}\, e^{\frac{1}{x+1}} =$$

$$= \frac{5x+7}{(x+1)^4}\, e^{\frac{1}{x+1}}$$

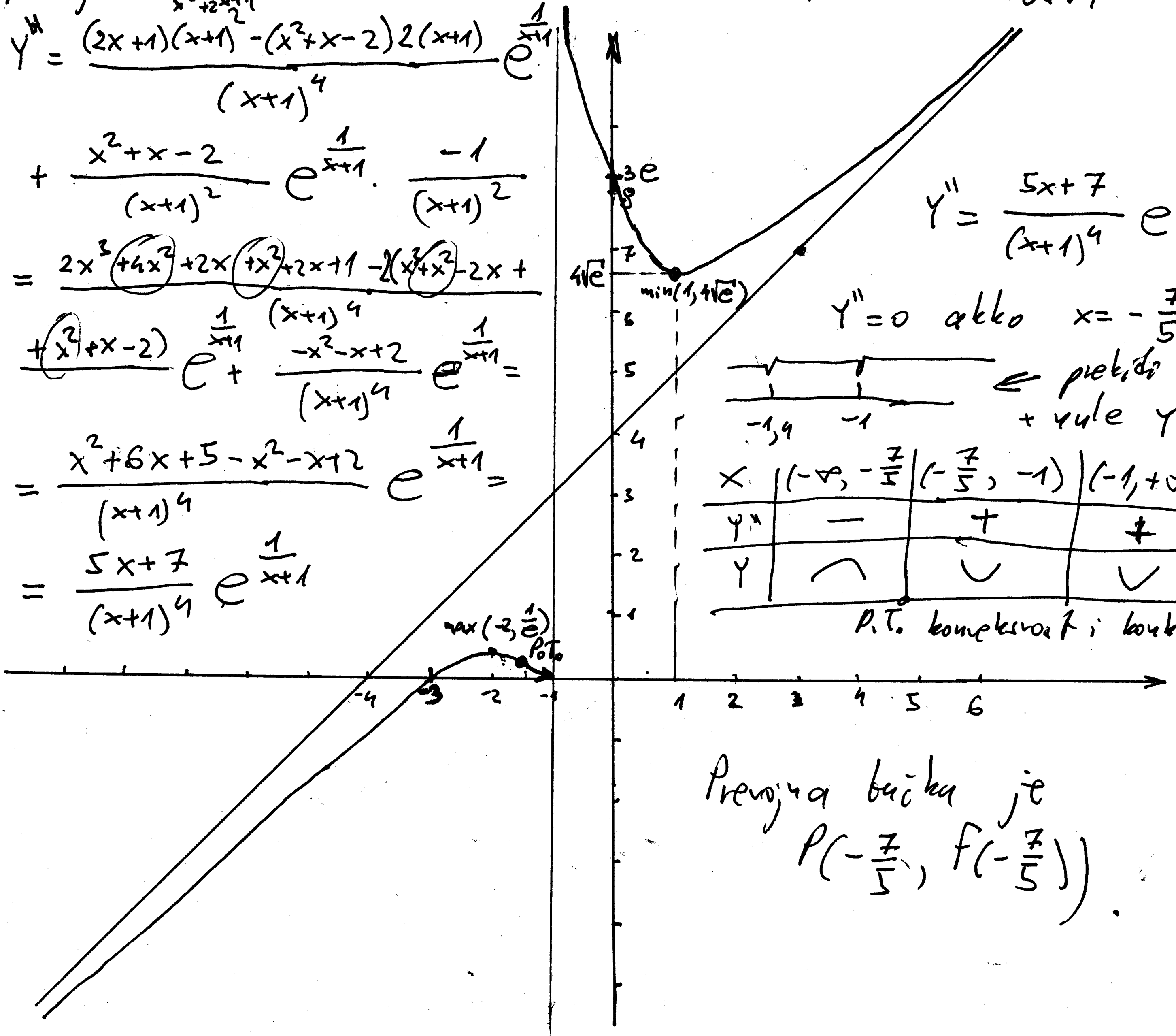
$$y'' = \frac{5x+7}{(x+1)^4}\, e^{\frac{1}{x+1}}$$

$y''=0$ akko $x = -\frac{7}{5} \approx -1,4$

← prekidi y'' + nule y

x	$(-\infty,-\frac{7}{5})$	$(-\frac{7}{5},-1)$	$(-1,+\infty)$
y''	−	+	+
y	∩	∪	∪

P.T. konveksnost i konkavnost



Prevojna tačka je $P(-\frac{7}{5}, f(-\frac{7}{5}))$.

Ispitati f-ju i nacrtati joj grafik $y=2x\ln(e-\frac{2}{x})$ bez analize znaka prvog i drugog izvoda.

Rj. definiciono područje

$x\neq 0$; $e-\frac{2}{x}>0$

$\frac{2}{x}<e$

ovo je tačno za sve $x<0$

ako je $x>0$

$\frac{2}{x}<e \quad /\cdot x$

$2<ex$

$x>\frac{2}{e}$

$\mathcal{D}: x\in(-\infty,0)\cup(\frac{2}{e},+\infty)$

$\frac{2}{e}\approx 0,7358$

parnost (neparnost), periodičnost

kako $\mathcal{D}$ nije simetrično f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$y=0 \Rightarrow 2x\ln(e-\frac{2}{x})=0 \Rightarrow x=0$ ili

$0\notin\mathcal{D}$

f-ja nema nulu

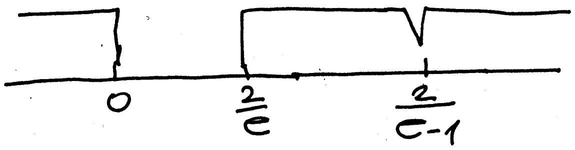
$\ln(e-\frac{2}{x})=0$

$e-\frac{2}{x}=1$

$\frac{2}{x}=e-1 \quad /\cdot x\ (x\neq 0)$

$(e-1)x=2$

$x=\frac{2}{e-1}\approx 1,1640$



← prekidi Y + nule Y

x	$(-\infty,0)$	$(\frac{2}{e},\frac{2}{e-1})$	$(\frac{2}{e-1},+\infty)$
$2x$	$-$	$+$	$+$
$\ln(e-\frac{2}{x})$	$+$	$-$	$+$
Y	$-$	$-$	$+$

znak f-je

$\frac{2}{e-\frac{1}{2}}\in(\frac{2}{e},\frac{2}{e-1})$

$\ln(e-\frac{2}{\frac{2}{e-\frac{1}{2}}})=\ln(e-\frac{2e-1}{2})$

$=\ln(\frac{1}{2})<0$

ponašanje na krajevima intervala definisanosti i asimptote

tačke u kojima f-ja nije definisana $x=0$, $x=\frac{2}{e}$

$$\lim_{x\to-0}f(x)=\lim_{x\to-0}2x\ln(e-\frac{2}{x})\left(=2\cdot(-0)\cdot\infty\right)=2\lim_{x\to-0}\frac{\ln(e-\frac{2}{x})}{\frac{1}{x}}\left(\frac{+\infty}{\infty}\right)\overset{L.P.}{=}$$

$$=2\lim_{x\to-0}\frac{\frac{1}{e-\frac{2}{x}}\cdot(-2)(\frac{1}{x})'}{(\frac{1}{x})'}=-4\lim_{x\to0-}\frac{1}{e\boxed{-\frac{2}{x}}_{\to+\infty}}=-4\cdot0=0$$

$$\lim_{x\to\frac{2}{e}+0}f(x)=\lim_{x\to\frac{2}{e}+0}2x\ln(e-\frac{2}{x})=2(\frac{2}{e}+0)\ln(e-\frac{2}{\frac{2}{e}+0})=$$

$$=2(\frac{2}{e}+0)\ln(e-(e-0))=2(\frac{2}{e}+0)\ln(+0)=-\infty$$

$\frac{2}{98}=2,5 \quad \frac{2}{0,7}=2,8$

$e\approx2,7183$

$\Rightarrow x=\frac{2}{e}$ je V.A.

$$\lim_{x\to\mp\infty}f(x)=\lim_{x\to\mp\infty}2x(\ln(e-\frac{2}{x}))=\mp\infty\cdot1=\mp\infty \Rightarrow$$ f-ja nema H.A.

