Pismeni ispit iz Matematike za ekonomiste, 07.07.2010.

GRUPA A

1. Riješiti matričnu jednačinu $\left(AXB\right)^{-1} = B^{-1}\left(X^{-1} + B\right)$, ako je

$$A = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

2. Ispitati funkciju i nacrtati njen grafik: $y = \dfrac{\ln^2 x + 1}{x^2}$.
3. Izračunati integral $I = \int \dfrac{x^4}{x^4 + x^2 - 6} dx.$
4. Riješiti diferencijalnu jednačinu $y' + \dfrac{y}{4x} + y^3 e^{\sqrt{x}} = 0$ ako je $y(1) = 1$.

GRUPA B

1. Naći sve vrijednosti korijena $\sqrt[6]{-27}$.
2. Ispitati funkciju i nacrtati njen grafik: $y = \dfrac{x^2 + 10}{x^2 + 4x + 4}$.
3. Izračunati površinu figure određene linijama: $y = 8x - 2x^2$, $3x + y = 0$, $3x - y - 12 = 0$.
4. Naći ekstreme funkcije $z = x + y - \dfrac{3}{2}\ln\left(x^2 + y^2 + 1\right)$.

GRUPA C

1. Dokazati matematičkom indukcijom tvrdnju $5 \left| \left(n^5 - n\right) \right., n \in \mathbb{N}$..
2. Ispitati funkciju i nacrtati njen grafik: $y = e^{\frac{x}{1-x}} - 1$.
3. Izračunati integral $I = \int \dfrac{dx}{x^5 - x^2}$.
4. Riješiti diferencijalnu jednačinu $y^3 y' + 3xy^2 + 2x^3 = 0$.

GRUPA D

1. Dati su vektori $\mathbf{a} = \left(3m + 3, 1, m + 5\right)$, $\mathbf{b} = \left(3m - 4, 3m - 2, -2\right)$, $\mathbf{c} = \left(3 - 3m, 2 - 3m, 1\right)$. Odrediti sve vrijednosti parametra *m* tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra *m* napisati vektor $\mathbf{a}$ kao linearnu kombinaciju vektora $\mathbf{b}$ i $\mathbf{c}$.
2. Ispitati funkciju i nacrtati njen grafik: $y = \dfrac{x^3 - 2}{2x^2}$.
3. Izračunati površinu figure određene linijama: $y = -\dfrac{1}{2}x + 2$, $y = \sqrt{x - 1}$, $y = 0$.
4. Naći uslovne ekstreme funkcije $z = \left(x - 3\right)^2 + \left(y - 4\right)^2$ uz uslov $x^2 + y^2 = \dfrac{25}{4}$.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Dokazati matematičkom indukcijom tvrdnju

$5 \mid (n^5 - n)$, $n \in \mathbb{N}$.

Rj. $5 \mid (k^5 - k)$, $k \in \mathbb{N}$ (ovo čitamo: pet dijeli $k^5 - k$ gdje je k neki prirodan broj) čitaj $k^5 - k$ je djeljivo sa 5)

BAZA INDUKCIJE

$k=1$: $5 \mid (1^5 - 1)$ tj. $5 \mid 0$ 5 dijeli 0 tj. $0 = 5 \cdot s$ gdje je s neki broj iz $\mathbb{N}$.

Tvrdnja je tačna za $k=1$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

KORAK INDUKCIJE

Pretpostavimo da je tvrdnja $5 \mid (k^5 - k)$ tačna za sve brojeve od 1 do n. Na osnovu ove pretpostavke dokažimo da $5 \mid (n+1)^5 - (n+1)$

$$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + \underline{1} - n - \underline{1} =$$

$$= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n - n =$$

$$= \underbrace{(n^5 - n)}_{\text{ovo je prema pretpostavci djeljivo sa 5}} + \underbrace{5(n^4 + 2n^3 + 2n^2 + n)}_{\text{ovo je djeljivo sa 5 (vidi se)}}$$

Prema tome $5 \mid (n+1)^5 - (n+1)$ što je i trebalo pokazati

ZAKLJUČAK

Tvrdnja je tačna za sve prirodne brojeve.

(#) Nađi sve vrijednosti korijena $\sqrt[6]{-27}$.

