Pismeni dio ispita iz Matematike, 23.06.2010.

I GRUPA

1. Riješiti matričnu jednačinu $A\cdot X^{-1}\cdot B = B\cdot A,$ ako je $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$
2. Ispitati funkciju i nacrtati joj grafik: $y = xe^{\frac{1}{2}\left(1-\frac{1}{x^2}\right)}.$
3. Izračunati površinu manje figure koja je određena linjama $x^2+y^2=16,\ x^2=12(y-1).$
4. Naći ekstreme funkcije $z=e^{-2x^2}\left(x-y^2\right).$

II GRUPA

1. Izračunati x ako je treći član u razvoju binoma $\left(x^{\log x}+x\right)^5$ jednak 100.
2. Ispitati funkciju i nacrtati joj grafik: $y=\dfrac{x^4-9x^2+12}{3x}.$
3. Izračunati površinu figure koja je određena linjama $y=\dfrac{16}{x^2},\ y=17-x^2$ u prvom kvadrantu.
4. Naći ekstreme funkcije $z=x^3-5xy+5y^2+7x-15y$.

III GRUPA

1. Odrediti vrijednost parametra k tako da sistem
$$8z-3x-6y=kx$$
$$2x+y+4z=ky$$
$$4x+3y+z=kz$$
ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.
2. Ispitati funkciju i nacrtati joj grafik: $\ln\dfrac{x^2-3x+2}{x^2+1}$.
3. Izračunati integral $I=\int_0^2 \ln\dfrac{x+4}{4-x}dx.$
4. Riješiti diferencijalnu jednačinu $y-xy'=a\left(1+x^2y'\right),\ a=const.$

IV GRUPA

1. Ako je $z=\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2},$ izračunati sve vrijednosti korjena $\sqrt[3]{\left(z+\dfrac{1}{z}+i\right)^5}$.
2. Ispitati funkciju $y=\dfrac{ax+b}{x^2+x+1}$ i nacrtati joj grafik ako se zna da ona ima ekstrem u tački $T\left(1,\dfrac{2}{3}\right).$
3. Izračunati integral $I=\int_0^{\frac{\pi}{2}}\cos x\sqrt{3\sin^2 x+2\cos^2 x}\ dx.$
4. Riješiti diferencijalnu jednačinu $y'=\dfrac{3x^2}{x^3+y+1}$.

(Za sve uočene greške pisati na **infoarrt@gmail.com**)

Izračunati x ako je treći član u razvoju binoma $(x^{\log x} + x)^5$ jednak 100.

Rj. $$(x^{\log x} + x)^5 = \sum_{k=0}^{5} \binom{5}{k} (x^{\log x})^{5-k} (x)^k$$

treći član je za $k=2$ tj. $\binom{5}{2} (x^{\log x})^3 x^2 = 100$.

$$\frac{5 \cdot \cancel{4}^{2}}{1 \cdot \cancel{2}} x^{3\log x} \cdot x^2 = 100 \quad /:10$$

$$x^{3\log x + 2} = 10 \quad /\log$$

$$\log x^{3\log x + 2} = 1$$

$$(3\log x + 2)\log x = 1$$

$$3\log^2 x + 2\log x - 1 = 0$$

$\log x = t$

$$3t^2 + 2t - 1 = 0$$

$$D = 4 + 12 = 16$$

$$t_{1,2} = \frac{-2 \pm 4}{6}$$

$$t_1 = \frac{-2-4}{6} = -1 \quad t_2 = \frac{2}{6} = \frac{1}{3}$$

$\log x = -1$

$\log x = (-1)\log 10$

$\log x = \log 10^{-1}$

$x = \frac{1}{10}$ jedno rješenje

$\log x = \frac{1}{3}$

$\log x = \log 10^{\frac{1}{3}}$

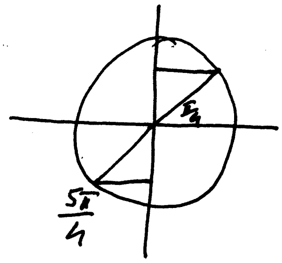
$x = \sqrt[3]{10}$ drugo rješenje

Ako je $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, izračunati sve vrijednosti korijena $\sqrt[3]{\left(z+\frac{1}{z}+i\right)^5}$.

R.: $z=\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $z+\frac{1}{z}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{1}{\frac{1-i\sqrt{3}}{2}}=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2}{1-i\sqrt{3}}\cdot\frac{(1+i\sqrt{3})}{(1+i\sqrt{3})}$

$=\frac{1}{2}-i\frac{\sqrt{3}}{2}+\frac{2+2i\sqrt{3}}{1+3}=\frac{1}{2}-\cancel{i\frac{\sqrt{3}}{2}}+\frac{1}{2}+\cancel{i\frac{\sqrt{3}}{2}}=1$

$z+\frac{1}{z}+i=1+i$ Uvedimo oznaku $w=z+\frac{1}{z}+i=1+i$



$|w|=\sqrt{2}$

$$\left.\begin{array}{l}\cos\varphi=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \sin\varphi=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \operatorname{tg}\varphi=1\end{array}\right\}\Rightarrow\varphi=45^\circ=\frac{\pi}{4}\text{ rad}$$

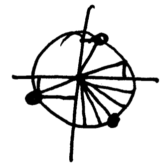
$w=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$w^5=(\sqrt{2})^5\left(\cos 5\cdot\frac{\pi}{4}+i\sin 5\cdot\frac{\pi}{4}\right)=4\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$w^n=c$ gdje je c kompleksan broj ima tačno n rješenje

$w_k=\sqrt[n]{|c|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)$, φ najmanji pozitivan ugao iz $[0,2\pi)$
$k=1,2,\dots,n$

Mi treba da nađemo $\sqrt[3]{\left(z+\frac{1}{z}+i\right)^5}$ tj. $\sqrt[3]{w^5}$



$v_1=\sqrt[3]{4\sqrt{2}}\left(\cos\frac{\frac{5\pi}{4}+0}{3}+i\sin\frac{5\pi}{12}\right)=32^{\frac{1}{6}}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)=$

$=\sqrt[6]{32}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$

$v_2=\sqrt[6]{32}\left(\cos\frac{\frac{5\pi}{4}+2\pi}{3}+i\sin\frac{\frac{5\pi+8\pi}{4}}{3}\right)=\sqrt[6]{32}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)$

$v_3=\sqrt[6]{32}\left(\cos\frac{\frac{5\pi}{4}+4\pi}{3}+i\sin\frac{\frac{5\pi}{4}+\frac{16\pi}{4}}{3}\right)=\sqrt[6]{32}\left(\cos\frac{21\pi}{12}+i\sin\frac{21\pi}{12}\right)$

Napišimo rješenja v_1, v_2 i v_3 u obliku $a+ib$:

$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}$

$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$

Kako je $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

to je $$v_1 = \sqrt[6]{32}\left(\frac{\sqrt{6}-\sqrt{2}}{4} + i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$\cos\frac{13\pi}{12} = -\cos\frac{\pi}{12} = -\frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin\frac{13\pi}{12} = -\sin\frac{\pi}{12} = -\frac{\sqrt{6}-\sqrt{2}}{4}$

$$v_2 = \sqrt[6]{32}\left(-\frac{\sqrt{6}+\sqrt{2}}{4} - i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$\cos\frac{21\pi}{12} = \cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin\frac{21\pi}{4} = \sin\frac{7\pi}{4} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$$v_3 = \sqrt[6]{32}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$$

v_1, v_2 i v_3 su tražena rješenja

(#) Riješiti matričnu jednačinu $A \cdot X^{-1} \cdot B = B \cdot A$, ako je $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Rj.

$$A X^{-1} B = B \cdot A \quad / \cdot A^{-1} \text{ sa lijeve strane}$$

$$X^{-1} B = A^{-1} B \cdot A \quad / \cdot B^{-1} \text{ sa desne strane}$$

$$X^{-1} = A^{-1} B \cdot A \cdot B^{-1} \quad /^{-1}$$

$$X = B A^{-1} B^{-1} A$$

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^{T} \qquad \det A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{kof} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$A_{11} = 1 \qquad A_{21} = -1$

$A_{12} = 0 \qquad A_{22} = 1$

$$A_{kof}^{T} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T}$$

$B_{11} = 1$

$B_{12} = -1$

$B_{21} = 0$

$B_{22} = 1$

$$B_{kof} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad B_{kof}^{T} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$B \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X = B A^{-1} \cdot B^{-1} A =$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$ traženo rješenje

(#) Odrediti vrijednost parametra k tako da sistem

$$8z - 3x - 6y = kx$$
$$2x + y + 4z = ky$$
$$4x + 3y + z = kz$$

ima beskonačno mnogo rješenja. Zatim naći ta rješenja za najveću dobijenu vrijednost parametra k.

R: Nepoznate sa desne strane prebacimo na lijevu i grupirajmo vrijednosti uz x, y i z.

$$(-3-k)x - 6y + 8z = 0$$
$$2x + (1-k)y + 4z = 0$$
$$4x + 3y + (1-k)z = 0$$

$$\begin{vmatrix} -3-k & -6 & 8 \\ 2 & 1-k & 4 \\ 4 & 3 & 1-k \end{vmatrix} = 0$$

$I_k + III_k$:
$$\begin{vmatrix} 5-k & -6 & 8 \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

Ovo je homogeni sistem linearnih jednačina. Trivijalno rješenje je $(0,0,0)$. Sistem ima beskonačno mnogo rješenja ako je $D=0$.