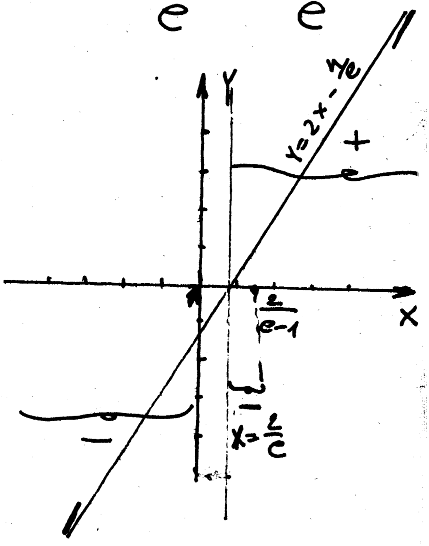
Tražimo kosu asimptotu u obliku $Y=kx+n$

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}2\ln\left(e-\overset{\nearrow 0}{\frac{2}{x}}\right)=2$$

$$n=\lim_{x\to\infty}\left[f(x)-kx\right]=\lim_{x\to\infty}\left[2x\ln\left(e-\frac{2}{x}\right)-2x\right]=2\lim_{x\to\infty}x\left(\ln\left(e-\frac{2}{x}\right)-1\right)\ (=\infty\cdot 0)$$

$$=2\lim_{x\to\infty}\frac{\ln\left(e-\frac{2}{x}\right)-1}{\frac{1}{x}}\ \left(=\frac{0}{0}\right)\overset{LoPi}{=}2\lim_{x\to\infty}\frac{\frac{1}{e-\frac{2}{x}}\cdot(-2)\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}=2\cdot\frac{-2}{e}=\frac{-4}{e}$$

$Y=2x-\frac{4}{e}$ je kosa asimptota $\left(-\frac{4}{e}\approx-1{,}4715\right)$



nakon ovog koraka počinjemo sa skiciranjem grafa

rast i opadanje

$$Y'=\left(2x\ln\left(e-\frac{2}{x}\right)\right)'=2\ln\left(e-\frac{2}{x}\right)+2x\frac{\frac{2}{x^2}}{e-\frac{2}{x}}=$$

$$=2\ln\left(e-\frac{2}{x}\right)+4\frac{\frac{1}{x}}{e-\frac{2}{x}}=2\ln\left(e-\frac{2}{x}\right)+\frac{4}{ex-2}$$

(u zadatku se kaže bez analize prvog i drugog izvoda)

ekstremi f-je

$Y'=0\Rightarrow\dots$

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(2\ln\left(e-\frac{2}{x}\right)+\frac{4}{ex-2}\right)'=$$

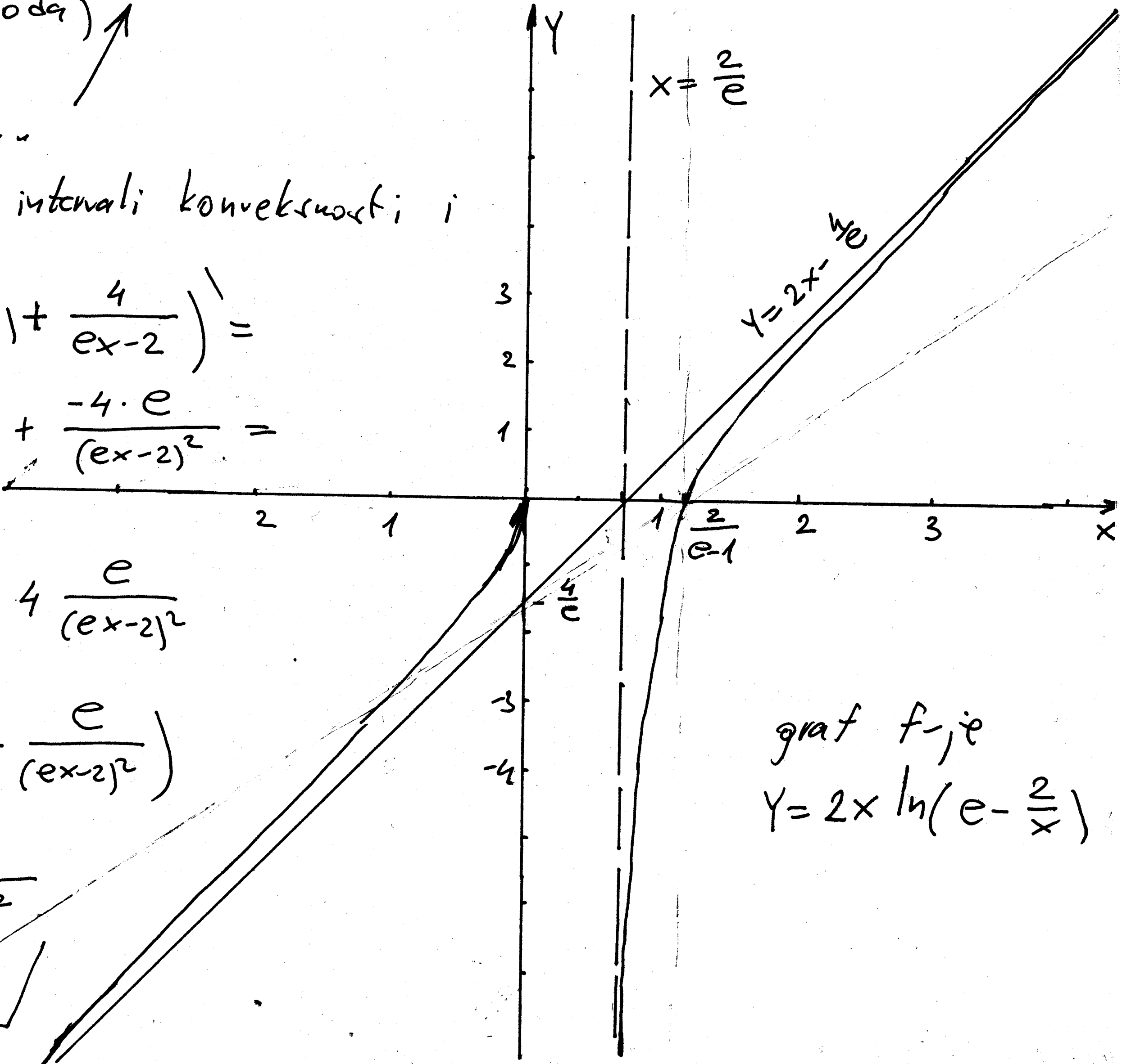
$$=2\frac{-2\left(\frac{1}{x}\right)'}{e-\frac{2}{x}}+\frac{-4\cdot e}{(ex-2)^2}=$$

$$=+4\frac{\frac{1}{x^2}}{\frac{ex-2}{x}}-4\frac{e}{(ex-2)^2}$$

$$=4\left(\frac{1}{x(ex-2)}-\frac{e}{(ex-2)^2}\right)$$

$$=-\frac{8}{x^3\left(e-\frac{2}{x}\right)^2}$$

ako sredimo



graf f-je $Y=2x\ln\left(e-\frac{2}{x}\right)$

(#) Ispitati f-ju i nacrtati joj grafik $y = \frac{1}{x} e^{-\frac{1}{x^2}}$.

R:

definiciono područje

$x \neq 0 \quad D: x \in (-\infty, 0) \cup (0, +\infty)$

nule, presjek grafa sa Y-osom, znak f-je

$y = 0$ akko $\frac{1}{x} e^{-\frac{1}{x^2}} = 0$

$y > 0$ za svako $x \in D$

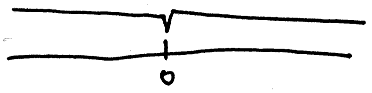
f-ja ne siječe Y-osu (zato što $0 \notin D$)

parnost (neparnost), periodičnost

$$f(-x) = \frac{1}{-x} e^{-\frac{1}{(-x)^2}} = -\frac{1}{x} e^{-\frac{1}{x^2}} = -f(x)$$

f-ja je neparna

f-ja nije periodična



← prekidi y + nule y'

	x	$(0, +\infty)$
	$\frac{1}{x}$	+
znak:	$e^{-\frac{1}{x^2}}$	+
	y	+

kako je f-ja simetrična u odnosu na koordinatni početak (zato što je neparna) dovoljno ju je ispitati na intervalu od 0 do +∞.

ponašanje na krajevima intervala definisanosti i asimptote

za x=0 f-ja ima prekid

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1}{x} e^{-\frac{1}{x^2}} \left(= -\infty \cdot e^{-\infty} = -\infty \cdot 0\right) = \lim_{x\to 0-} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} \left(= \frac{-\infty}{\infty}\right) \overset{L.P.}{=}$$

$$= \lim_{x\to 0-} \frac{-\frac{1}{x^2}}{e^{\frac{1}{x^2}} \cdot \frac{-2}{x^3}} = \lim_{x\to 0-} \left(-\frac{1}{x^2}\right) \cdot \left(-\frac{x^3}{2}\right) \cdot e^{-\frac{1}{x^2}} = \lim_{x\to 0-} \frac{1}{2} \cdot \frac{x}{e^{\frac{1}{x^2}}} \left(= \frac{0}{\infty}\right) = 0$$

kako je graf simetričan u odnosu na $(0,0) \Rightarrow \lim_{x\to 0+} f(x) = 0$

$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{1}{x} e^{-\frac{1}{x^2}} (= 0 \cdot 1) = 0 \Rightarrow y=0$ je H.A.

(graf simetričan $\lim_{x\to -\infty} f(x) = 0 \Rightarrow y=0$ je H.A.)