Rj. Označimo sa $z = \sqrt[6]{-27}$

$$z^6 = -27$$

Teorema Jednačina $z^n = w$, gdje je w po volji odabran kompleksan broj različit od 0 ima tačno n različitih rješenja koji su oblika

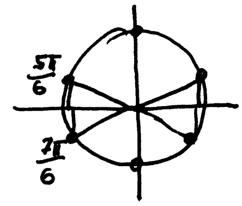
$$z_k = \sqrt[n]{|w|}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)$$

gdje je φ najmanji pozitivan ugao iz intervala $[0, 2\pi)$ takav da $w = |w|(\cos\varphi + i\sin\varphi)$, a $k = 0, 1, 2, \ldots, n-1$.

U našem slučaju $w = -27 \Rightarrow |w| = \sqrt{(-27)^2 + 0^2} = 27$

$w = a + ib$ $\quad (|w| = \sqrt{a^2 + b^2})$

$$\left.\begin{aligned} \cos\varphi &= \frac{-27}{27}\left(= \frac{a}{|w|}\right) = -1 \\ \sin\varphi &= \frac{b}{|w|} = \frac{0}{27} = 0 \end{aligned}\right\} \Rightarrow \varphi = \pi$$



$$w = -27 = 27(\cos\pi + i\sin\pi)$$

$$z_0 = \sqrt[6]{27}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = (3^3)^{\frac{1}{6}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_1 = \sqrt[6]{27}\left(\cos\frac{\pi + 2\pi}{6} + i\sin\frac{\pi + 2\pi}{6}\right) = \sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i\sqrt{3}$$

$$z_2 = \sqrt[6]{27}\left(\cos\frac{\pi + 4\pi}{6} + i\sin\frac{\pi + 4\pi}{6}\right) = \sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{3}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_3 = \sqrt[6]{27}\left(\cos\frac{\pi + 6\pi}{6} + i\sin\frac{\pi + 6\pi}{6}\right) = \sqrt{3}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$z_4 = \sqrt[6]{27}\left(\cos\frac{\pi + 8\pi}{6} + i\sin\frac{\pi + 8\pi}{6}\right) = \sqrt{3}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -i\sqrt{3}$$

$$z_5 = \sqrt[6]{27}\left(\cos\frac{\pi + 10\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

Sve vrijednosti korijena $\sqrt[6]{-27}$ su: $\frac{3}{2} + i\frac{\sqrt{3}}{2}$, $i\sqrt{3}$, $-\frac{3}{2} + i\frac{\sqrt{3}}{2}$, $-\frac{3}{2} - i\frac{\sqrt{3}}{2}$, $-i\sqrt{3}$ i $\frac{3}{2} - i\frac{\sqrt{3}}{2}$.

Riješiti matričnu jednačinu $(AXB)^{-1} = B^{-1}(X^{-1}+B)$

ako je $A=\begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Rj.

$$(AXB)^{-1} = B^{-1}(X^{-1}+B)$$

$$B^{-1}X^{-1}A^{-1} = B^{-1}X^{-1} + B^{-1}\cdot B \quad / \cdot B \text{ sa lijeve strane}$$

$$X^{-1}A^{-1} = X^{-1} + B$$

$$X^{-1}A^{-1} - X^{-1} = B$$

$$X^{-1}(A^{-1}-I) = B \quad / \cdot (A^{-1}-I)^{-1} \text{ sa desne strane}$$

$$X^{-1} = B(A^{-1}-I)^{-1} \quad /^{-1}$$

$$X = (A^{-1}-I)\cdot B^{-1}$$

$$\det A = \begin{vmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{vmatrix} \overset{I_k+II_k}{=} \begin{vmatrix} -1 & -4 & 5 \\ -1 & -3 & 1 \\ -2 & -5 & -1 \end{vmatrix} \overset{II_v-I_v}{\underset{III_v-I_v\cdot 2}{=}}$$

$$= \begin{vmatrix} -1 & -4 & 5 \\ 0 & 1 & -4 \\ 0 & 3 & -11 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -4 \\ 3 & -11 \end{vmatrix} = (-1)(-11+12) = -1$$

$$A^{-1} = \frac{1}{\det A}\cdot A_{kof}^{T}$$

$$A_{11} = (-1)^2\begin{vmatrix} -3 & 1 \\ -5 & -1 \end{vmatrix} = 3+5 = 8$$

$$A_{12} = (-1)^3\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -(-2-3) = 5$$

$$A_{13} = (-1)^4\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix} = -10+9 = -1$$

