Grupa B (05.07.2011.)

1. $\binom{n}{k}+\binom{n}{k+1}=\frac{n!}{k!(n-k)!}+\frac{n!}{(k+1)!(n-k-1)!}$

$=\frac{n!(k+1)+n!(n-k)}{(k+1)!(n-k)!}=$

$=\frac{n!(k+1+n-k)}{(k+1)!(n-k)!}=\frac{n!(n+1)}{(k+1)!(n-k)!}$

$=\frac{(n+1)!}{(k+1)!(n-k)!}=\binom{n+1}{k+1}$

2. $y=\frac{x^2-4}{x}e^{-\frac{5}{3x}}$

D.p. $x\neq 0 \Rightarrow x\in(-\infty,0)\cup(0,+\infty)$

Nule: $x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

Znak:

	$-\infty$	-2	0	2	$+\infty$
x^2-4	$+$	$-$	$-$	$+$	
x	$-$	$-$	$+$	$+$	
y	$-$	$+$	$-$	$+$	

$\lim_{x\to 0_-}\frac{x^2-4}{x}\cdot e^{-\frac{5}{3x}}=\frac{-4}{0_-}\cdot e^{\frac{-5}{0_-}}=+\infty\cdot e^{+\infty}=+\infty$

$\lim_{x\to 0_+}\frac{x^2-4}{x}\cdot e^{-\frac{5}{3x}}=\frac{-4}{0_+}\cdot e^{-\frac{5}{0_+}}=-\infty\cdot 0$

$=\lim_{x\to 0_+}\frac{x-\frac{4}{x}}{e^{\frac{5}{3x}}}=\left(\frac{-\infty}{+\infty}\right)\overset{L.P.}{=}\lim_{x\to 0_+}\frac{1+\frac{4}{x^2}}{e^{\frac{5}{3x}}\cdot\left(-\frac{5}{3x^2}\right)}=$

$$= \lim_{x\to 0+} \frac{\frac{x^2+4}{\cancel{x^2}}}{e^{\frac{5}{3x}}\cdot\left(-\frac{5}{3\cancel{x^2}}\right)} = \frac{4}{-\infty} = 0$$

V.A $x=0$ (lijeva)

$$\lim_{x\to+\infty} \frac{x^2-4}{x}\cdot e^{-\frac{5}{3x}} = \lim_{x\to+\infty}\left(x-\cancel{\frac{4}{x}}\right)e^{-\cancel{\frac{5}{3x}}} = +\infty$$

$$\lim_{x\to-\infty} \frac{x^2-4}{x}e^{-\frac{5}{3x}} = \lim_{x\to-\infty}\left(x-\cancel{\frac{4}{x}}\right)e^{-\cancel{\frac{5}{3x}}} = -\infty$$

Nema H.A

K.A $y=kx+n$

$$k=\lim_{x\to\pm\infty} \frac{x^2-4}{x^2}e^{-\cancel{\frac{5}{3x}}} = 1$$

$$n=\lim_{x\to\pm\infty}\left(\frac{x^2-4}{x}e^{-\frac{5}{3x}}-x\right) =$$

$$=\lim_{x\to\pm\infty}\left[x\cdot\left(e^{-\frac{5}{3x}}-1\right)-\cancel{\frac{4}{x}}e^{-\frac{5}{3x}}\right]$$

$$=\lim_{x\to\pm\infty}\frac{e^{-\frac{5}{3x}}-1}{\frac{1}{x}} \overset{L.P.}{=} \lim_{x\to\pm\infty}\frac{e^{-\cancel{\frac{5}{3x}}}\cdot\frac{5}{3\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}} = -\frac{5}{3}$$

$$y=x-\frac{5}{3}$$

x	0	$\frac{5}{3}$
y	$-\frac{5}{3}$	0

$$y' = \left(\frac{x^2-4}{x}\, e^{-\frac{5}{3x}}\right)' = \left[\left(x-\frac{4}{x}\right) e^{-\frac{5}{3x}}\right]'$$

$$= \left(1+\frac{4}{x^2}\right) e^{-\frac{5}{3x}} + \left(x-\frac{4}{x}\right)\cdot e^{-\frac{5}{3x}}\cdot \frac{5}{3x^2}$$

$$= e^{-\frac{5}{3x}}\left(1+\frac{4}{x^2}+\frac{5}{3x}-\frac{20}{3x^3}\right)$$

$$= e^{-\frac{5}{3x}}\cdot \frac{3x^3+5x^2+12x-20}{3x^3}$$

$$(3x^3+5x^2+12x-20):(x-1) = 3x^2+8x+20$$

$$\begin{array}{l}
-3x^3 \mp 3x^2 \\
\hline
= \quad 8x^2+12x-20 \\
\quad -8x^2 \pm 8x \\
\hline
\quad = \quad 20x-20 \\
\quad\quad -20x \pm 20 \\
\hline
\quad\quad = \quad =
\end{array}$$