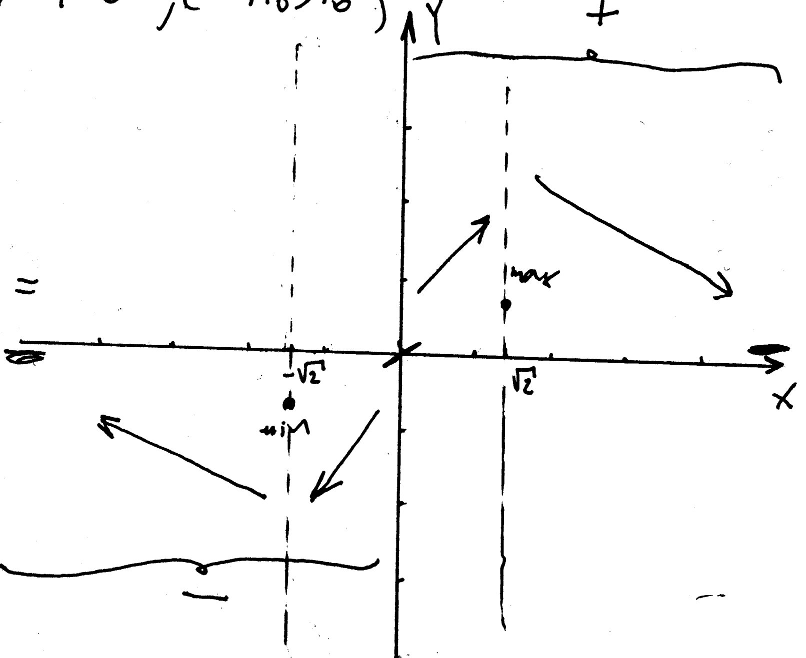
Nakon ovog koraka počinjemo sa skiciranjem grafa f-je

rast i opadanje

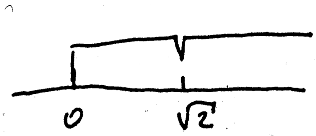
$$y' = \left(\frac{1}{x} e^{-\frac{1}{x^2}}\right)' = \left(\frac{1}{x}\right)' e^{-\frac{1}{x^2}} + \frac{1}{x}\left(e^{-\frac{1}{x^2}}\right)' =$$

$$= \frac{-1}{x^2} e^{-\frac{1}{x^2}} + \frac{1}{x} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} =$$

$$= \left(-\frac{1}{x^2} + \frac{2}{x^4}\right) e^{-\frac{1}{x^2}} = -\frac{x^2-2}{x^4} e^{-\frac{1}{x^2}}$$



$y'=0$ akko $x^2-2=0$ tj. $x_{1,2}=\pm\sqrt{2}$



← prekidi f-je y + nule y'

$\sqrt{2}\approx 1,4142$

x	$(0,\sqrt{2})$	$(\sqrt{2},+\infty)$
y'	+	−
y	↗	↘

max

ekstremi f-je

$y'=0$ akko $x_{1,2}=\pm\sqrt{2}$

$x_1=\sqrt{2}$ je stacionarna tačka. Iz tabele rasta i opadanja vidimo da u njoj f-ja ima ekstrem i to max

$$f(\sqrt{2})=\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}=\frac{1}{\sqrt{2e}}\approx 0,4289$$

$(\sqrt{2},\frac{1}{\sqrt{2e}})$ je tačka maksimuma

$(-\sqrt{2},-\frac{1}{\sqrt{2e}})$ je tačka minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\left(-\frac{1}{x^2}+\frac{2}{x^4}\right)e^{-\frac{1}{x^2}}\right)'=\left(\frac{2}{x^3}-\frac{8}{x^5}\right)e^{-\frac{1}{x^2}}+\left(-\frac{1}{x^2}+\frac{2}{x^4}\right)e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}=$$

$$=\left(\frac{2}{x^3}-\frac{8}{x^5}-\frac{2}{x^5}+\frac{4}{x^7}\right)e^{-\frac{1}{x^2}}=\left(\frac{2}{x^3}-\frac{10}{x^5}+\frac{4}{x^7}\right)e^{-\frac{1}{x^2}}=\frac{2x^4-10x^2+4}{x^7}e^{-\frac{1}{x^2}}$$

$$y''=2\cdot\frac{x^4-5x^2+2}{x^7}\cdot e^{-\frac{1}{x^2}}$$

$y''=0$ akko $x^4-5x^2+2=0$

$x^2=t \quad t^2-5t+2=0$

$D=25-8=17$

$t_{1,2}=\frac{5\pm\sqrt{17}}{2}$

$t_1=\frac{5-\sqrt{17}}{2}\approx 0,4384$

$t_2=\frac{5+\sqrt{17}}{2}\approx 4,5616$

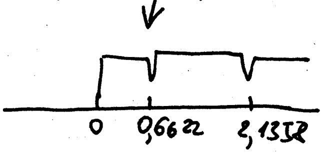
$x_{1,2}=\pm\sqrt{\frac{5-\sqrt{17}}{2}}$.

$x_{3,4}=\pm\sqrt{\frac{5+\sqrt{17}}{2}}$

$x_1=\sqrt{0,4384}\approx 0,6622$

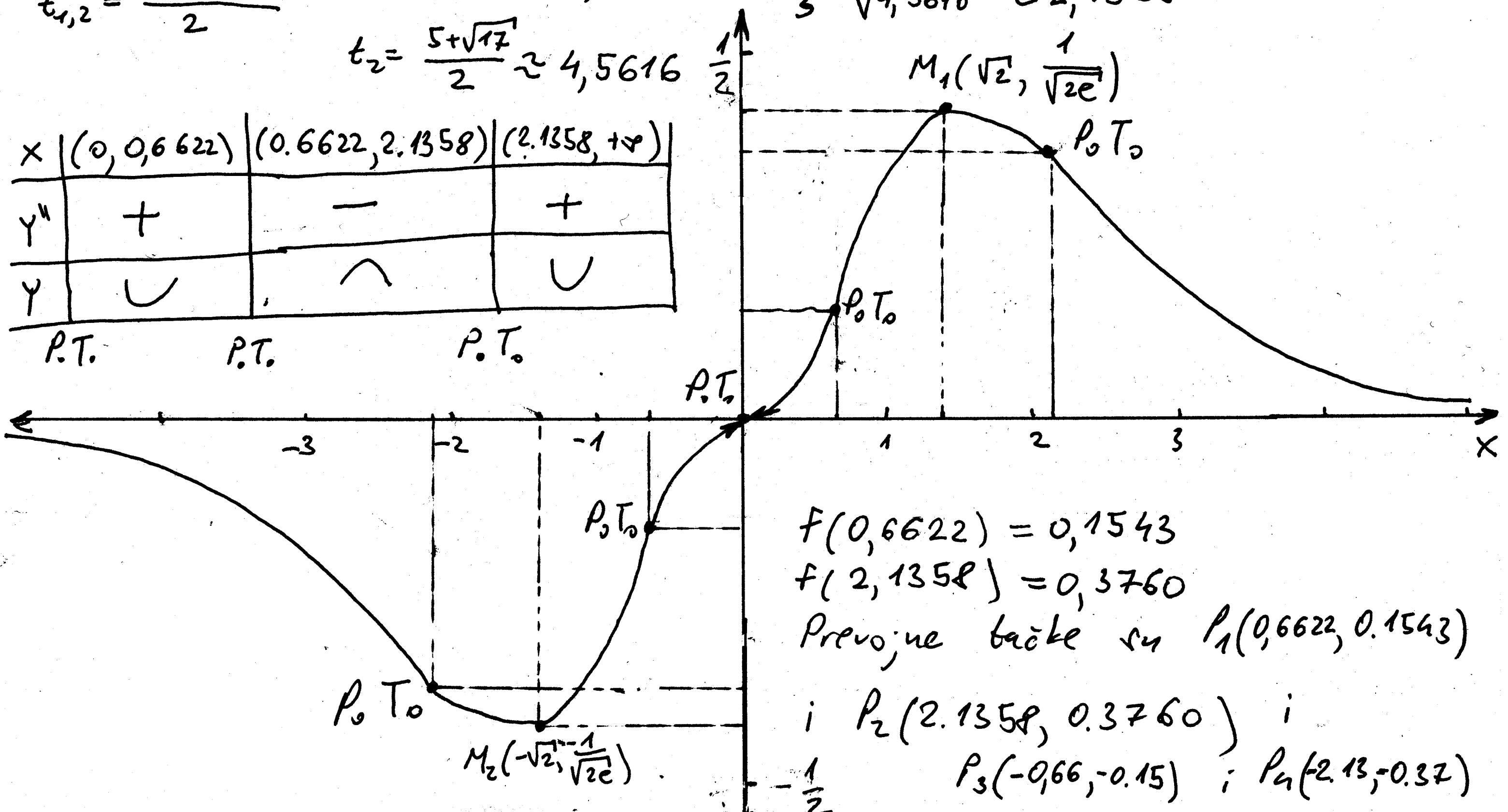
$x_3=\sqrt{4,5616}\approx 2,1358$

prekidi od y + nule y''



x	(0,0,6622)	(0.6622,2.1358)	(2.1358,+∞)
y''	+	−	+
y	∪	∩	∪

P.T. P.T. P.T.



