Grupa D EF ispit 5. juli 2011.

1. $$\sum_{j=1}^{n} \frac{j^2}{4j^2-1} = \frac{n^2+n}{4n+2} \quad (n \in N)$$

$$n=1 \Rightarrow \frac{1^2}{4\cdot 1^2-1} = \frac{1^2+1}{4\cdot 1+2}$$

$$\frac{1}{3} = \frac{2}{6} \quad \text{— tačno}$$

Pretpostavka:

$$n=k \Rightarrow \sum_{j=1}^{k} \frac{j^2}{4j^2-1} = \frac{k^2+k}{4k+2}$$

Tvrdimo da je data jednakost tačna i za $n=k+1$, tj.

$$\sum_{j=1}^{k+1} \frac{j^2}{4j^2-1} = \frac{(k+1)^2+(k+1)}{4(k+1)+2}$$

Dokaz: $$\sum_{j=1}^{k+1} \frac{j^2}{4j^2-1} = \sum_{j=1}^{k} \frac{j^2}{4j^2-1} + \frac{(k+1)^2}{4(k+1)^2-1} =$$

$$\overset{(P)}{=} \frac{k^2+k}{4k+2} + \frac{(k+1)^2}{4(k+1)^2-1} =$$

$$= \frac{k^2+k}{2(2k+1)} + \frac{(k+1)^2}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)[k(2k+3)+2(k+1)]}{2(2k+1)(2k+3)} = \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k^2+4k+k+2)}{2(2k+1)(2k+3)} = \frac{(k+1)[2k(k+2)+(k+2)]}{2(2k+1)(2k+3)}$$

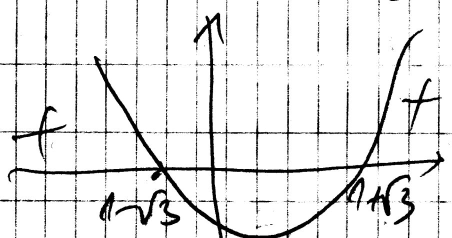
$$= \frac{(k+1)(k+2)(2k+1)}{2(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$= \frac{(k+1)(k+1+1)}{4k+6} = \frac{(k+1)^2+(k+1)}{4(k+1)+2}.$$

2. $y = \ln(x^2-2x-2)$

Def područje: $x^2-2x-2>0$

$$x_{1,2} = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$$



$$\boxed{x\in(-\infty, 1-\sqrt{3})\cup(1+\sqrt{3}, +\infty)}$$

Nula: $x^2-2x-2=1$

$$x^2-2x-3=0$$

$$x_{1,2} = \frac{2\pm\sqrt{4+12}}{2} = \frac{2\pm 4}{2}$$

$x_1=3,\ x_2=-1$ - pripadaju D.p.

Znak:

	$-\infty$	-1	$1-\sqrt{3}$	$1+\sqrt{3}$	3	$+\infty$
y	+	0 −			− 0	+

Vertikalne asimptote:

$$\lim_{x\to(1-\sqrt{3})-} \ln(x^2-2x-2) = \ln 0_+ = -\infty$$
$$\lim_{x\to(1+\sqrt{3})+} \ln(x^2-2x-2) = \ln 0_+ = -\infty$$

$x=1-\sqrt{3}$ - lijeva

$x=1+\sqrt{3}$ - desna

$\lim_{x\to+\infty} \ln(x^2-2x-2) = \ln(+\infty) = +\infty$

– nema hor. asimptote

Kosa: $y = kx + n$

$k = \lim_{x\to+\infty} \frac{\ln(x^2-2x-2)}{x} = \frac{\infty}{\infty} =$

$= \lim_{x\to+\infty} \frac{\frac{2x-2}{x^2-2x-2}}{1} = \lim_{x\to+\infty} \frac{2x}{x^2} =$

$= \lim_{x\to+\infty} \frac{2}{x} = 0$ – nema K.A.