$I_v - III_v$:
$$\begin{vmatrix} 0 & -9 & 7+k \\ 6 & 1-k & 4 \\ 5-k & 3 & 1-k \end{vmatrix} = 0$$

$$6\left[\overbrace{(-9)(1-k)}^{-9+9k} - \overbrace{3(7+k)}^{-21-3k}\right] + (5-k)\left[(-9)\cdot 4 - \overbrace{(7+k)(1-k)}^{\substack{7-6k-k^2 \\ 7-7k+k-k^2}}\right] = 0$$

$$(-6)(6k-30) + (5-k)(\overbrace{-36-7}^{-43}+6k+k^2) = 0$$

$$-36k + 180 + (-215) + 30k + 5k^2 + 43k - 6k^2 - k^3 = 0$$

$$-k^3 - k^2 + 37k - 35 = 0 \quad / \cdot (-1)$$

$$k^3 + k^2 - 37k + 35 = 0$$
$$k^3 - k^2 + 2k^2 - 2k - 35k + 35 = 0$$
$$k^2(1-k) + 2k(k-1) - 35(k-1) = 0$$

$$(k-1)(k^2+2k-35) = 0$$
$$(k-1)(k+7)(k-5) = 0$$
$$k_1 = 1, \quad k_2 = -7, \quad k_3 = 5$$

Za $k=5$ imamo:

$$8x + 6y - 8z = 0 \quad \dots(1)$$
$$2x - 4y + 4z = 0 \quad \dots(2)$$
$$4x + 3y - 4z = 0 \quad \dots(3)$$

$(2)+(3)$: $6x - y = 0$
$\Rightarrow y = 6x$

$(2) \Rightarrow 2x - 24x + 4z = 0$
$4z = 22x$
$z = \frac{11x}{2}$

$(1) \equiv (3)$ jer se (3) dobija dijeljenjem (1) sa 2.

Za $k=5$ sistem ima rješenje $(t, 6t, \frac{11t}{2})$ gdje je $t \in \mathbb{R}$ proizvoljno.

(#) Ispitati f-ju i nacrtati joj grafik $y=\frac{x^4-9x^2+12}{3x}$.

R: definiciono područje

$D: x\neq 0$

$x\in\mathbb{R}\setminus\{0\}$

parnost (neparnost), periodičnost

$$f(-x)=\frac{(-x)^4-9(-x)^2+12}{3(-x)} = -\frac{x^4-9x^2+12}{3x} = -f(x)$$

f-ja je neparna $\xrightarrow{\text{simetrična u odnosu na } (0,0)}$ (dovoljno ju je ispitati za $x>0$)

f-ja nije periodična

znak f-je: 0, $1{,}2758$, $2{,}71$ ← prekidi f-je i nule f-je

x	$(0, 1{,}27)$	$(1{,}27,\ 2{,}71)$	$(2{,}71, +\infty)$
y	$+$	$-$	$+$

znak f-je

nule, presjek sa y-osom i znak f-je

$y=0$ akko $x^4-9x^2+12=0$

$x^2=t$

$t^2-9t+12=0$

$D=81-48=33$

$$t_{1,2}=\frac{9\pm\sqrt{33}}{2}$$

$x^2=\frac{9-\sqrt{33}}{2}$ $\quad x^2=\frac{9+\sqrt{33}}{2}$

$x_1\approx -1{,}2758$ $\quad x_2\approx -2{,}7152$

$x_3\approx 1{,}2758$ $\quad x_4\approx 2{,}7152$

$f(0)$ nije definisano

f-ja ne siječe y-osu

ponašanje na krajevima intervala definisanosti i asimptote

Za $x=0$ f-ja ima prekid

$$\lim_{x\to 0+} f(x)=\lim_{x\to +0}\frac{x^4-9x^2+12}{3x}=\frac{12}{0+}=+\infty$$

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{x^4-9x^2+12}{3x}=\frac{12}{0-}=-\infty$$

$\Rightarrow$ $x=0$ je V.A.

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^4-9x^2+12}{3x}\,\frac{/:x}{/:x}=\lim_{x\to\pm\infty}\frac{x^3-9x+\frac{12}{x}}{3}=\pm\infty$$

$\Rightarrow$ f-ja nema H.A.

tražimo kosu asimptotu u obliku $y=kx+n$,

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^4-9x^2+12}{3x^2}\,\frac{/:x^2}{/:x^2}=\infty$$

f-ja nema kosu asimptotu

Nakon ovog koraka počinjemo skicirati graf f-je.

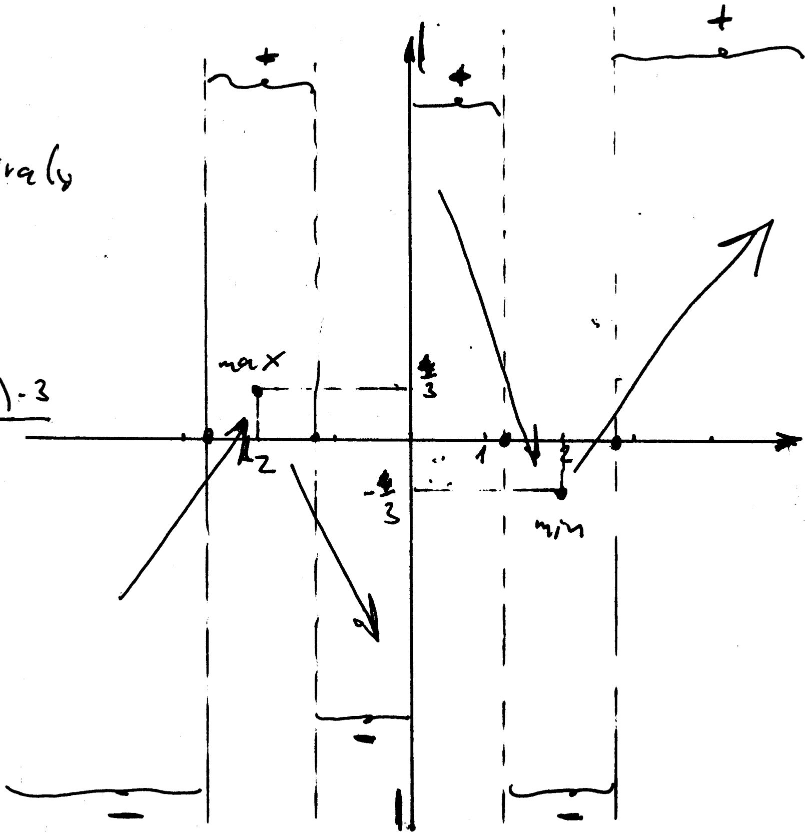
rast i opadanje

$$y'=\left(\frac{x^4-9x^2+12}{3x}\right)'=\frac{(4x^3-18x)3x-(x^4-9x^2+12)\cdot 3}{9x^2}$$

$$=\frac{12x^4-54x^2-3x^4+27x^2+36}{9x^2}=$$

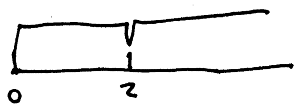
$$=\frac{9x^4-27x^2-36}{9x^2}=\frac{x^4-3x^2-4}{x^2}$$

$$y'=x^2-3-\frac{4}{x^2}$$



$y' = 0$ akko $x^4 - 3x^2 - 4 = 0$

$t = x^2$



← prebidi f-je y + nule f-je y'

$t^2 - 3t - 4 = 0$

$D = 9 + 16 = 25$

$t_{1,2} = \dfrac{3 \pm 5}{2}$

$t_1 = -1 \quad t_2 = 4$

$\Downarrow$

$x^2 = 4$

$x_1 = -2 \quad x_2 = 2$

x	(0, 2)	(2, +∞)
y'	−	+
y	↘	↗

min

$f(2) = \dfrac{16 - 36 + 12}{6}$

$f(2) = -\dfrac{8}{6} = -\dfrac{4}{3}$

ekstremi f-je

Na osnovu tabele rasta i opadanja i simetričnosti grafa f-ja ima minimum u $(2, -\frac{4}{3})$ i maksimum u $(-2, \frac{4}{3})$.

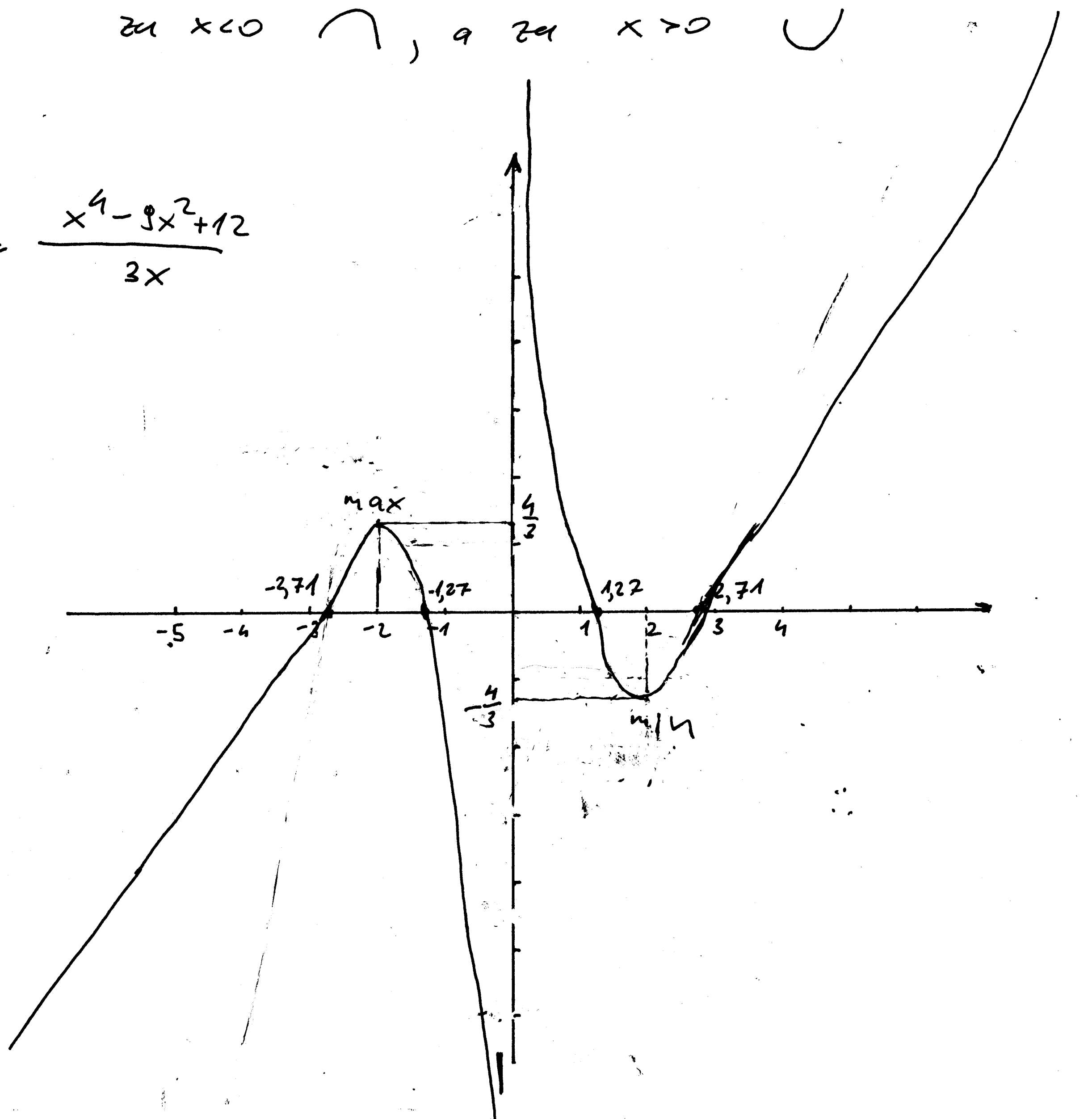
prevojne tačke i intervali konveksnosti i konkavnosti