$$y' = e^{-\frac{5}{3x}}\ \frac{(x-1)(3x^2+8x+20)}{3x^3}$$

$y' = 0 \Rightarrow x = 1$

	$-\infty$ … 0	0 … 1	1 … $+\infty$
$x-1$	$-$	$-$ 0	$+$
$3x^3$	$-$ 0	$+$	$+$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$ min	$\nearrow$

$$y(1) = \frac{1-4}{1}\cdot e^{-\frac{5}{3\cdot 1}}$$

$$y(1) = -3\, e^{-\frac{5}{3}} = -\frac{3}{\sqrt[3]{e^5}}$$

$$y'' = e^{-\frac{5}{3x}}\cdot \frac{5}{3x^2}\cdot \frac{3x^3+5x^2+12x-20}{3x^3} + e^{-\frac{5}{3x}}\cdot \frac{1}{3}\cdot$$

$$\frac{(9x^2+10x+12)\cdot x^3-(3x^3+5x^2+12x-20)\cdot 3x^2}{x^6}$$

$$= e^{-\frac{5}{3x}}\left(\frac{5\cdot(3x^3+5x^2+12x-20)}{9x^5}+\frac{\cancel{x^2}(\cancel{9x^3}+10x^2+12x-\cancel{9x^3}-15x^2-36x+60)}{3x^{\cancel{6}4}}\right)$$

$$= e^{-\frac{5}{3x}}\cdot\frac{\cancel{15x^3}+25x^2+60x-100-\cancel{15x^3}-72x^2+180x}{9x^5}$$

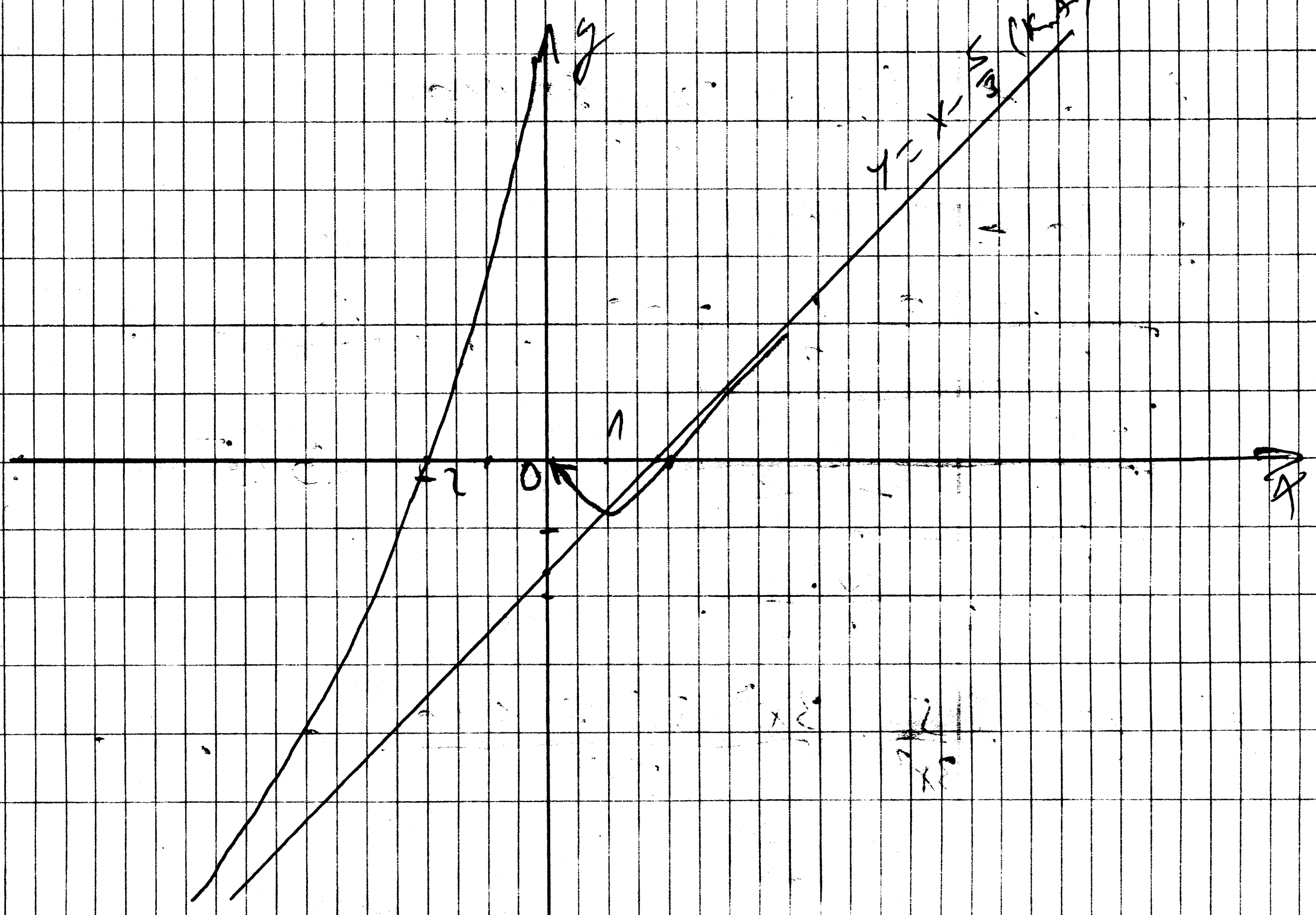
$$= e^{-\frac{5}{3x}}\cdot\frac{-47x^2+240x-100}{9x^5}$$