$f(0,6622)=0,1543$

$f(2,1358)=0,3760$

Prevojne tačke su $P_1(0,6622, 0.1543)$ i $P_2(2.1358, 0.3760)$ i $P_3(-0,66,-0.15)$ i $P_4(-2.13,-0.37)$

(#) Ispitati f-ju i nacrtati joj grafik $y=\ln\frac{1+x^3}{1-x^3}$

Rj. deficiono područje

$$\frac{1+x^3}{1-x^3}>0 \quad \text{tj.} \quad \frac{(1+x)\overbrace{(1-x+x^2)}^{>0\ \forall x\in\mathbb{R}}}{(1-x)\underbrace{(1+x+x^2)}_{>0\ \forall x\in\mathbb{R}}}>0$$

$1+x=0$ za $x=-1$

$1-x=0$ za $x=1$

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
$1+x$	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
y	$-$	$+$	$-$

$\mathcal{D}: x\in(-1,1)$

parnost (neparnost), periodičnost

$$F(-x)=\ln\frac{1+(-x)^3}{1-(-x)^2}=\ln\frac{1-x^3}{1+x^3}=$$

$$=\ln\left(\frac{1+x^3}{1-x^3}\right)^{-1}=-\ln\frac{1+x^3}{1-x^3}$$

f-ja je neparna, f-ja nije periodična

nule, presjek grafa sa y-osom, znak f-je

nula: $y=0$ akko $\frac{1+x^3}{1-x^3}=1$ tj. $\frac{1+x^3}{1-x^3}-1=0$

$$\frac{1+x^3-(1-x^3)}{1-x^3}=0 \Rightarrow \frac{2x^3}{1-x^3}=0 \Rightarrow x=0$$

$(0,0)$ je presjek sa y-osom i nula f-je

$\ln\frac{1+x^3}{1-x^3}>0$ tj. $\frac{1+x^3}{1-x^2}>1$ $\quad y>0$ za $x\in(0,1)$

$\ln\frac{1+x^3}{1-x^3}>\ln 1$ $\quad \frac{2x^3}{1-x^3}>0$ $\quad y<0$ za $x\in(-1,0)$ $\quad$ znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ i $x=1$ f-ja nije definisana

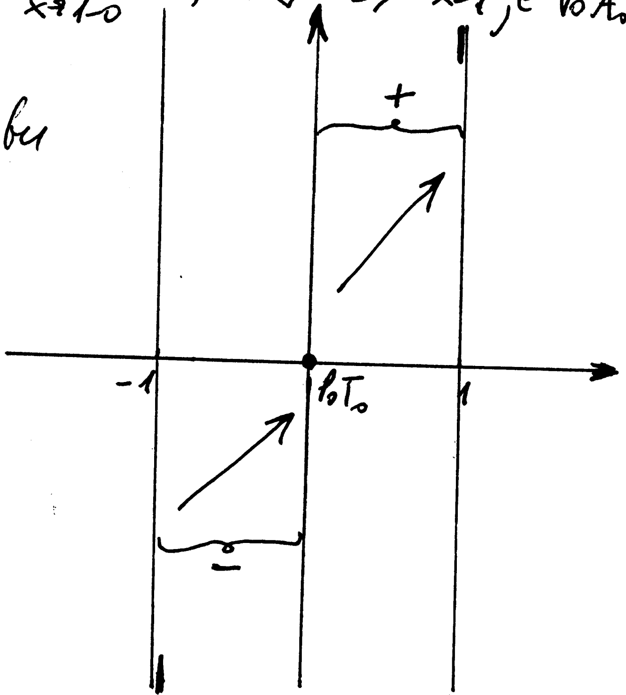
$$\lim_{x\to-1+0}f(x)=\lim_{x\to-1+0}\ln\frac{1+x^3}{1-x^3}=\ln\lim_{x\to-1+0}\frac{1+x^3}{1-x^3}=\ln\frac{1+(-1+0)^3}{1-(-1+0)^2}=\ln\frac{1-1+0}{1+1-0}=\ln\frac{+0}{2-0}$$

$$=\ln(+0)=-\infty \Rightarrow x=-1 \text{ je V.A.}$$

Kako je f-ja neparna možemo bez računanja zaključiti da je $\lim_{x\to1-0}f(x)=+\infty \Rightarrow x=1$ je V.A.

Kako je $\mathcal{D}:(-1,1)$ f-ja nema horizontalnu ni kosu asimptotu

Nakon ovog koraka počinjemo sa skiciranjem grafa f-je:



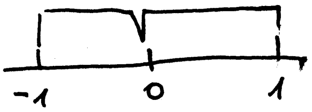
rast i opadanje

$$y'=\left(\ln\frac{1+x^3}{1-x^3}\right)'=\frac{1}{\frac{1+x^3}{1-x^3}}\cdot\left(\frac{1+x^3}{1-x^3}\right)'=$$

$$=\frac{\cancel{1-x^3}}{1+x^3}\cdot\frac{3x^2(1-x^3)-(1+x^3)\cdot(-3)x^2}{(1-x^3)^{\cancel{2}}}$$

$$y' = \frac{3x^2(1-x^3+1+x^3)}{1-x^6} = \frac{6x^2}{1-x^6}$$

$y'=0$ akko $x=0$



↙ prekidi y
+ nule y'

x	(-1,0)	(0,1)
y'	+	+
y	↗	↗

rast i opadanje

ekstremi f-je

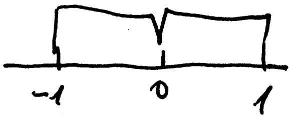
Na osnovu tabele rasta i opadanja vidimo da f-ja nema ekstrema.

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{6x^2}{1-x^6}\right)' = \frac{12x(1-x^6)-6x^2\cdot(-6)x^5}{(1-x^6)^2} = 6x\,\frac{2-2x^6+6x^6}{(1-x^6)^2}$$

$$y'' = \frac{6x(2+4x^6)}{(1-x^6)^2}, \quad y''=0 \text{ akko } 6x(2+4x^6)=0$$

$$x=0 \text{ ili } \underbrace{2+4x^6}_{>0\ \forall x}=0$$

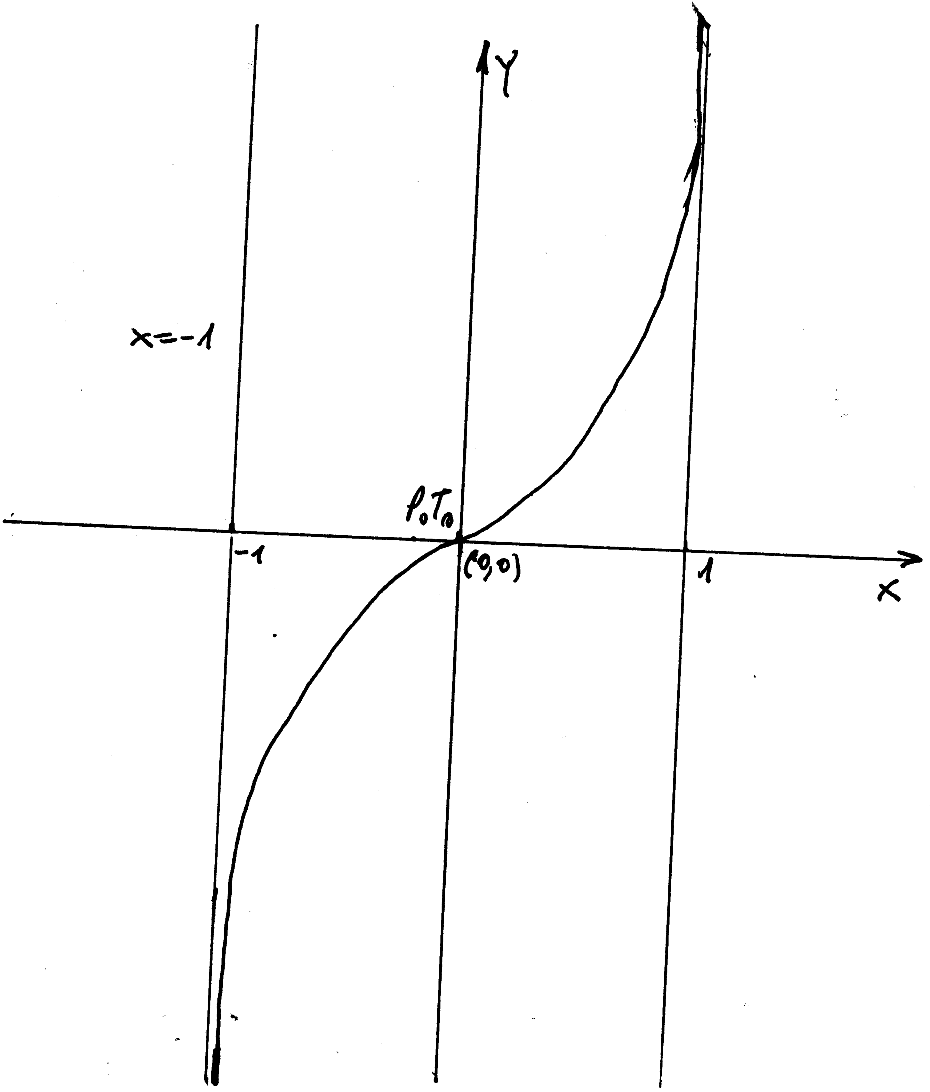


prekidi y
+ nule y''

x	(-1,0)	(0,1)
y''	−	+
y	∩	∪

P.T.