$y'' = \left(x^2 - 3 - \dfrac{4}{x^2}\right)' = 2x - 4(-2)x^{-3} = 2x + \dfrac{8}{x^3}$

$y'' = \dfrac{2x^4 + 8}{x^3}$ kako je $2x^4 + 8 > 0 \ \forall x \Rightarrow$ f-ja nema prevojnih tački

za $x < 0$ $\cap$, a za $x > 0$ $\cup$

f-ja $y = \dfrac{x^4 - 9x^2 + 12}{3x}$



(#) Ispitati f-ju $y=\frac{ax+b}{x^2+x+1}$ i nacrtati joj grafik ako se zna da ona ima ekstrem u tački $T(1,\frac{2}{3})$.

Rj. $f(x)=\frac{ax+b}{x^2+x+1}$

$$y'=\frac{a(x^2+x+1)-(ax+b)(2x+1)}{(x^2+x+1)^2}$$

$f(1)=\frac{2}{3} \Rightarrow \frac{a+b}{3}=\frac{2}{3}$

$a+b=2$

$$y'=\frac{a(x^2+x+1)-(2ax^2+ax+2bx+b)}{(x^2+x+1)^2}$$

$$y'=\frac{-ax^2-2bx+a-b}{(x^2+x+1)^2}$$

U stacionarnoj tački f-ja može imati ekstrem

$y'=0 \Rightarrow -ax^2-2bx+a-b=0$

$x=1$

$-a-2b+a-b=0$

$-3b=0$

$b=0, \quad a=2$

$$y=\frac{2x}{x^2+x+1}$$

$$y'=\frac{-2x^2+2}{(x^2+x+1)^2}$$

$$y'=(-2)\,\frac{x^2-1}{(x^2+x+1)^2}$$

definiciono područje

$x^2+x+1\neq 0$

f-ja je definisana za $\forall x$

parnost (neparnost), periodičnost

$f(-x)=\frac{-2x}{x^2-x+1}$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa x-osom, znak f-je

$y=0 \Rightarrow 2x=0 \Rightarrow x=0$

$(0,0)$ je presjek sa y-osom i nula f-je

kako je $x^2+x+1>0$ $\forall x$ to je

$y>0$ za $x>0$

$y<0$ za $x<0$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekid $\Rightarrow$ f-ja nema vertikalnu asimptotu

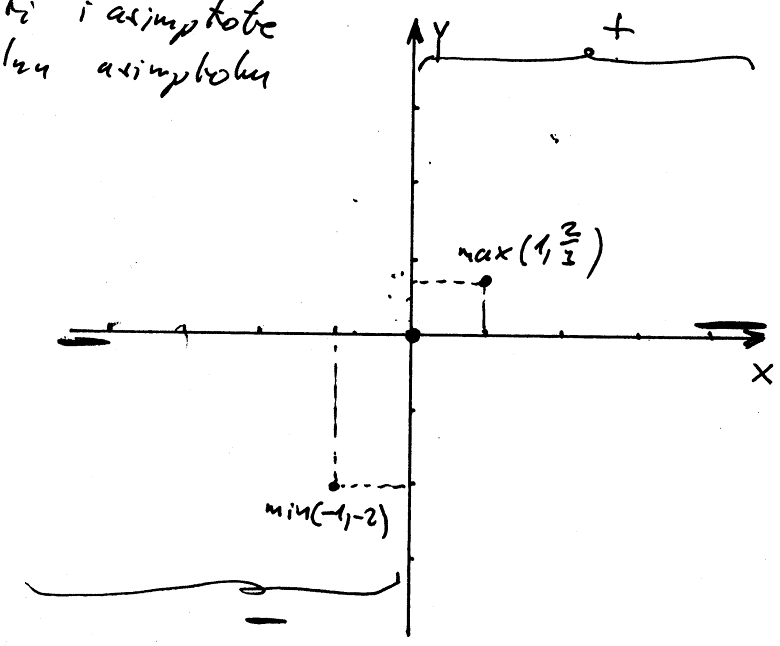
$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{2x \;/:x}{x^2+x+1 \;/:x}=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{2x}{x^2+x+1}=0$$

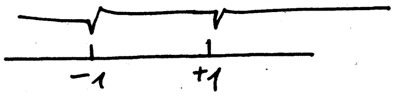
$\Rightarrow$ $x=0$ je H.A.

F-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik f-je.



rast i opadanje



$$Y' = (-2)\,\frac{x^2-1}{(x^2+x+1)^2}$$

x	$(-\infty,-1)$	$(-1,1)$	$(1,+\infty)$
Y'	$-$	$+$	$-$
Y	$\searrow$	$\nearrow$	$\searrow$

min max

$$Y' = 0 \Rightarrow x = \pm 1$$

ekstremi f-je

$$f(-1) = \frac{-2}{1-1+1} = -2$$

F-ja ima minimum u tački $P(-1,-2)$ i maksimum u tački $(1, \frac{2}{3})$

$$f(1) = \frac{2}{1+1+1} = \frac{2}{3}$$

prevojne tačke i intervali konveksnosti i konkavnosti

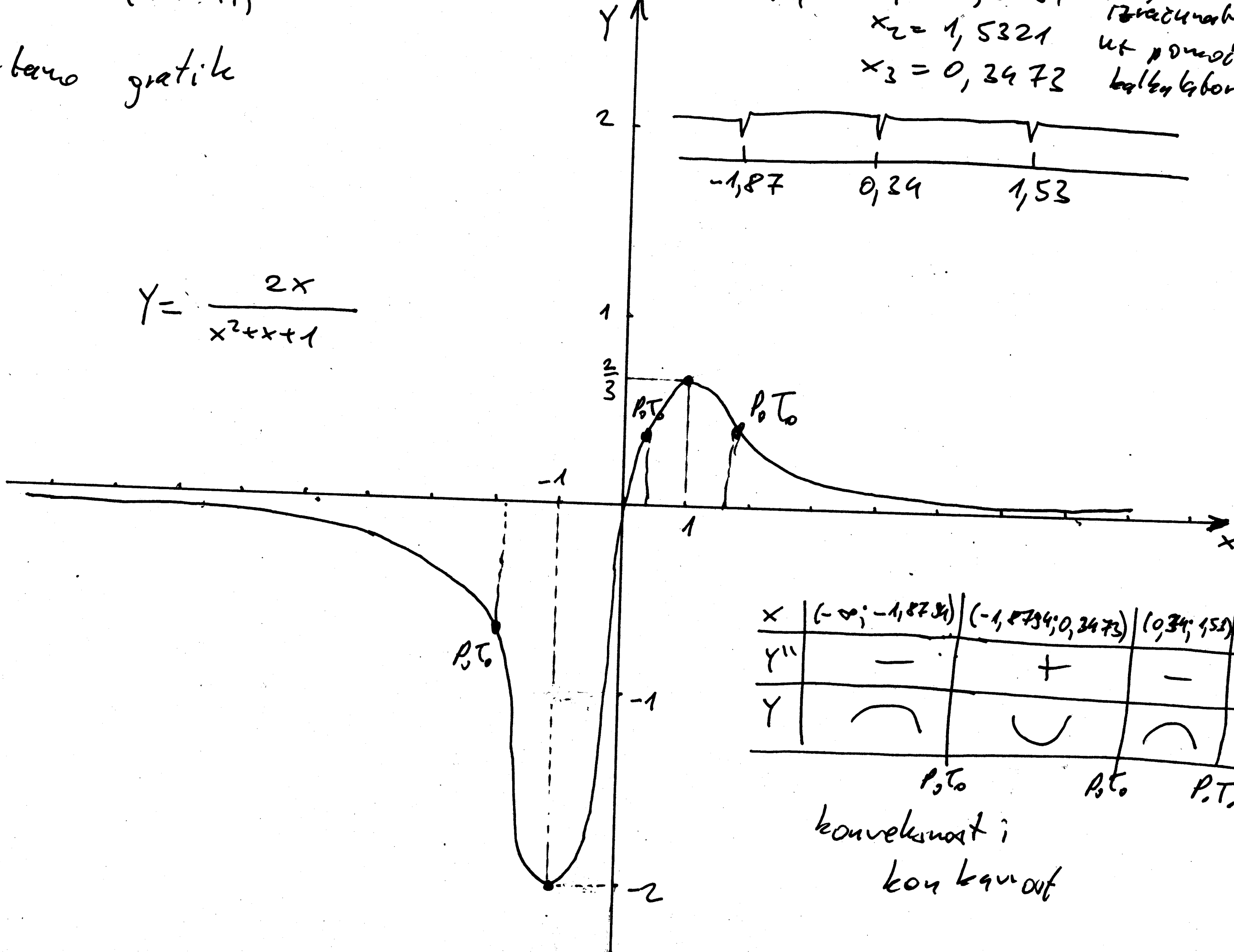
$$Y'' = (-2)\left(\frac{x^2-1}{(x^2+x+1)^2}\right)' = (-2)\,\frac{2x(x^2+x+1)^{\cancel{2}\,1} - (x^2-1)\,2\cancel{(x^2+x+1)}(2x+1)}{(x^2+x+1)^{\cancel{4}\,3}}$$