$y' = \frac{2x-2}{x^2-2x-2}$

$y' = 0 \Rightarrow 2x = 2 \Rightarrow x = 1 \notin D.p \Rightarrow$ nema ekstrema

$x \in (-\infty, 1-\sqrt{3}) \Rightarrow y' < 0 \Rightarrow y \searrow$ (opada)

$x \in (1+\sqrt{3}, +\infty) \Rightarrow y' > 0 \Rightarrow y \nearrow$ (raste)

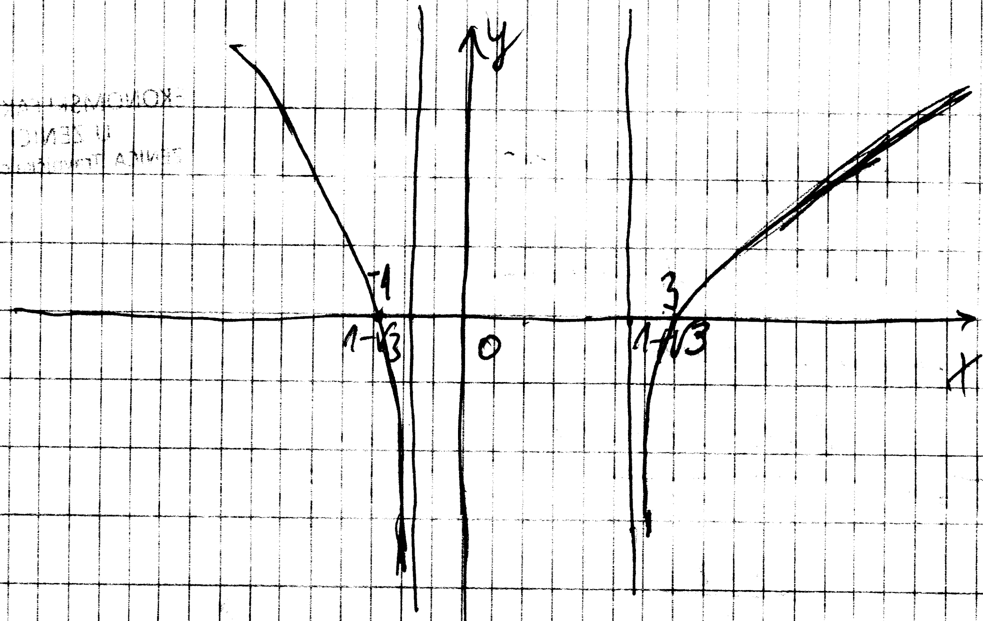
$y'' = \left(\frac{2x-2}{x^2-2x-2}\right)' = \frac{2(x^2-2x-2)-(2x-2)\cdot(2x-2)}{(x^2-2x-2)^2}$

$= \frac{2x^2-4x-4-(4x^2-4x-4x+4)}{(x^2-2x-2)^2}$

$= \frac{2x^2-\cancel{4x}-4-4x^2+\cancel{4x}+4x-4}{(x^2-2x-2)^2}$

$= \frac{-2x^2+4x-8}{(x^2-2x-2)^2} = \frac{-2(x^2-2x+4)}{(x^2-2x-2)^2} < 0 \Rightarrow y \cup$

jer je $x^2-2x+4 = (x-1)^2+3 > 0$ za sve x



3. $$I=\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx = \left| \begin{matrix} x=t^6 \\ dx=6t^5dt \end{matrix} \right| = \int \frac{\sqrt{t^6}\cdot 6t^5dt}{(1+\sqrt[3]{t^6})^2} =$$