(0,0) je prevojna tačka.



(#) Izračunati integral $\int \frac{x}{(x^2-2x+2)^2}\,dx$.

Rj.

$$x^2-2x+2 = x^2-2x+1+1 = (x-1)^2+1$$

$$I=\int \frac{x}{(x^2-2x+2)^2}\,dx = \int \frac{x}{\left((x-1)^2+1\right)^2}\,dx = \left| \begin{array}{l} x-1=t \\ dx=dt \\ x=t+1 \end{array} \right| = \int \frac{t+1}{(t^2+1)^2}\,dt =$$

$$= \underbrace{\int \frac{t}{(t^2+1)^2}\,dt}_{I_1} + \boxed{\int \frac{dt}{(t^2+1)^2}}\, I_2$$

$$I_1=\int \frac{t\,dt}{(t^2+1)^2} = \left| \begin{array}{l} t^2+1=s \\ 2t\,dt=ds \\ t\,dt=\frac{1}{2}ds \end{array} \right| = \frac{1}{2}\int \frac{ds}{s^2} = \frac{1}{2}\cdot\frac{s^{-1}}{-1}+c = -\frac{1}{2}\cdot\frac{1}{s}+c$$

$$= -\frac{1}{2}\cdot\frac{1}{t^2+1}+c$$

$$I_2=\int \frac{dt}{(t^2+1)^2} = \int \frac{1+t^2-t^2}{(t^2+1)^2}\,dt = \int \frac{t^2+1}{(t^2+1)^2}\,dt - \int \frac{t^2}{(t^2+1)^2}\,dt =$$

$$= \underbrace{\int \frac{dt}{1+t^2}}_{I_3} - \underbrace{\int \frac{t^2}{(t^2+1)^2}\,dt}_{I_4}$$

$$I_3 = \int \frac{dt}{1+t^2} = \operatorname{arc\,tg} t + c$$

$$I_4 = \int \frac{t^2}{(t^2+1)^2}\,dt = \left| \begin{array}{ll} u=t & dv=\frac{t}{(t^2+1)^2}\,dt \\ du=dt & v=-\frac{1}{2}\cdot\frac{1}{t^2+1} \end{array} \right| =$$

$$= -\frac{1}{2}\cdot\frac{t}{t^2+1} + \frac{1}{2}\int \frac{dt}{t^2+1} = -\frac{1}{2}\cdot\frac{t}{t^2+1} + \frac{1}{2}\operatorname{arc\,tg} t + c$$

$$I = -\frac{1}{2}\cdot\frac{1}{t^2+1} + \operatorname{arc\,tg} t + \frac{1}{2}\frac{t}{t^2+1} - \frac{1}{2}\operatorname{arc\,tg} t + c$$

$$= \frac{1}{2}\cdot\frac{t-1}{t^2+1} + \frac{1}{2}\operatorname{arc\,tg} t + c = \frac{1}{2}\cdot\frac{x-2}{x^2-2x+2} + \frac{1}{2}\operatorname{arctg}(x-1)+c$$

traženo rješenje

(#) Izračunati integral $\int x^3\sqrt{1+a^2x^2}\,dx$, $(a>0)$.

Rj.

$$\int x^3\sqrt{1+a^2x^2}\,dx = \int x^2\cdot x\cdot\sqrt{1+a^2x^2}\,dx = \left|\begin{array}{l} 1+a^2x^2=t^2 \\ a^2\cdot 2x\,dx=2t\,dt \\ x\,dx=\frac{1}{a^2}t\,dt \\ a^2x^2=t^2-1 \\ x^2=\frac{1}{a^2}(t^2-1) \end{array}\right|$$

$$= \int \frac{1}{a^2}(t^2-1)\cdot\frac{1}{a^2}t\cdot t\,dt =$$

$$= \frac{1}{a^4}\int (t^4-t^2)\,dt = \frac{1}{a^4}\cdot\frac{1}{5}t^5 - \frac{1}{a^4}\cdot\frac{1}{3}t^3 =$$

$$= \frac{1}{5a^4}\sqrt{(1+a^2x^2)^5} - \frac{1}{3a^4}\sqrt{(1+a^2x^2)^3} + C$$

(#) Izračunati integral $\int x\sqrt{1-x^4}\,dx$

Rj.

$$\int x\sqrt{1-x^4}\,dx=\int x\sqrt{(1-x^2)(1+x^2)}\,dx=\left|\begin{array}{l}x^2=t\\ 2x\,dx=dt\\ x\,dx=\frac{1}{2}dt\end{array}\right|=\frac{1}{2}\int\sqrt{(1-t)(1+t)}\,dt=$$

$$=\frac{1}{2}\int\sqrt{1-t^2}\,dt=\frac{1}{2}\int\frac{1-t^2}{\sqrt{1-t^2}}\,dt=\frac{1}{2}\left[\int\frac{dt}{\sqrt{1-t^2}}-\int\frac{t^2}{\sqrt{1-t^2}}\,dt\right]$$

$$\int\frac{t^2}{\sqrt{1-t^2}}\,dt=\left|\begin{array}{ll}u=t & dv=\frac{t}{\sqrt{1-t^2}}\,dt\\ du=dt & v=\int\frac{t}{\sqrt{1-t^2}}\,dt=\left|\begin{array}{l}1-t^2=s^2\\ -2t\,dt=2s\,ds\\ t\,dt=-s\,ds\end{array}\right|=-\int\frac{s\,ds}{s}=-\int ds\\ & =-s=-\sqrt{1-t^2}\end{array}\right|=$$

$$=-t\sqrt{1-t^2}+\int\sqrt{1-t^2}\,dt$$ Sad imamo:

$$\underline{\underline{\frac{1}{2}\int\sqrt{1-t^2}\,dt}}=\frac{1}{2}\arcsin t+\frac{1}{2}t\sqrt{1-t^2}\underline{\underline{-\frac{1}{2}\int\sqrt{1-t^2}\,dt}}$$

$$\int\sqrt{1-t^2}\,dt=\frac{1}{2}t\sqrt{1-t^2}+\frac{1}{2}\arcsin t$$

vratimo smjene $$\frac{1}{2}\int\sqrt{1-t^2}=\frac{1}{2}\left(\frac{1}{2}t\sqrt{1-t^2}+\frac{1}{2}\arcsin t\right)$$

$$\int x\sqrt{1-x^4}\,dx=\frac{1}{4}x^2\sqrt{1-x^4}+\frac{1}{4}\arcsin x^2+C$$

Izračunati integral $\int\limits_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int\limits_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left|\begin{array}{l} x=t^2 \\ dx=2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array}\right| = \int\limits_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int\limits_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int\limits_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int\limits_1^2 \frac{t^2-4t+5}{t^2-4t+5}\,dt +$$

$$+2\int\limits_1^2 \frac{6t-5}{t^2-4t+5}\,dt = 2\int\limits_1^2 dt + 2\cdot 3\int\limits_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int\limits_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}\,dt = \int\limits_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5}\,dt = \int\limits_1^2 \frac{2t-4}{t^2-4t+5}\,dt + \frac{7}{3}\int\limits_1^2 \frac{dt}{t^2-4t+5}$$

$$\int\limits_1^2 dt = t\Big|_1^2 = 2-1 = 1\ , \quad \int\limits_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left|\begin{array}{l} t^2-4t+5=s \\ (2t-4)dt=ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array}\right| = \int\limits_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1$$

$$=\ln 1-\ln 2 = -\ln 2$$

$$\int\limits_1^2 \frac{dt}{t^2-4t+5} = \int\limits_1^2 \frac{dt}{(t-2)^2+1} = \left|\begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array}\right| = \int\limits_{-1}^0 \frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int\limits_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1 + 6\left(-\ln 2 + \frac{7}{3}\cdot\frac{\pi}{4}\right) = 2-6\ln 2+\frac{7\pi}{2} \approx 8{,}8367$$