$$Y'' = (-2)\,\frac{2x^3+\cancel{2x^2}+2x-4x^3+4x-\cancel{2x^2}+2}{(x^2+x+1)^3} = (-2)\,\frac{-2x^3+x+2}{(x^2+x+1)^3} = (-2)\,\frac{(-2)(x^3-3x-1)}{(x^2+x+1)^3}$$

$$Y'' = 4\,\frac{x^3-3x-1}{(x^2+x+1)^3}$$

korjeni od x^3-3x-1 su $x_1 = -1,8734$, $x_2 = 1,5321$, $x_3 = 0,3473$ (vrijednosti izračunate uz pomoć kalkulatora)

crtamo grafik



$$Y = \frac{2x}{x^2+x+1}$$

x	$(-\infty; -1,8734)$	$(-1,8734; 0,3473)$	$(0,34; 1,53)$	$(1,53; \ldots)$
Y''	$-$	$+$	$-$	$+$
Y	$\frown$	$\smile$	$\frown$	$\smile$

P.T. P.T. P.T.

konveksnost i konkavnost

(#) Ispitati f-ju i nacrtati joj grafik $y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$

Rj. definiciono područje

$x \neq 0$

$\mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

parnost (neparnost), periodičnost

$f(-x) = -x e^{\frac{1}{2}(1-\frac{1}{(-x)^2})} = -x e^{\frac{1}{2}(1-\frac{1}{x^2})} = -f(-x)$

f-ja je neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$f(0)$ nije definisano

f-ja ne sječe y-osu

$y \neq 0, \ \forall x \in \mathcal{D}$

$(e^{\frac{1}{2}(1-\frac{1}{x^2})} > 0 \ \forall x)$

f-ja nema nulu

x	$(-\infty, 0)$	$(0, +\infty)$
y	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=0$ f-ja ima prekid

$\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0-)\cdot e^{\frac{1}{2}(1-\infty)} = (0-)e^{-\infty} = \frac{0-}{e^{\infty}} = 0$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (0+)e^{-\infty} = 0$

f-ja nema vertikalnu asimptotu

$\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = (-\infty)\cdot e^{\frac{1}{2}} = -\infty$

$\lim\limits_{x\to \infty} f(x) = \lim\limits_{x\to \infty} x e^{\frac{1}{2}(1-\frac{1}{x^2})} = \infty \cdot e^{\frac{1}{2}} = \infty$

f-ja nema horizontalnu asimptotu

tražimo kosu asimptotu u obliku

$y = kx + n$

$k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} e^{\frac{1}{2}(1-\frac{1}{x^2})} = e^{\frac{1}{2}} = \sqrt{e}$

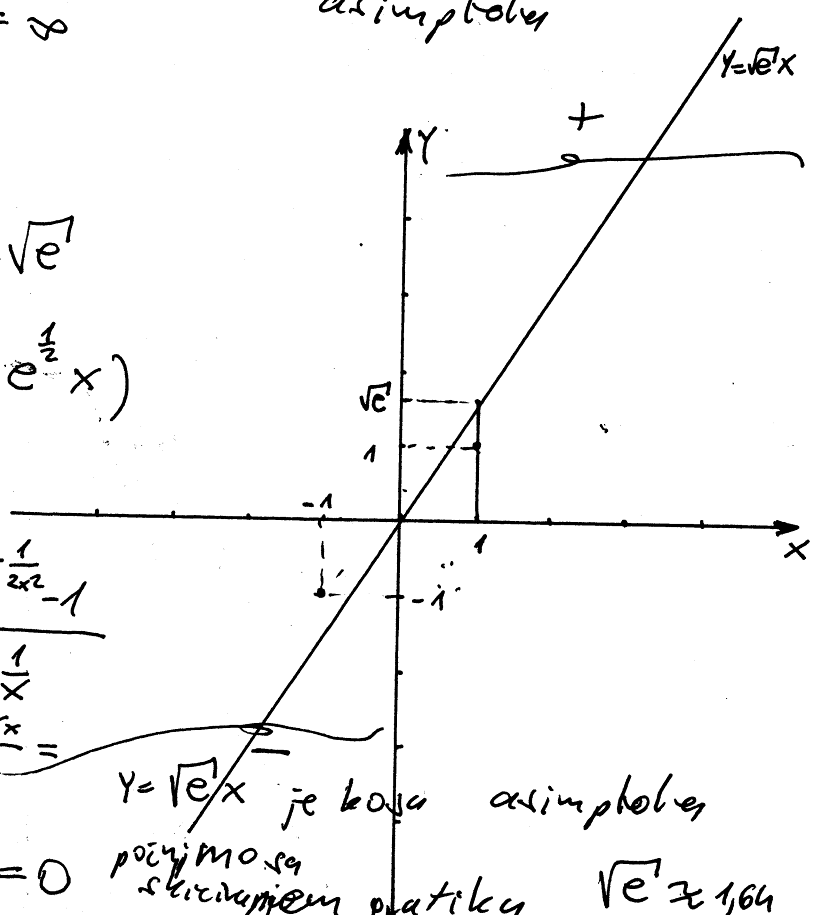
$n = \lim\limits_{x\to\infty} [f(x) - k\cdot x] = \lim\limits_{x\to\infty} (x e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}} x)$

$= \lim\limits_{x\to\infty} x(e^{\frac{1}{2}(1-\frac{1}{x^2})} - e^{\frac{1}{2}}) =$

$= \lim\limits_{x\to\infty} e^{\frac{1}{2}} x (e^{\frac{-1}{2x^2}} - 1) = \sqrt{e} \lim\limits_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} - 1}{\frac{1}{x}}$

$(= \frac{0}{0}) \overset{L.P.}{=} \sqrt{e} \lim\limits_{x\to\infty} \frac{e^{-\frac{1}{2x^2}} \cdot (-\frac{1}{2})(-2)\frac{1}{x^3}}{\frac{-1}{x^2}} =$

$= \sqrt{e} \lim\limits_{x\to\infty} \frac{-e^{-\frac{1}{2x^2}}}{x} = \sqrt{e} \cdot \frac{-1}{\infty} = 0$



$y = \sqrt{e}x$ je kosa asimptota

počinjemo sa skiciranjem grafika $\sqrt{e} \approx 1{,}64$

$$\left(-\frac{1}{x^2}\right)' = (-x^{-2})' = 2x^{-3} = \frac{2}{x^3}$$

rast i opadanje

$$y' = \left(x e^{\frac{1}{2}(1-\frac{1}{x^2})}\right)' = e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \left(\frac{1}{2}\left(1-\frac{1}{x^2}\right)\right)' =$$

$$= e^{\frac{1}{2}(1-\frac{1}{x^2})} + x e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3} = e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1+\frac{1}{x^2}\right)$$

$y' = 0$ akko $1 + \frac{1}{x^2} = 0$

$\frac{x^2+1}{x^2} = 0$

$y' > 0 \ \forall x \Rightarrow$ f-ja uvijek raste

f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left[e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(1+\frac{1}{x^2}\right)\right]' = e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{x^3}\left(1+\frac{1}{x^2}\right) + e^{\frac{1}{2}(1-\frac{1}{x^2})} \cdot \frac{-2}{x^3} =$$

$$= e^{\frac{1}{2}(1-\frac{1}{x^2})}\left(\frac{1}{x^3} + \frac{1}{x^5} - \frac{2}{x^3}\right) = \left(\frac{1}{x^5} - \frac{1}{x^3}\right) e^{\frac{1}{2}(1-\frac{1}{x^2})}$$

$f(1) = 1 e^{\frac{1}{2}0} = 1$

$y'' = 0$ akko $\frac{1-x^2}{x^5} = 0 \Rightarrow 1 - x^2 = 0$

$x = \pm 1$

prekidi od y + nule od y''