$y'=0 \Rightarrow -47x^2+240x-100=0$

$D=240^2-4\cdot 47\cdot 100$

$D=57600-18800=38800=20^2\cdot 97$

$$x_{1,2}=\frac{-240\pm 20\sqrt{97}}{-94}=\frac{120\pm 10\sqrt{97}}{47}\to \text{P.П}$$



3. $I = \int \cos^6 x \, dx = \int (\cos^2 x)^3 \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx$

$= \int \frac{1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x}{3} \, dx$

$= \frac{1}{3}\left(\int dx + 3\int \cos 2x \, dx + 3\underbrace{\int \cos^2 2x \, dx}_{I_1} + \underbrace{\int \cos^3 2x \, dx}_{I_2}\right)$

$I_1 = \int \frac{1+\cos 4x}{2} \, dx = \frac{1}{2}\left(\int dx + \int \cos 4x \, dx\right)$

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C = \frac{1}{2}x + \frac{1}{8}\sin 4x + C$

$I_2 = \int \cos 2x \cdot \cos^2 2x \, dx = \int \cos 2x \cdot (1 - \sin^2 2x) \, dx =$

$= \left| \begin{array}{l} \sin 2x = t \\ 2\cos 2x \, dx = dt \\ \cos 2x \, dx = \frac{1}{2} dt \end{array} \right| = \int (1 - t^2) \cdot \frac{1}{2} \, dt$

$= \frac{1}{2}\left(\int dt - \int t^2 \, dt\right) = \frac{1}{2}\left(t - \frac{t^3}{3}\right) + C$

$= \frac{1}{2}t - \frac{t^3}{6} + C = \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C$

$I = \frac{1}{3}\left(x + \frac{3}{2}\sin 2x + \frac{3}{2}x + \frac{3}{8}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right) + C$

$I = \frac{1}{3}\left(\frac{5x}{2} + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + C$

$= \frac{5x}{6} + \frac{2}{3}\sin 2x + \frac{1}{8}\sin 4x - \frac{1}{18}\sin^3 2x + C$

5. $F(x,y,\lambda) = 2x^2 + 12xy + y^2 + \lambda(x^2 + 4y^2 - 25)$

$F'_x = 4x + 12y + \lambda \cdot 2x = 0 \quad \ldots (1)$

$F'_y = 12x + 2y + \lambda \cdot 8y = 0 \quad \ldots (2)$

$x^2 + 4y^2 = 25 \quad \ldots (3)$

$(1) \Rightarrow 2\lambda x = -(4x + 12y) \quad / : x$

$2\lambda = -\left(4 + 12 \cdot \frac{y}{x}\right) \quad \ldots (4)$

$(2) \Rightarrow 8\lambda y = -(12x + 2y) \quad / : 4y$

$2\lambda = -\left(\frac{3x}{y} + \frac{1}{2}\right) \quad \ldots (5)$

$(4),(5) \Rightarrow 4 + 12 \cdot \frac{y}{x} = 3 \cdot \frac{x}{y} + \frac{1}{2}$

$\frac{y}{x} = t \Rightarrow 4 + 12t = \frac{3}{t} + \frac{1}{2} \quad / \cdot 2t$