$$= 6\int \frac{t^3\cdot t^5\,dt}{(1+t^2)^2} = 6\int \frac{t^8}{(t^2+1)^2}\,dt$$

$$t^8:(t^4+2t^2+1) = t^4-2t^2+3-\frac{4t^2+3}{t^4+2t^2+1}$$

$$\begin{array}{l} \underline{\pm t^8 \pm 2t^6 \pm t^4} \\ = -2t^6 - t^4 \\ \underline{\mp 2t^6 \mp 4t^4 \mp 2t^2} \\ = 3t^4+2t^2 \\ \underline{\pm 3t^4 \pm 6t^2 \pm 3} \\ = -4t^2-3 \end{array}$$

$$I = 6\int\left(t^4-2t^2+3-\frac{4t^2+3}{t^4+2t^2+1}\right)dt =$$

$$=6\cdot\left[\frac{t^5}{5}-2\cdot\frac{t^3}{3}+3t-\int\frac{4(t^2+1)-1}{(t^2+1)^2}\,dt\right]=$$

$$=\frac{6t^5}{5}-4t^3+18t-6\cdot\left[4\int\frac{dt}{t^2+1}-\underbrace{\int\frac{dt}{(t^2+1)^2}}_{I_1}\right]$$

$$I_1 = \int \frac{t^2+1-t^2}{(t^2+1)^2}\,dt = \int \frac{dt}{t^2+1} - \underbrace{\int t \cdot \frac{t\,dt}{(t^2+1)^2}}_{I_2}$$

$$= \operatorname{arctg} t - I_2$$

$$I_2 = \left| \begin{array}{ll} u = t & dv = \frac{t\,dt}{(t^2+1)^2} \\ du = dt & v = \int \frac{t\,dt}{(t^2+1)^2} = \left| \begin{array}{l} t^2+1 = z \\ t\,dt = \frac{1}{2}dz \end{array} \right| = \frac{1}{2}\int \frac{dz}{z^2} = -\frac{1}{2z} \\ & v = -\frac{1}{2(t^2+1)} \end{array} \right|$$

$$= -\frac{t}{2(t^2+1)} + \frac{1}{2}\int \frac{dt}{t^2+1} = -\frac{t}{2(t^2+1)} + \frac{1}{2}\operatorname{arctg} t + C$$

$$I = \frac{6t^5}{5} - 4t^3 + 18t - 6\left[4\cdot \operatorname{arctg} t - \operatorname{arctg} t - \frac{t}{2(t^2+1)} + \frac{1}{2}\operatorname{arctg} t\right] + C$$

$$= \frac{6t^5}{5} - 4t^3 + 18t - 21\operatorname{arctg} t + \frac{3t}{t^2+1} + C_1, \quad t = \sqrt[6]{x}$$

4.

$$z = (8x^2 - 6xy + 3y^2)e^{2x+3y}$$

$$z'_x = (16x - 6y)e^{2x+3y} + (8x^2 - 6xy + 3y^2)e^{2x+3y}\cdot 2$$

$$z'_y = (-6x + 6y)e^{2x+3y} + (8x^2 - 6xy + 3y^2)\cdot e^{2x+3y}\cdot 3$$

$$z'_x = 0 \Rightarrow 16x - 6y + 2(8x^2 - 6xy + 3y^2) = 0 \quad /:2$$

$$z'_y = 0 \Rightarrow -6x + 6y + 3(8x^2 - 6xy + 3y^2) = 0 \quad /:3$$

$$(*) \quad \left.\begin{array}{l} 8x - 3y + (8x^2 - 6xy + 3y^2) = 0 \\ -2x + 2y + (8x^2 - 6xy + 3y^2) = 0 \end{array}\right\} -$$

$$10x - 5y = 0 \quad /:5$$

$2x-y=0 \Rightarrow 2x=y$ – uvrstimo u (*)