-1 0 1

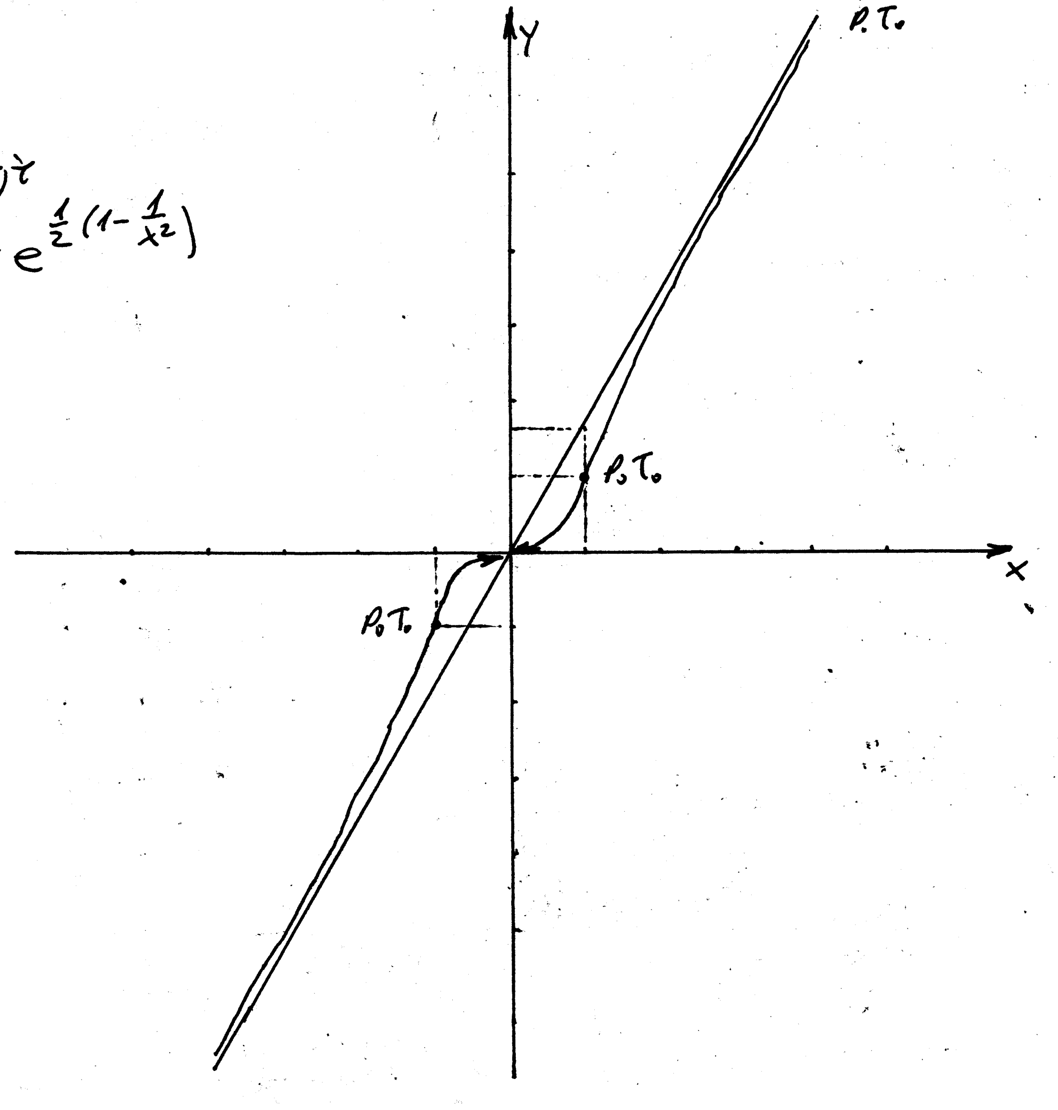
	$(0,1)$	$(1,+\infty)$
y''	+	−
y	$\cup$	$\cap$

P.T.

(1,1) i (-1,-1) su prevojne tačke

graf. f-je

$y = x e^{\frac{1}{2}(1-\frac{1}{x^2})}$



Ispitati f-ju i nacrtati joj grafik $y = \ln \frac{x^2-3x+2}{x^2+1}$.

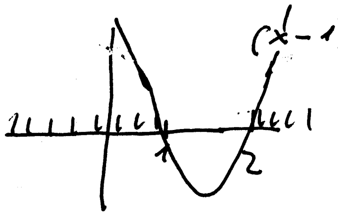
Rj: definiciono područje

Kako je $x^2+1>0 \quad \forall x \in \mathbb{R}$

to iz $\frac{x^2-3x+2}{x^2+1} > 0 \Rightarrow$

treba da bude $x^2-3x+2>0$

$(x-1)(x-2)>0$



$\mathcal{D}: x \in (-\infty, 1) \cup (2, +\infty)$

nule, presjek sa y-osom, znak

$y=0 \Rightarrow \ln \frac{x^2-3x+2}{x^2+1} = 0$

$\Rightarrow \frac{x^2-3x+2}{x^2+1} = 1 \quad / \cdot x^2+1$

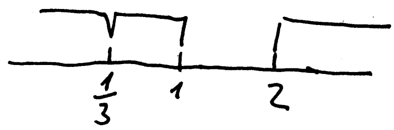
$\cancel{x^2}-3x+2 = \cancel{x^2}+1$

$3x = 1 \Rightarrow x = \frac{1}{3}$

$(\frac{1}{3}, 0)$ je nula f-je

$y(0) = \ln 2 \approx 0{,}6931$

$(0, \ln 2)$ je presjek sa y-osom



← prekidi y + nule y

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

x	$(-\infty, \frac{1}{3})$	$(\frac{1}{3}, 1)$	$(2, +\infty)$
y	+	−	−

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja ima prekid za $x=1$ i $x=2$

$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} \ln \frac{x^2-3x+2}{x^2+1} = \ln \frac{(1-0)^2-3(1-0)+2}{(1-0)^2+1} = \ln(0_+) = -\infty \Rightarrow$

$\Rightarrow x=1$ je V. A. (sa lijeve str.)

$\lim_{x\to 2+0} f(x) = \lim_{x\to 2+0} \ln \frac{x^2-3x+2}{x^2+1} = \ln(0+) = -\infty$

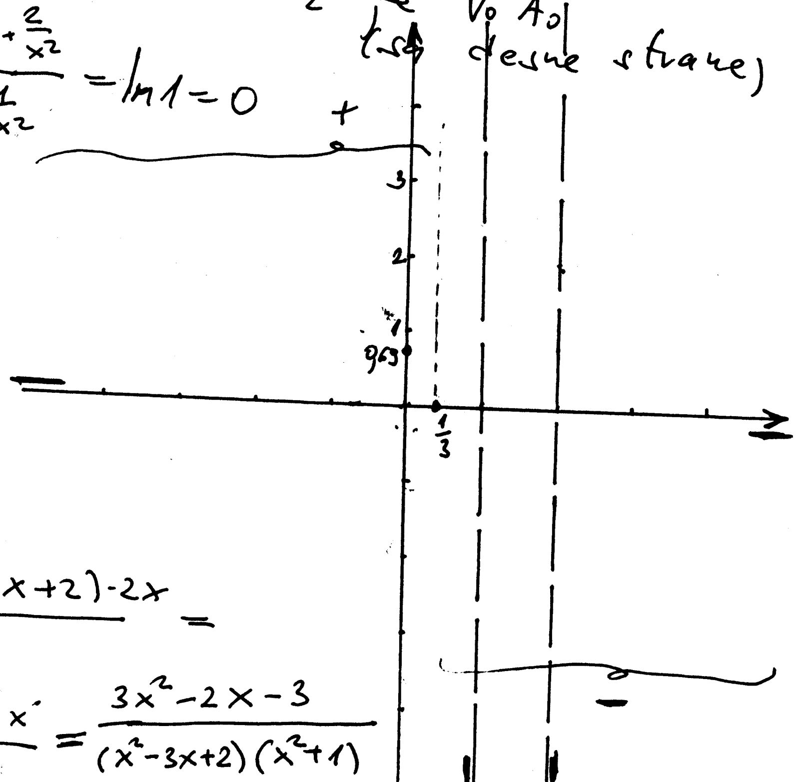
$\Rightarrow x=2$ je V. A. (sa desne strane)

$\lim_{x\to\mp\infty} f(x) = \lim_{x\to\mp\infty} \ln \frac{x^2-3x+2}{x^2+1} = \lim_{x\to\mp\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}} = \ln 1 = 0$

$\Rightarrow y=0$ je H. A.

K. A. nema.

počinjemo sa skiciranjem grafa



rast i opadanje

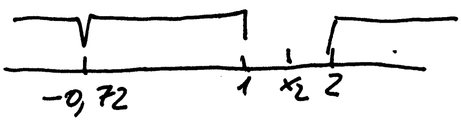
$y' = \frac{x^2+1}{x^2-3x+2} \cdot \left(\frac{x^2-3x+2}{x^2+1}\right)'$

$y' = \frac{\cancel{x^2+1}}{x^2-3x+2} \cdot \frac{(2x-3)(x^2+1)-(x^2-3x+2)\cdot 2x}{(x^2+1)^{\cancel{2}}} =$

$= \frac{\cancel{2x^3}+2x-3x^2-3-\cancel{2x^3}+6x^2-4x}{(x^2-3x+2)(x^2+1)} = \frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}$

$y' = 0 \Rightarrow 3x^2 - 2x - 3 = 0 \Rightarrow x_{1,2} = \dfrac{2 \pm \sqrt{4+36}}{6}$