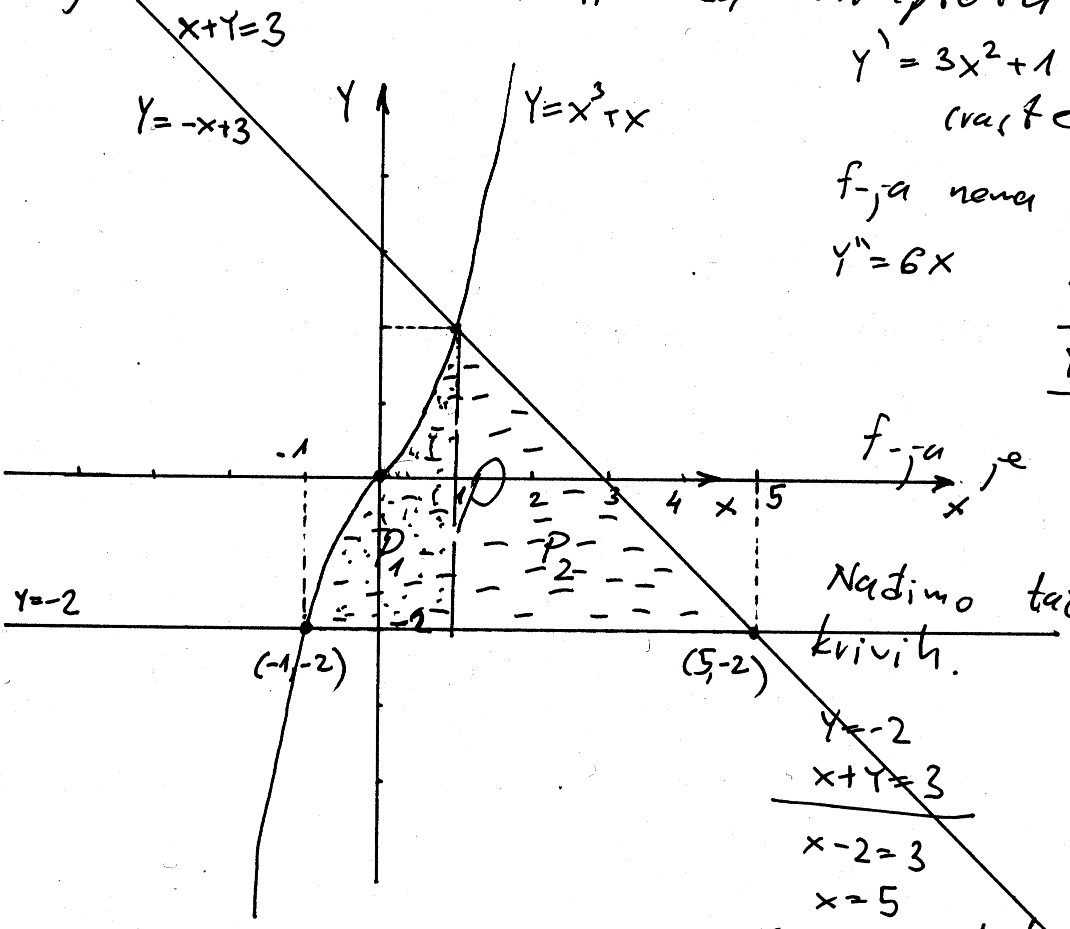
traženo rešenje

(#) Izračunati površinu figure koja je određena linijama $y=-2,\ y=x^3+x,\ x+y=3.$

Rj. $y=-2,\ x+y=3$ su prave linije i njih nije teško nacrtati.
Problem za crtanje predstavlja kriva $y=x^3+x$.
Ispitajmo f-ju $y=x^3+x$. $\mathcal{D}: x\in\mathbb{R}$

$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna

$A(0,0)$ je nula f-je i presjek sa y-osom
f-ja nema prekida $\Rightarrow$ f-ja nema vertikalnu asimptotu
f-ja nema horizontalnu ni kosu asimptotu



$y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)

f-ja nema ekstrem

$y''=6x$

x	$(0,+\infty)$
y''	+
y	∪

(0,0) je prevojna tačka

f-ja je ovog oblika

Nađimo tačke presjeka datih krivih.

$$\frac{\begin{array}{l} y=-2 \\ x+y=3 \end{array}}{\begin{array}{l} x-2=3 \\ x=5 \end{array}}$$

(5,-2) je tačka presjeka

$$\frac{\begin{array}{l} y=-2 \\ y=x^3+x \end{array}}{\begin{array}{l} -2=x^3+x \\ x^3+x+2=0 \end{array}}$$

$x=-1: \quad -1-1+2=0$

$$\begin{array}{l} (x^3+x+2):(x+1)=x^2-x+2 \\ \underline{-\ x^3+x^2} \\ \quad -x^2+x+2 \\ \underline{-\ -x^2-x} \\ \qquad 2x+2 \\ \qquad \underline{2x+2} \\ \qquad // \ // \end{array}$$

$$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.

(-1,-2) je tačka presjeka datih krivih

$Y = x^3 + x$

$x + Y = 3$

$Y = x^3 + x$

$Y = -x + 3$

$-x + 3 = x^3 + x$

$x^3 + 2x - 3 = 0$

$x = 1: \quad 1^3 + 2 \cdot 1 - 3 = 3 - 3 = 0$

$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$

$x^3 - x^2$

$x^2 + 2x - 3$

$- \; x^2 - x$

$3x - 3$

$- \; 3x - 3$

$// \quad //$

$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{>0 \ \forall x}(x - 1)$

(1, 2) je presječna tačka krivih

$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2)\right] dx = \int_{-1}^{1} (x^3 + x + 2)\, dx = \frac{1}{4} x^4 \Big|_{-1}^{1} + \frac{1}{2} x^2 \Big|_{-1}^{1} + 2x \Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x + 3) - (-2)\right] dx = \int_{1}^{5} (-x + 5)\, dx = -\frac{x^2}{2} \Big|_{1}^{5} + 5x \Big|_{1}^{5} =$$

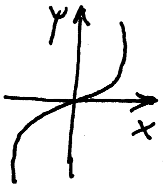
$$= -\frac{1}{2}(25 - 1) + 5 \cdot 4 = -\frac{1}{2} \cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$ površina figure

(#) Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

Rj. Grafički nije teško predstaviti prave $y=-x$ i $y=3x-2$. Problem predstavlja kriva $y=\sqrt[3]{x}$.

Ako znamo da kriva $y=x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x=y^3$ što je ekvivalentno sa $y=\sqrt[3]{x}$.

Pronađimo tačke presjeka datih krivih.

$y=-x$

$y=3x-2$

$-x=3x-2$

$-4x=-2$

$x=\frac{1}{2} \Rightarrow y=-\frac{1}{2}$

$y=-x$

$y=\sqrt[3]{x}$

$y=-x$

$y^3=x$

$-x^3=x$

$x^3+x=0$

$x(x^2+1)=0$

$x=0 \Rightarrow y=0$

$y=3x-2$

$y=\sqrt[3]{x}$

$\sqrt[3]{x}=3x-2$

$(3x-2)^3=x$

$27x^3-3\cdot(3x)^2\cdot 2+$

$+3\cdot 3x\cdot(-2)^2+(-2)^3=x$

$27x^3-54x^2+36x-8=x$

$27x^3-54x^2+35x-8=0$

pokušajmo riješiti sistem na drugi način

$\sqrt[3]{x}=3x-2$

$x=t^3$

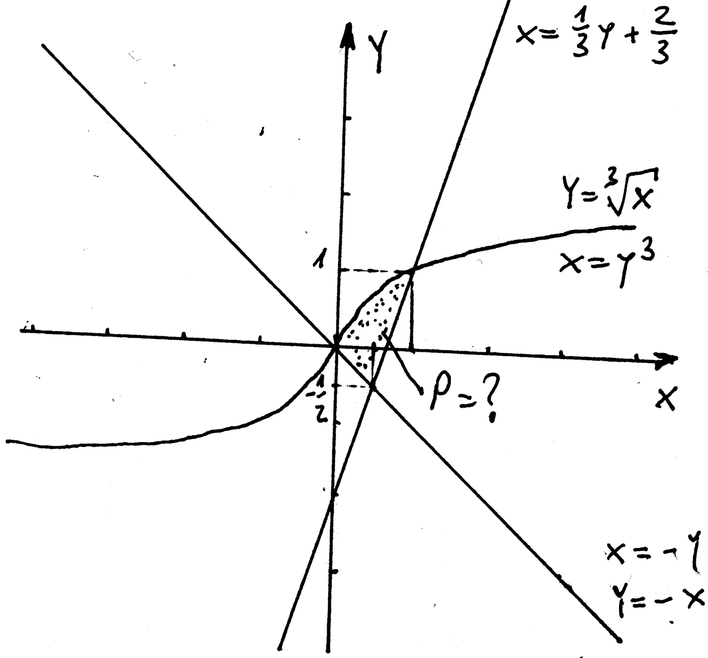
$3t^3-2=t$

$3t^3-t-2=0$

$t=1:\ 3-1-2=0$

$x=t^3=1 \Rightarrow y=1$

$\boxed{3x=y+2}$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-(-y)\right]dy+\int_{0}^{1}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-y^3\right]dy=$$

$$=\int_{-\frac{1}{2}}^{0}\left(\frac{4}{3}y+\frac{2}{3}\right)dy+\int_{0}^{1}\left(-y^3+\frac{1}{3}y+\frac{2}{3}\right)dy=$$

$$=\frac{4}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{1}{2}}^{0}+\frac{2}{3}y\Big|_{-\frac{1}{2}}^{0}-\frac{1}{4}y^4\Big|_{0}^{1}+\frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{0}^{1}+$$

$$+\frac{2}{3}y\Big|_{0}^{1}=\frac{\cancel{2}}{3}\cdot\left(-\frac{1}{\cancel{4}_{2}}\right)+\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}-\frac{1}{4}+\frac{1}{6}+$$

$$+\frac{2}{3}=-\frac{\cancel{1}}{\cancel{6}}+\frac{1}{3}-\frac{1}{4}+\frac{\cancel{1}}{\cancel{6}}+\frac{2}{3}=\frac{3}{4}$$

(#) Nađi ekstreme f-je $z = \frac{2x+2y-1}{\sqrt{x^2+y^2+1}}$.