$8t + 24t^2 = 6 + t$

$24t^2 + 7t - 6 = 0$

$D = 49 + 576 = 625$

$t_{1,2} = \frac{-7 \pm 25}{48} \Rightarrow t_1 = \frac{18}{48} = \frac{3}{8}, \quad t_2 = \frac{-32}{48} = -\frac{2}{3}$

$\mathrm{I} \begin{cases} \frac{y}{x} = \frac{3}{8} \\ x^2 + 4y^2 = 25 \end{cases} \qquad \mathrm{II} \begin{cases} \frac{y}{x} = -\frac{2}{3} \\ x^2 + 4y^2 = 25 \end{cases}$

$\mathrm{I} \Rightarrow y = \frac{3}{8}x \Rightarrow x^2 + 4 \cdot \frac{9}{64}x^2 = 25 \Rightarrow x^2\left(1 + \frac{9}{16}\right) = 25$

$x^2 \cdot \frac{25}{16} = 25 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4 \Rightarrow y = \pm \frac{3}{8} \cdot 4 = \pm \frac{3}{2}$

$\underbrace{M_1\left(4, \frac{3}{2}\right), M_2\left(-4, -\frac{3}{2}\right)}_{\text{stac.-tačke}} \quad 2\lambda = -\left(4 + 12 \cdot \frac{3}{8}\right) = -\frac{17}{2}$

$\Rightarrow \lambda = -\frac{17}{4}$

II) $\Rightarrow y = -\frac{2}{3}x \Rightarrow x^2 + 4\cdot\frac{4}{9}x^2 = 25 \Rightarrow x^2\left(1+\frac{16}{9}\right) = 25$

$x^2\cdot\frac{25}{9} = 25 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$y = -\frac{2}{3}\cdot(\pm 3) = \mp 2$

$\underbrace{M_3(3,-2),\ M_4(-3,2)}_{\text{stac. tačke}},\quad 2\lambda = -\left(4+12\cdot\left(-\frac{2}{3}\right)\right) \Rightarrow \lambda = 2$

$F''_{xx} = 4+2\lambda$

$F''_{xy} = 12$

$F''_{yy} = 2+8\lambda$

Za tačke M_1 i M_2:

$A = 4+2\cdot\left(-\frac{17}{4}\right) = 4-\frac{17}{2} = -\frac{9}{2}$

$B = 12$

$C = 2+8\cdot\left(-\frac{17}{4}\right) = 2-34 = -32$

$D = AC-B^2 = -\frac{9}{2}\cdot(-32) - 12^2 = 144-144 = 0$

Za tačke M_3 i M_4:

$A = 4+2\cdot 2 = 8$

$B = 12$

$C = 2+8\cdot 2 = 18$

$D = 18\cdot 8 - 12^2 = 144 - 144 = 0.$

Pošto se ne može ništa zaključiti standardnim načinom, prijeđemo na drugi način.

$$x^2+4y^2=25 \Rightarrow \frac{x^2}{25}+\frac{y^2}{\frac{25}{4}}=1 \Rightarrow x=5\cos t,\ y=\frac{5}{2}\sin t,\ 0\le t\le 2\pi$$

$$\Rightarrow z=50\cos^2 t+150\cos t\sin t+\frac{25}{4}\sin^2 t$$

$$z'(t)=-100\cos t\sin t+150\cos 2t+\frac{25}{2}\sin t\cos t$$

$$=25\left(6\cos 2t-\frac{7}{4}\sin 2t\right)$$

$$z'(t)=0 \Rightarrow \operatorname{tg} 2t=\frac{24}{7} \Rightarrow \cos^2 2t=\frac{1}{1+\operatorname{tg}^2 2t}=\frac{49}{625}$$

$$\Rightarrow \cos 2t=\pm\frac{7}{25}$$

$$\cos 2t=\frac{7}{25} \Rightarrow 2\cos^2 t-1=\frac{7}{25} \Rightarrow \cos^2 t=\frac{16}{25}$$

$$\cos t=\pm\frac{4}{5} \Rightarrow \sin t=\pm\frac{3}{5}$$

Odavde se dobiju stacionarne tačke M_1 i M_2

$$\cos 2t=-\frac{7}{25} \Rightarrow 2\cos^2 t-1=-\frac{7}{25} \Rightarrow \cos^2 t=\frac{9}{25}$$

$$\cos t=\pm\frac{3}{5} \Rightarrow \sin t=\mp\frac{4}{5}$$

Odavde se dobiju tačke M_3 i M_4.

Analizom znaka izvoda funkcije $z(t)$, zaključujemo da u tačkama M_1 i M_2 imamo maksimum, a u tačkama M_3 i M_4 minimum.