$x_{1,2} = \dfrac{2 \pm 2\sqrt{10}}{6} = \dfrac{1 \pm \sqrt{10}}{3}$



$x_1 = \dfrac{1+\sqrt{10}}{3} \approx 1,387 \notin \mathcal{D}$

$x_2 = \dfrac{1-\sqrt{10}}{3} \approx -0,721 \in \mathcal{D}$

x	$\left(-\infty, \frac{1-\sqrt{10}}{3}\right)$	$\left(\frac{1-\sqrt{10}}{3}, 1\right)$	$(2, +\infty)$
y'	+	−	+
y	↗	↘	↗

max

ekstremi f-je

$f\left(\frac{1-\sqrt{10}}{3}\right) \approx 1,016$

F-ja ima maksimum u tački $(-0,72; 1,02)$

prevojne tačke i intervali konveksnosti i konkavnosti

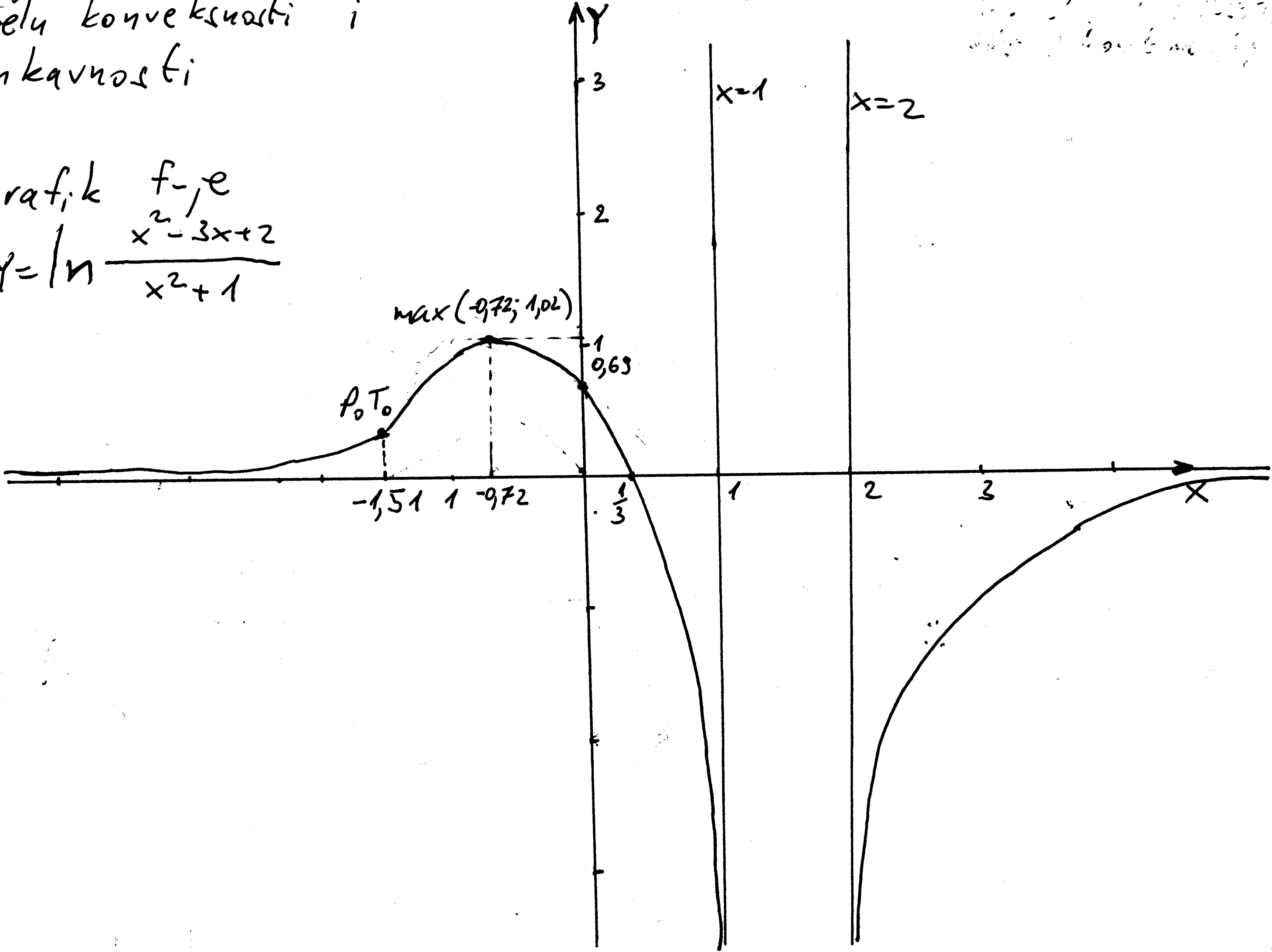
$y'' = \left(\dfrac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}\right)' = \underset{\dots}{\text{ZA VJEŽBU}} = \dfrac{-6x^5+15x^4-30x^2+30x-13}{(x^2-3x+2)^2(x^2+1)^2}$

$y'' = 0$ akko $x = -1,5166$ (izračunato uz pomoć kalkulatora)

Kako je brojnik u y'' previše složen nije potrebno praviti tabelu konveksnosti i konkavnosti

grafik f-je

$y = \ln \dfrac{x^2-3x+2}{x^2+1}$



(#) Izračunati integral $I=\int_0^2 \ln\frac{x+4}{4-x}\,dx.$

Rj.

$$I=\int_0^2 \ln\frac{x+4}{4-x}\,dx = \left|\begin{array}{l} u=\ln\frac{x+4}{4-x} \\ du=\frac{1}{\frac{x+4}{4-x}}\cdot\underbrace{\left(\frac{x+4}{4-x}\right)'}_{\frac{4-x-(x+4)\cdot(-1)}{(4-x)^2}}dx=\frac{\cancel{4-x}}{x+4}\cdot\frac{8\,dx}{(4-x)^{\cancel{2}}}=\frac{8}{4^2-x^2}dx \\ dv=dx \\ v=x \end{array}\right| = x\ln\frac{x+4}{4-x}\Big|_0^2-\int_0^2 x\cdot\frac{8}{16-x^2}\,dx=$$

$$=2\ln\frac{6}{2}-0+4\int_0^2\frac{-2x}{16-x^2}\,dx=\left|\begin{array}{l}16-x^2=t\\ -2x\,dx=dt\end{array}\right|=$$

$$=2\ln 3+4\ln|16-x^2|\Big|_0^2=2\ln 3+4(\ln 12-\ln 16)=$$

$$=\ln 3^2+4\ln\frac{12}{16}=\ln 9+4\ln\frac{3}{4}=\ln 9+\ln\left(\frac{3}{4}\right)^4=$$

$$=\ln 9\cdot\left(\frac{3}{4}\right)^4$$ traženo rješenje $$=\ln\frac{3^6}{4^4}$$

(#) Izračunati integral $I=\int_0^{\pi/2}\cos x\sqrt{3\sin^2x+2\cos^2x}\,dx$.

$$I=\int_0^{\pi/2}\cos x\sqrt{3\sin^2x+2\cos^2x}\,dx=\int_0^{\pi/2}\cos x\sqrt{3\sin^2x+2(1-\sin^2x)}\,dx=$$

$$=\int_0^{\pi/2}\cos x\sqrt{\sin^2x+2}\,dx=\left|\begin{matrix}\sin x=t & x=0\Rightarrow t=0\\ \cos x\,dx=dt & x=\frac{\pi}{2}\Rightarrow t=1\end{matrix}\right|$$

$$=\int_0^1\sqrt{t^2+2}\,dt=\int_0^1\frac{t^2+2}{\sqrt{t^2+2}}\,dt\overset{(*)}{=}$$

Metoda Ostrogradskog

$$\int\frac{x^2+2}{\sqrt{x^2+2}}\,dx=(ax+b)\sqrt{x^2+2}+\lambda\int\frac{dx}{\sqrt{x^2+2}}\quad /'$$

$$\frac{x^2+2}{\sqrt{x^2+2}}=a\sqrt{x^2+2}+(ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+2}}+\frac{\lambda}{\sqrt{x^2+2}}\quad /\sqrt{x^2+2}$$

$$x^2+2=a(x^2+2)+(ax+b)x+\lambda$$

$$a+a=1\Rightarrow 2a=1\Rightarrow a=\frac{1}{2}$$
$$b=0$$
$$2a+\lambda=2\Rightarrow 1+\lambda=2$$
$$\lambda=1$$

$$\int\frac{x^2+2}{\sqrt{x^2+2}}\,dx=\frac{1}{2}x\sqrt{x^2+2}+\int\frac{dx}{\sqrt{x^2+2}}$$
$$=\frac{1}{2}x\sqrt{x^2+2}+\ln|x+\sqrt{x^2+2}|+C$$

$$\overset{(*)}{=}\frac{1}{2}t\sqrt{t^2+2}\Big|_0^1+\ln|t+\sqrt{t^2+2}|\Big|_0^1=\frac{1}{2}\sqrt{3}+\ln|1+\sqrt{3}|-\ln|\sqrt{2}|=$$

$$=\frac{\sqrt{3}}{2}+\ln\frac{1+\sqrt{3}}{\sqrt{2}}$$

(#) Izračunati površinu figure koja je određena linijama $y = \frac{16}{x^2}$, $y = 17 - x^2$ u prvom kvadrantu.