Rj.

$$\frac{\partial z}{\partial x} = \frac{2\cdot\sqrt{x^2+y^2+1} - (2x+2y-1)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+1}}}{x^2+y^2+1} =$$

$$= \frac{2\cdot(x^2+y^2+1) - x(2x+2y-1)}{(x^2+y^2+1)\sqrt{x^2+y^2+1}} = \frac{\underline{2x^2}+2y^2+2\,\underline{-2x^2}-2yx+x}{\sqrt{(x^2+y^2+1)^3}}$$

$$= \frac{2y^2-2xy+x+2}{\sqrt{(x^2+y^2+1)^3}}$$

$$\frac{\partial z}{\partial y} = \frac{2\sqrt{x^2+y^2+1} - (2x+2y-1)\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+1}}}{x^2+y^2+1} = \frac{2x^2\boxed{+2y^2}+2-2xy\boxed{-2y^2}+y}{(x^2+y^2+1)\sqrt{x^2+y^2+1}}$$

$$= \frac{2x^2-2xy+y+2}{\sqrt{(x^2+y^2+1)^3}}$$

$\frac{\partial z}{\partial x} = 0 \qquad 2y^2-2xy+x+2=0 \quad (a)$

$\frac{\partial z}{\partial y} = 0 \qquad 2x^2-2xy+y+2=0 \quad (b)$

$(a)-(b)$; $2y^2-2x^2+x-y=0$

$2(y-x)(y+x)-(y-x)=0$

$(y-x)(2y-2x-1)=0$

$y=x$ ili $2y-2x-1=0$

$2y-2x=1$

$y-x=\frac{1}{2}$

$y=x+\frac{1}{2}$

Za $y=x$ imamo $2x^2-2x^2+x+2=0$

$x=-2 \Rightarrow y=-2$

Za $y=x+\frac{1}{2}$ imamo $2(x+\frac{1}{2})^2-2x\cdot(x+\frac{1}{2})+x+2=0$

$2(x^2+x+\frac{1}{4})-x(2x+1)+x+2=0$

$\underline{\underline{2x^2}}+2x+\frac{1}{2}-\underline{\underline{2x^2}}\,\underline{-x}\,\underline{+x}+2=0$

$2x=-\frac{5}{2} \Rightarrow x=-\frac{5}{4} \Rightarrow y=-\frac{3}{4}$

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(-\frac{5}{4},-\frac{3}{4})$.

$$\frac{\partial^2 z}{\partial x^2} = \frac{(-2y+1)\sqrt{(x^2+y^2+1)^3} - (2y^2-2xy+x+2)\frac{3}{\cancel{2}}\sqrt{(x^2+y^2+1)}\cdot\cancel{2}x}{(x^2+y^2+1)^3}$$

$$= \frac{[(-2y+1)(x^2+y^2+1)-3x(2y^2-2xy+x+2)]\sqrt{x^2+y^2+1}}{(x^2+y^2+1)^3} =$$

$$= \frac{\left[\boxed{-2x^2y}\enclose{circle}{-2y^3}\underline{\underline{-2y}}\boxed{+x^2}+y^2+1\underline{-6xy^2}\boxed{+6x^2y}\boxed{-3x^2}\underline{-6x}\right]\sqrt{x^2+y^2+1}}{(x^2+y^2+1)^3}$$

$$= \frac{(-2y^3+4x^2y-6xy^2-2x^2-6x-2y+y^2+1)\sqrt{x^2+y^2+1}}{(x^2+y^2+1)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(4y-2x)\sqrt{(x^2+y^2+1)^3} - (2y^2-2xy+x+2)\cdot\frac{3}{\cancel{2}}\sqrt{(x^2+y^2+1)}\cdot\cancel{2}y}{(x^2+y^2+1)^3} \begin{matrix} / \cdot \sqrt{x^2+y^2+1} \\ \\ / \cdot \sqrt{x^2+y^2+1} \end{matrix}$$

$$= \frac{(4y-2x)(x^2+y^2+1)^2 - 3y(2y^2-2xy+x+2)(x^2+y^2+1)}{(x^2+y^2+1)^3\sqrt{x^2+y^2+1}} =$$

$$= \frac{-(2x^3-4x^2y+3xy+2x+2y^3+2y)}{(x^2+y^2+1)^2\sqrt{x^2+y^2+1}}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2x^3+6x^2y-x^2-4xy^2+2x+2y^2+6y-1}{\sqrt{(x^2+y^2+1)^5}}$$

$M_1(-2,-2)$

$A = \frac{5}{27}$, $B = -\frac{4}{27}$, $C = \frac{5}{27}$

$D = AC - B^2 = \frac{25}{27^2} - \frac{16}{27^2} > 0$ ⇒ f-ja ima ekstrem

$A > 0$ ⇒ f-ja ima minimum

$z_{min}(-2,-2) = -3$

Za $M_2(-\frac{5}{4}, -\frac{3}{4})$

$A = \frac{4\sqrt{8}}{25}$, $B = -\frac{4\sqrt{8}}{125}$, $C = \frac{212\sqrt{8}}{625}$, $D = AC - B^2 \approx 0{,}4260 > 0$

⇒ f-ja ima ekstrem, $A > 0$ ⇒ f-ja ima minimum

$z_{min}(-\frac{5}{4}, -\frac{3}{4}) = -\sqrt{8} = -2\sqrt{2}$

(#) Nađi uslovne ekstreme f-je $z=2x+4y$ ako je $\frac{2}{x}+\frac{4}{y}=3$.

Rj. Formirajmo Lagranžovu f-ju $F(x,y,\lambda)=2x+4y+\lambda\left(\frac{2}{x}+\frac{4}{y}-3\right)$.

$$\frac{\partial F}{\partial x}=2+2\lambda\cdot\frac{(-1)}{x^2}$$

$\left(\frac{1}{x}\right)'=(x^{-1})'=(-1)(x^{-2})$ $\qquad$ $(x^{-2})'=(-2)x^{-3}=\frac{-2}{x^3}$

$$\frac{\partial F}{\partial y}=4+4\lambda\cdot\frac{(-1)}{y^2}$$

$$\frac{\partial F}{\partial \lambda}=\frac{2}{x}+\frac{4}{y}-3$$

Formirajmo sistem

$$4-\frac{4\lambda}{y^2}=0 \quad /:4$$
$$2-\frac{2\lambda}{x^2}=0 \quad /:2$$
$$\frac{2}{x}+\frac{4}{y}=3$$

$1-\frac{\lambda}{x^2}=0 \qquad 1=\frac{\lambda}{x^2} \quad (1)$

$1-\frac{\lambda}{y^2}=0 \qquad 1=\frac{\lambda}{y^2} \quad (2)$

$\frac{2}{x}+\frac{4}{y}=3 \qquad \frac{2}{x}+\frac{4}{y}=3 \quad (3)$

(1) i (2) $\Rightarrow \frac{\lambda}{x^2}=\frac{\lambda}{y^2} \Rightarrow x^2=y^2$

tj. $x=\pm y$

za $x=y$ iz (3) $\frac{2}{x}+\frac{4}{x}=3$

$\frac{6}{x}=3 \Rightarrow x=2 \Rightarrow y=2$

za $x=-y$ iz (3)

$\frac{2}{x}-\frac{4}{x}=3 \Rightarrow -\frac{2}{x}=3$

$3x=-2 \Rightarrow x=-\frac{2}{3}$

$\Rightarrow y=\frac{2}{3}$

za $M_1(2,2) \Rightarrow 2-2\lambda\cdot\frac{1}{4}=0$

$\lambda=4$

Za $M_2\left(-\frac{2}{3},\frac{2}{3}\right) \Rightarrow 2-2\lambda\cdot\frac{9}{4}=0 \Rightarrow \lambda=\frac{4}{9}$

Stacionarne tačke su $M_1(2,2)$ za $\lambda=4$ i $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$ za $\lambda=\frac{4}{9}$.