$8x-3\cdot 2x+8x^2-6x\cdot 2x+3\cdot 4x^2=0$

$2x+8x^2-\cancel{12x^2}+\cancel{12x^2}=0$

$2x(1+4x)=0 \Rightarrow x_1=0,\ x_2=-\frac{1}{4}$

$y_1=0,\ y_2=-\frac{1}{2}$

$M_1(0,0),\ M_2\left(-\frac{1}{4},-\frac{1}{2}\right)$ – stacionarne točke

$z''_{xx}=\left[(16x^2-12xy+6y^2+16x-6y)e^{2x+3y}\right]'_x$

$=(32x-12y+16)\cdot e^{2x+3y}+(16x^2-12xy+6y^2+16x-6y)\cdot$
$\cdot e^{2x+3y}\cdot 2$

$=e^{2x+3y}(32x-12y+16+32x^2-24xy+12y^2+32x-12y)$

$=e^{2x+3y}(32x^2+12y^2-24xy+64x-24y+16)$

$z''_{xy}=(-12x+12y-6)e^{2x+3y}+(16x^2-12xy+6y^2+16x-6y)\cdot e^{2x+3y}\cdot 3$

$=e^{2x+3y}(-12x+12y-6+48x^2-36xy+18y^2+48x-18y)$

$=e^{2x+3y}(48x^2+18y^2-36xy+36x-6y-6)$

$z''_{yy}=\left[(24x^2-18xy+9y^2-6x+6y)e^{2x+3y}\right]'_y$

$=(-18x+18y+6)e^{2x+3y}+(24x^2-18xy+9y^2-6x+6y)\cdot e^{2x+3y}\cdot 3$

$=e^{2x+3y}(-18x+18y+6+72x^2-54xy+27y^2-18x+18y)$

$=e^{2x+3y}(72x^2+27y^2-54xy-36x+36y+6)$

Za tačku $M_1(0,0)$:

$A = e^0 \cdot 16 = 16$

$B = e^0 \cdot (-6) = -6$

$C = e^0 \cdot 6 = 6$

$D = 96 - 36 = 60 > 0,\ A > 0 \Rightarrow$ minimum u M_1!

$\Rightarrow f_{min}(0,0) = 0$.

Za tačku $M_2(-\frac{1}{4}, -\frac{1}{2})$:

$A = e^{-\frac{1}{2}-\frac{3}{2}}\left[32\cdot\frac{1}{16} + 12\cdot\frac{1}{4} - 24\cdot\frac{1}{8} + 64\cdot(-\frac{1}{4}) - 24\cdot(-\frac{1}{2}) + 16\right]$

$= e^{-2}\cdot(2 + \cancel{3} - \cancel{3} - \cancel{16} + 12 + \cancel{16}) = 14e^{-2} = \frac{14}{e^2}$

$B = e^{-\frac{1}{2}-\frac{3}{2}}\left[48\cdot\frac{1}{16} + 18\cdot\frac{1}{4} - 36\cdot\frac{1}{8} + 36\cdot(-\frac{1}{4}) - 6\cdot(-\frac{1}{2}) - 6\right]$

$= e^{-2}\cdot(\cancel{3} + \cancel{\frac{9}{2}} - \cancel{\frac{9}{2}} - 9 + \cancel{3} - \cancel{6}) = -9e^{-2} = -\frac{9}{e^2}$

$C = e^{-2}\cdot\left[72\cdot\frac{1}{16} + 27\cdot\frac{1}{4} - 54\cdot\frac{1}{8} - 36(-\frac{1}{4}) + 36\cdot(-\frac{1}{2}) + 6\right]$

$= e^{-2}\cdot(\frac{9}{2} + \cancel{\frac{27}{4}} - \cancel{\frac{27}{4}} + 9 - 18 + 6)$

$= e^{-2}\cdot(-3 + \frac{9}{2}) = e^{-2}\cdot\frac{3}{2} = \frac{3}{2e^2}$

$D = \frac{14}{e^2}\cdot\frac{3}{2e^2} - \left(-\frac{9}{e^2}\right)^2 = \frac{21}{e^4} - \frac{81}{e^4} < 0$

U tački M_2 nije ekstrem