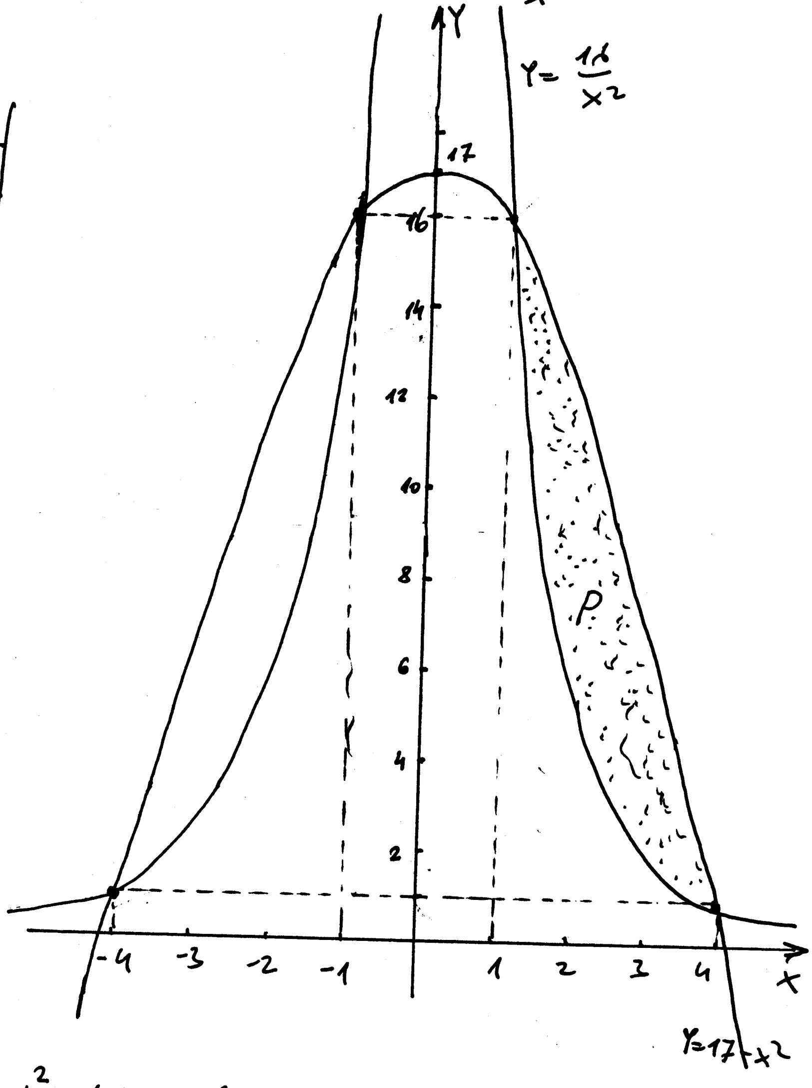
Rj. skicirajmo grafike f-ja

od ranije znamo da f-je oblika $y = ax^2 + bx + c$ izgledaju ovako:

$\cap$ ili $\cup$ (u zavisnosti od $a<0$ ili $a>0$).

Da bi skicirali ove dvije f-je problem predstavlja $y = \frac{16}{x^2}$.

Ispitajmo, ukratko ovu f-ju



$\mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

f-ja je parna

uvijek pozitivna

x	$(-\infty, 0)$	$(0, +\infty)$
y'	+	−
y	↗	↘

$\lim_{x \to \pm\infty} \frac{16}{x^2} = 0$

$\lim_{x \to \pm 0} \frac{16}{x^2} = +\infty$

$y' = (-2)\, 16\, x^{-3}$

$y' = \frac{-32}{x^3}$ nema ekstrem

$y'' = -32 \cdot (-3)\, x^{-4}$

$y'' = \frac{32 \cdot 3}{x^4}$

f-ja je uvijek $\cup$

Primjetimo da je i f-ja $y = 17 - x^2$ parna.

Pronađimo presječne tačke ove dvije f-je.

$$\begin{aligned} y &= \frac{16}{x^2} \\ y &= 17 - x^2 \end{aligned}$$

$$\begin{aligned} x^2 y &= 16 \\ x^2 &= 17 - y \end{aligned}$$

$$\begin{aligned} x^2 &= \frac{16}{y} \\ x^2 &= 17 - y \end{aligned}$$

$\frac{16}{y} = 17 - y \quad / \cdot y$

$-y^2 + 17y - 16 = 0 \quad / \cdot (-1) \Rightarrow y^2 - 17y + 16 = 0$

$D = 289 - 64 = 225$

$y_{1,2} = \frac{17 \pm 15}{2} \qquad y_1 = \frac{2}{2} = 1 \qquad y_2 = \frac{32}{2} = 16$

$y_1 = 1 \Rightarrow x_1 = \pm 4, \qquad y_2 = 16 \Rightarrow x_2 = \pm 1$

Presječne tačke krivih su $(-4,1)$, $(4,1)$, $(-1,16)$, $(1,16)$

$-\frac{3}{4}$

$$P = \int_1^4 \left[(17 - x^2) - \frac{16}{x^2} \right] dx = 17x \Big|_1^4 - \frac{1}{3} x^3 \Big|_1^4 - 16 \frac{x^{-1}}{-1} \Big|_1^4 = 17(4-1) - \frac{1}{3} \cdot (64 - 1) + 16\left(\frac{1}{4} - 1\right)$$

$$= 51 - \frac{63}{3} - 12 = 39 - 21 = 18$$

#) Izvačunati površinu manje figure ~~koja~~ je određena linijama

$x^2+y^2=16, \quad x^2=12(y-1).$

Rj. Skicirajmo ove dvije krive.

$x^2+y^2=16$ je krug poluprečnika 4 sa centrom u tački $(0,0)$.

$x^2=12(y-1)$

$x^2=12y-12$

$12y=x^2+12$

$y=\frac{1}{12}x^2+1$

ovo je parabola oblika ∪

Nađimo presjek ove dvije krive

$$\frac{\begin{array}{c} x^2+y^2=16 \\ x^2=12(y-1) \end{array}}{12(y-1)+y^2=16}$$

$y^2+12y-12-16=0$

$y^2+12y-28=0$

$D=144+112$

$D=256$

$y_{1,2}=\frac{-12\pm16}{2}$

$y_1=\frac{-28}{2}=-14$

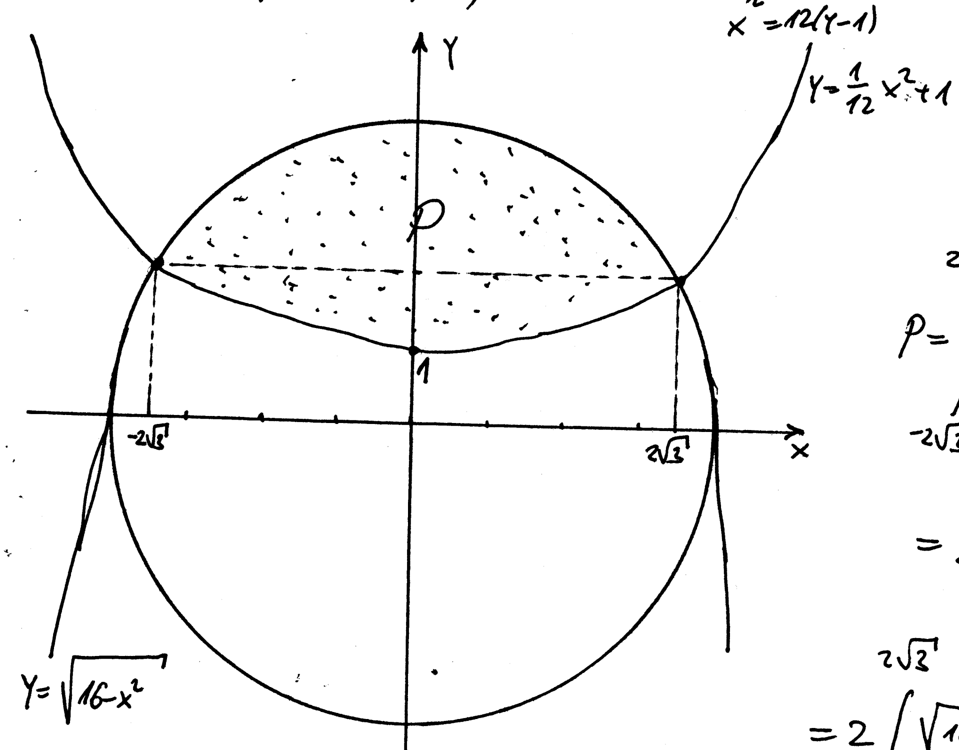
$y_2=\frac{4}{2}=2$

$y_1=-14 \Rightarrow x^2=12\cdot(-15)$ ništa

$y_2=2 \Rightarrow x^2=12$

$x=\pm2\sqrt{3}$

Presječne tačke krivih su $(-2\sqrt{3},2)$; $(2\sqrt{3},2)$



$x^2+y^2=16$

$y^2=16-x^2$

$y=\pm\sqrt{16-x^2}$

$$P=\int_{-2\sqrt{3}}^{2\sqrt{3}}\left[\sqrt{16-x^2}-\left(\frac{1}{12}x^2+1\right)\right]dx=$$

$$=2\int_0^{2\sqrt{3}}\left(\sqrt{16-x^2}-\frac{1}{12}x^2-1\right)dx=$$

$$=2\int_0^{2\sqrt{3}}\sqrt{16-x^2}-2\int_0^{2\sqrt{3}}\left(\frac{1}{12}x^2+1\right)dx=I_1-I_2$$

Metodom Ostrogradskog

$$\int\sqrt{16-x^2}\,dx=\int\frac{16-x^2}{\sqrt{16-x^2}}dx=(ax+b)\sqrt{16-x^2}+\lambda\int\frac{dx}{\sqrt{16-x^2}} \quad \text{(ZA VJEŽBU)} \Rightarrow a=\frac{1}{2},\ b=0,\ \lambda=8$$

$$\int_0^{2\sqrt{3}}\sqrt{16-x^2}\,dx=\frac{1}{2}x\sqrt{16-x^2}\Big|_0^{2\sqrt{3}}+8\arcsin\frac{x}{4}\Big|_0^{2\sqrt{3}}=\sqrt{3}\cdot2+8\cdot\arcsin\frac{\sqrt{3}}{2}=2\sqrt{3}+\frac{8\pi}{3}$$

$$\int_0^{2\sqrt{3}}\left(\frac{1}{12}x^2+1\right)dx=\frac{1}{12}\cdot\frac{1}{3}x^3\Big|_0^{2\sqrt{3}}+x\Big|_0^{2\sqrt{3}}=\frac{\cancel{8\cdot3}\sqrt{3}^{\,2}}{\cancel{12\cdot3}_{\,3}}+2\sqrt{3}=\frac{8\sqrt{3}}{3}$$

$$P=4\sqrt{3}-\frac{16\pi}{3}-\frac{16\sqrt{3}}{3}=\frac{16\pi}{3}-\frac{4\sqrt{3}}{3}$$

tražena površina

(#) Naći ekstreme f-je $z = x^3 - 5xy + 5y^2 + 7x - 15y$.