$\frac{\partial^2 F}{\partial x^2}=\frac{4\lambda}{x^3}$,

$\frac{\partial^2 F}{\partial x\partial y}=0$

$\frac{\partial^2 F}{\partial y^2}=\frac{8\lambda}{y^3}$

Za $M_1(2,2)$, $\lambda=4$

$A=\frac{16}{8}=2$, $B=0$, $C=\frac{32}{8}=4$, $D=AC-B^2=8>0$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2)=4+8=12$

Za $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$, $\lambda=\frac{4}{9}$, $A=\frac{\frac{16}{9}}{-\frac{8}{27}}=-\frac{16\cdot 27}{8\cdot 9}=-2\cdot 3=-6$

$B=0$, $C=\frac{\frac{32}{9}}{\frac{8}{27}}=\frac{32\cdot 27}{8\cdot 9}=4\cdot 3=12$, $D=AC-B^2=-72<0 \Rightarrow$

$\Rightarrow$ f-ja u tački M_2 nema ekstremnu vrijednost

(#) Nadi ekstreme f-je $z = \frac{4}{x} + \frac{4}{y} + (x+y)^2$.

R-j

$$\frac{\partial z}{\partial x} = 4 \cdot (-1)\, x^{-2} + 2(x+y) = \frac{-4}{x^2} + 2x + 2y$$

$$\frac{\partial z}{\partial y} = 4 \cdot (-1)\, y^{-2} + 2(x+y) = \frac{-4}{y^2} + 2x + 2y$$

D: $x \neq 0$
$y \neq 0$
definiciono područje

$$-4 \cdot \frac{1}{x^2} + 2x + 2y = 0$$

$$-4\, \frac{1}{y^2} + 2x + 2y = 0$$

$$2x + 2y = \frac{4}{x^2}$$

$$2x + 2y = \frac{4}{y^2}$$

$$\frac{4}{x^2} = \frac{4}{y^2}$$

$$x^2 = y^2$$

$$x = \pm y$$

a) $x = y$

$$-\frac{4}{x^2} + 4x = 0 \quad /:4$$

$$x - \frac{1}{x^2} = 0 \quad / \cdot x^2 \ (x \neq 0)$$

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0 \quad \Rightarrow \quad \begin{matrix} x = 1 \\ y = 1 \end{matrix}$$

M(1,1) je stacionarna tačka

b) $x = -y$

$$-\frac{4}{x^2} = 0$$

ova jednačina
nema rješenje

$$\frac{\partial z}{\partial x^2} = \frac{8}{x^3} + 2$$

$$\frac{\partial z}{\partial x \partial y} = 2$$

$$\frac{\partial z}{\partial y^2} = \frac{8}{y^3} + 2$$

M(1,1)

$A = 10$
$B = 2$
$C = 10$

$D = AC - B^2 > 0$

f-ja ima ekstrem

$A > 0$

f-ja ima minimum

$$z_{min}(1,1) = 4 + 4 + 4 = 12$$

(#) Riješiti diferencijalnu jednačinu $(x^2y+x^2)dx+(x^4y-y)dy=0.$

Rj: $(x^2y+x^2)dx+(x^4y-y)dy=0$

$x^2(y+1)dx+(x^4-1)y\,dy=0$

$x^2(y+1)dx=-(x^4-1)y\,dy$

$\frac{y}{y+1}dy=-\frac{x^2}{x^4-1}dx$ diferencijalni račun sa razdvojenim promjenjivim //

$$\int\frac{y}{y+1}dy=-\int\frac{x^2}{x^4-1}dx$$

$$\int\frac{y^{+1-1}}{y+1}dy=\int dy-\int\frac{dy}{y+1}=y-\ln|y+1|+C$$

$$\frac{x^2}{x^4-1}=\frac{x^2}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2+1}\quad /\cdot(x^4-1)$$

$$x^2=A\underbrace{(x+1)(x^2+1)}_{x^3+x+x^2+1}+B\underbrace{(x-1)(x^2+1)}_{x^3+x-x^2-1}+C\underbrace{(x-1)(x+1)}_{x^2+x-x-1}$$

$$x^2=A(x^3+x^2+x+1)+B(x^3-x^2+x-1)+C(x^2-1)$$

$A+B=0$ (a) $\quad A=-B$

$A-B+C=1$ (b) $\quad$ (b): $-B-B+C=1 \Rightarrow -2B+C=1$

$A+B=0$ (c) $\quad$ (d): $-B-B-C=0 \Rightarrow -2B-C=0$ $\Big\}+ \Rightarrow -4B=1$, $B=-\frac{1}{4}$

$A-B-C=0$ (d)

$\Rightarrow A=\frac{1}{4} \qquad \frac{1}{4}+\frac{1}{4}+C=1 \Rightarrow C=\frac{1}{2}$

$$\int\frac{x^2}{x^4-1}dx=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{1}{4}\ln|x-1|-\frac{1}{4}|x+1|+\frac{1}{2}\operatorname{arctg}x+C$$

$$y-\ln|y+1|=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\operatorname{arc\,tg}x+C$$

rješenje diferencijalne jednačine

ⓒ# Riješiti diferencijalnu jednačinu

$$(5y+7x)\,dy+(8y+10x)\,dx=0.$$

Rj:

$$(5y+7x)\,dy+(8y+10x)\,dx=0$$

$$(5y+7x)\,dy=(-8y-10x)\,dx=0$$

$$\frac{dy}{dx}=\frac{-8y-10x \quad /:x}{5y+7x \quad /:x}$$

$$y'=\frac{-8\left(\frac{y}{x}\right)-10}{5\left(\frac{y}{x}\right)+7}$$

ovo je homogena diferencijalna jednačina, uvodimo smjenu $u=\frac{y}{x}$

$$y=u\cdot x \quad \Big/\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{-8u-10}{5u+7}$$

$$u'x=\frac{-8u-10}{5u+7}-u$$

$$u'x=\frac{-8u-10-u(5u+7)}{5u+7}$$

$$u'x=\frac{-8u-10-5u^2-7u}{5u+7}$$

$$\frac{du}{dx}x=\frac{\overbrace{-5u^2-15u-10}^{(-5)(u^2+3u+2)}}{5u+7}$$

$$\frac{du}{dx}x=(-5)\frac{(u+1)(u+2)}{5u+7}$$

$$\frac{(5u+7)\,du}{u^2+3u+2}=-5\,\frac{dx}{x} \quad \ldots(*) \;\Big/\!\!\int$$

$$\frac{5u+7}{u^2+3u+2}=\frac{5u+7}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2} \quad \Big/(u+1)(u+2)$$

$$5u+7=A(u+2)+B(u+1)$$

$$\begin{aligned} A+B&=5\\ -\;2A+B&=7\\ \hline \end{aligned} \qquad \begin{aligned} -A&=-2\\ A&=2 \end{aligned} \qquad B=3$$

$$\int\frac{5u+7}{u^2+3u+2}\,du=2\int\frac{du}{u+1}+3\int\frac{du}{u+2}$$

$$(*) \Rightarrow \quad 2\ln|u+1|+3\ln|u+2|=-5\ln|x|+\ln|C|$$

$$\ln(u+1)^2(u+2)^3=\ln(x^{-5}C)$$

$$(u+1)^2(u+2)^3=\frac{C}{x^5}$$

$$\left(\frac{y}{x}+1\right)^2\left(\frac{y}{x}+2\right)^3=\frac{C}{x^5}.$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu

$$(3y^2+3xy+x^2)dx=(x^2+2xy)dy$$

Rj:

$$(x^2+2xy)dy=(3y^2+3xy+x^2)dx \quad /:dx \quad /:(x^2+2xy)$$

$$\frac{dy}{dx}=\frac{3y^2+3xy+x^2}{x^2+2xy} \quad \begin{matrix}:x^2\\ :x^2\end{matrix}$$

$$y'=\frac{3(\frac{y}{x})^2+3\frac{y}{x}+1}{2\frac{y}{x}+1}$$

ovo je homogena difer. jedn.
uvodimo smjenu $u=\frac{y}{x}$

$$y=ux \quad /\frac{d}{dx}$$

$$y'=u'x+u$$

$$u'x+u=\frac{3u^2+3u+1}{2u+1}$$

$$u'x=\frac{3u^2+3u+1}{2u+1}-u$$

$$u'x=\frac{3u^2+3u+1-2u^2-u}{2u+1}$$

$$u'x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{du}{dx}x=\frac{u^2+2u+1}{2u+1}$$

$$\frac{2u+1}{u^2+2u+1}du=\frac{dx}{x} \quad \ldots(*)$$

$$\int\frac{2u+1}{u^2+2u+1}du=\int\frac{2u+2-1}{u^2+2u+1}du=$$

$$=\int\frac{2u+2}{u^2+2u+1}du-\int\frac{du}{u^2+2u+1}=$$

$$=\left|\begin{matrix}u^2+2u+1=t\\(2u+2)du=dt\end{matrix}\right|=\int\frac{dt}{t}-\int\frac{du}{(u+1)^2}=\left|\begin{matrix}u+1=s\\du=ds\end{matrix}\right|=$$

$$\ln|t|-\int\frac{ds}{s^2}=\ln|u^2+2u+1|-\frac{s^{-1}}{(-1)}+c=\ln(u+1)^2+\frac{1}{u+1}+c$$

$$(*) \Rightarrow \ln(u+1)^2+\frac{1}{u+1}=\ln|x|+c$$

$$\ln(\frac{y}{x}+1)^2+\frac{1}{\frac{y}{x}+1}=\ln|x|+c$$ rješenje diferencijalne jednačine