R: Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 3x^2 - 5y + 7 \qquad \frac{\partial z}{\partial y} = 10y - 5x - 15$$

$$\begin{aligned} 3x^2 - 5y + 7 &= 0 \quad /\cdot 2 \\ -5x + 10y - 15 &= 0 \\ \hline 6x^2 - 10y + 14 &= 0 \\ -5x + 10y - 15 &= 0 \quad + \\ \hline \end{aligned}$$

$$6x^2 - 5x - 1 = 0$$

$$D = 25 + 24 = 49$$

$$x_{1,2} = \frac{5 \pm 7}{2 \cdot 6}$$

$$x_1 = \frac{-2}{2 \cdot 6} = \frac{-1}{6}, \quad x_2 = \frac{12}{12} = 1$$

$$6\left(x + \frac{1}{6}\right)(x - 1) = 0$$

Za $x_1 = -\frac{1}{6} \Rightarrow -5 \cdot \left(-\frac{1}{6}\right) + 10y - 15 = 0$

$$10y = 15 - \frac{5}{6}$$

$$10y = \frac{90 - 5}{6} = \frac{85}{6}$$

$$y = \frac{\cancel{85}^{17}}{\cancel{10}_{2} \, 6} = \frac{17}{12}$$

$x_2 = 1 \Rightarrow -5 + 10y - 15 = 0$

$$10y = 20$$

$$y = 2$$

Stacionarne tačke su $(1,2)$ i $\left(-\frac{1}{6}, \frac{17}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -5 \qquad \frac{\partial^2 z}{\partial y^2} = 10$$

Za $M_1(1,2)$

$A = 6, \; B = -5, \; C = 10, \; D = AC - B^2 = 60 - 25 > 0$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$z_{min}(1,2) = 1 - 10 + 20 + 7 - 30 = 8 + 10 - 30 = 8 - 20 = -12$$

Za $M_2\left(-\frac{1}{6}, \frac{17}{12}\right)$

$A = -1, \; B = -5, \; C = 10, \; D = AC - B^2 = -10 - 25 = -35$

F-ja u ovoj tački nema ekstrem

(#) Nađi ekstreme f-je $z = e^{-2x^2}(x-y^2)$.

Rj. Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = e^{-2x^2}\cdot(-4)x(x-y^2) + e^{-2x^2}\cdot 1 = e^{-2x^2}(-4x^2+4xy^2+1)$$

$$\frac{\partial z}{\partial y} = e^{-2x^2}\cdot(-2)y = -2ye^{-2x^2}$$

$$e^{-2x^2}(-4x^2+4xy^2+1)=0$$
$$-2y\,e^{-2x^2}=0$$

e^{-2x^2} je uvijek pozitivno

$$-4x^2+4xy^2+1=0$$
$$-2y=0 \Rightarrow y=0$$

$$-4x^2+1=0$$
$$x^2=\frac{1}{4} \Rightarrow x_1=-\frac{1}{2},\ x_2=\frac{1}{2}$$

Stacionarne tačke su $M_1(-\frac{1}{2},0)$ i $M_2(\frac{1}{2},0)$

$$\frac{\partial^2 z}{\partial x^2} = e^{-2x^2}\cdot(-4x)(-4x^2+4xy^2+1) + e^{-2x^2}(-8x+4y^2) =$$
$$= e^{-2x^2}(16x^3-16x^2y^2-4x-8x+4y^2) = e^{-2x^2}(16x^3-16x^2y^2-12x+4y^2)$$
$$=4e^{-2x^2}(4x^3-4x^2y^2-3x+y^2)$$

$$\frac{\partial^2 z}{\partial x\partial y} = e^{-2x^2}(8xy) = 8xye^{-2x^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -2e^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2},0)$

$$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\left(-\frac{1}{8}\right)-4\cdot\frac{1}{4}\cdot 0-3\left(-\frac{1}{2}\right)+0\right) = 4e^{-\frac{1}{2}}\left(-\frac{1}{2}+\frac{3}{2}\right) = \frac{4}{\sqrt{e}}$$

$$B=0,\ C=-2e^{-2\cdot\frac{1}{4}} = -2e^{-\frac{1}{2}} = \frac{-2}{\sqrt{e}}$$

$$D = AC-B^2 = \frac{-8}{e} < 0$$

F-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2},0)$

$$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\frac{1}{8}-0-3\cdot\frac{1}{2}+0\right) =$$
$$=4e^{-\frac{1}{2}}\left(\frac{1}{2}-\frac{3}{2}\right) = \frac{-4}{\sqrt{e}}$$

$$B=0,\ C=-2e^{-2\cdot\frac{1}{4}} = \frac{-2}{\sqrt{e}}$$

$$D = AC-B^2 = \frac{8}{e} > 0 \Rightarrow$$ f-ja z u tački M_2 ima ekstrem

$$A<0 \Rightarrow z_{max}\left(\frac{1}{2},0\right) = e^{-2\cdot\frac{1}{4}}\left(\frac{1}{2}-0\right) = \frac{1}{2}\cdot e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}}$$

Riješiti diferencijalnu jednačinu $y - xy' = a(1 + x^2 y')$, $a = const.$

Rj. $y - xy' = a(1 + x^2 y')$, $a = const.$

$$y - xy' = a + ax^2 y'$$

$$ax^2 y' + xy' = y - a$$

$$(ax^2 + x) y' = y - a$$

$$y' = \frac{1}{ax^2 + x} \cdot (y - a)$$

Ovo je diferenc. jednačina sa razdvojenim promjenjivim

$$y' = \frac{dy}{dx}$$

$$\frac{dy}{y-a} = \frac{dx}{ax^2 + x}$$

$$\int \frac{dx}{x(ax+1)} = \int \frac{dy}{y-a}$$

$$\int \frac{dx}{x(ax+1)} = \int \left(\frac{1}{x} - \frac{a}{ax+1} \right) dx$$

$ax + 1 = t$

$a\,dx = dt$

$dx = \frac{1}{a} dt$

$$= \ln|x| - a \cdot \frac{1}{a} \ln|ax+1| + c$$

$$= \ln\left|\frac{x}{ax+1}\right| + c$$

$$\ln\left|\frac{x}{ax+1}\right| = \ln|y-a| + \ln c$$

$$\frac{x}{ax+1} = c(y-a)$$ Rešenje diferencijalne jednačine

(7) Riješiti diferencijalnu jednačinu $y' = \dfrac{3x^2}{x^3+y+1}$.

Rj.

$$y' = \frac{3x^2}{x^3+y+1}$$

$$\frac{dy}{dx} = \frac{3x^2}{x^3+y+1}$$

$$\frac{dx}{dy} = \frac{x^3+y+1}{3x^2}$$

$$x' = \frac{1}{3}x + \frac{1}{3}yx^{-2} + \frac{1}{3}x^{-2}$$

$$x' - \frac{1}{3}x = \left(\frac{1}{3}y + \frac{1}{3}\right)x^{-2}$$

ovo je Bernulijeva diferencijalna jednačina

Bernulijeva diferencijalna jednačina je oblika $y' + p(x)y = q(x)\cdot y^n$, $n \in \mathbb{R}$, $n \neq 0$, $n \neq 1$

Uvodimo smjenu

$$x = uv, \quad x' = u'v + uv'$$

$$u'v + uv' - \frac{1}{3}uv = \left(\frac{1}{3}y + \frac{1}{3}\right)(uv)^{-2}$$

$$u'v + u\left(v' - \frac{1}{3}v\right) = \left(\frac{1}{3}y + \frac{1}{3}\right)u^{-2}v^{-2}$$

a) $v' - \frac{1}{3}v = 0$

$$v' = \frac{1}{3}v$$

$$\frac{dv}{dy} = \frac{1}{3}v$$

$$\frac{dv}{v} = \frac{1}{3}dy$$

$$\ln v = \frac{1}{3}y$$

$$v = e^{\frac{1}{3}y}$$

b) $v = e^{\frac{1}{3}y} = e^{\frac{y}{3}}$

$$u'e^{\frac{y}{3}} = \frac{y+1}{3}u^{-2}e^{-\frac{2y}{3}} \quad \Big/ e^{-\frac{y}{3}}\cdot u^2$$

$$u^2u' = \frac{y+1}{3}e^{-y}$$

$$u^2\frac{du}{dy} = \frac{1}{3}ye^{-y} + \frac{1}{3}e^{-y}$$

$$u^2du = \frac{1}{3}ye^{-y}dy + \frac{1}{3}e^{-y}dy \quad \ldots(1)$$

$$\int e^{-y}dy = \left|\begin{matrix}-y = t\\ dy = -dt\end{matrix}\right| = \int e^t(-dt) = -\int e^t dt = -e^t + C = -e^{-y} + C$$

Kako je $\int ye^{-y}dy = \left|\begin{matrix}u = y & dv = e^{-y}dy\\ du = dy & v = -e^{-y}\end{matrix}\right| = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y} + C$

To je kad izračunamo integral od (1):

$$\frac{1}{3}u^3 = -\frac{1}{3}ye^{-y} - \frac{1}{3}e^{-y} + C_1 - \frac{1}{3}e^{-y} \quad /\cdot 3$$

$$u^3 = -ye^{-y} - 2e^{-y} + C$$

$$u = \sqrt[3]{-ye^{-y} - 2e^{-y} + C}$$

$$x = uv$$

$$x = e^{\frac{y}{3}}\sqrt[3]{-ye^{-y} - 2e^{-y} + C}$$

opšte rješenje diferencijalne jednačine

$$x^3 = e^y(-ye^{-y} - 2e^{-y} + C)$$

$$x^3 = -y - 2 + Ce^y$$