$$A_{21} = (-1)^3\begin{vmatrix} -4 & 5 \\ -5 & -1 \end{vmatrix} = -(4+25) = -29$$

$$A_{22} = (-1)^4\begin{vmatrix} 3 & 5 \\ 3 & -1 \end{vmatrix} = -3-15 = -18$$

$$A_{23} = (-1)^5\begin{vmatrix} 3 & -4 \\ 3 & -5 \end{vmatrix} = -(-15+12) = 3$$

$$A_{31} = 11 \qquad A_{32} = 7 \qquad A_{33} = -1$$

$$A_{kof} = \begin{bmatrix} 8 & 5 & -1 \\ -29 & -18 & 3 \\ 11 & 7 & -1 \end{bmatrix}. \qquad A^{-1} = (-1)\begin{bmatrix} 8 & -29 & 11 \\ 5 & -18 & 7 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix}.$$

$$\det B = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} \overset{II_v-I_v\cdot 2}{\underset{III_v-I_v\cdot 2}{=}} \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & -6 & -3 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -3 \end{vmatrix} = 9-36 = -27$$

$$B^{-1} = \frac{1}{\det B}\cdot B_{kof}^{T} = \frac{(-1)}{-27}\begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} = \frac{1}{27}\cdot 3\cdot\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

ZA VJEŽBU (slično)

$$B^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A^{-1} - I = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix}$$

$$X = (A^{-1} - I) \cdot B^{-1} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$= \frac{1}{9}\begin{bmatrix} 27 & 33 & -87 \\ 15 & 21 & -51 \\ -5 & -1 & 8 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 11/3 & -\frac{29}{3} \\ 5/3 & 7/3 & -\frac{17}{3} \\ -\frac{5}{9} & -\frac{1}{9} & \frac{8}{9} \end{bmatrix}$$ rješenje matrične jednačine

(#) Dati su vektori $\vec{a}=(3m+3,\ 1,\ m+5)$, $\vec{b}=(3m-4,\ 3m-2,\ -2)$ $\vec{c}=(3-3m,\ 2-3m,\ 1)$. Odrediti sve vrijednosti parametra m tako da ovi vektori budu linearno zavisni, pa za najveću dobijenu vrijednost parametra m napisati vektor $\vec{a}$ kao linearnu kombinaciju vektora $\vec{b}$ i $\vec{c}$.

Rj. Vektori $\vec{a}, \vec{b}$ i $\vec{c}$ su linearno zavisni ako postoje skalari α, β, γ, bar jedan različit od nule, takvi da

$$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=0.$$

$$\alpha(3m+3,\ 1,\ m+5)+\beta(3m-4,\ 3m-2,\ -2)+\gamma(3-3m,\ 2-3m,\ 1)=0$$

$$\begin{aligned}(3m+3)\alpha+(3m-4)\beta+(3-3m)\gamma&=0\\ \alpha+(3m-2)\beta+(2-3m)\gamma&=0\\ (m+5)\alpha-2\beta+\gamma&=0\end{aligned}$$

Ovaj (homogeni) sistem ima netrivijalna rješenja akko je $D=0$.

$$D=\begin{vmatrix}3m+3 & 3m-4 & 3-3m\\ 1 & 3m-2 & 2-3m\\ m+5 & -2 & 1\end{vmatrix} \overset{II_k+III_k}{=} \begin{vmatrix}3m+3 & -1 & 3-3m\\ 1 & 0 & 2-3m\\ m+5 & -1 & 1\end{vmatrix}$$

$$\overset{I_v-III_v}{=} \begin{vmatrix}2m-2 & 0 & 2-3m\\ 1 & 0 & 2-3m\\ m+5 & -1 & 1\end{vmatrix}=\begin{vmatrix}2m-2 & 2-3m\\ 1 & 2-3m\end{vmatrix} \overset{I_v-II_v}{=} \begin{vmatrix}2m-3 & 0\\ 1 & 2-3m\end{vmatrix}$$

$$=(2m-3)(2-3m) \qquad D=0 \text{ akko } m=\frac{3}{2} \text{ ili } m=\frac{2}{3}$$

$\frac{3}{2}>\frac{2}{3} \Rightarrow m=\frac{3}{2}$: $\vec{a}=\left(\frac{9}{2}+3,\ 1,\ \frac{3}{2}+5\right)=\left(\frac{15}{2},\ 1,\ \frac{13}{2}\right)$

$\vec{b}=\left(\frac{9}{2}-4,\ \frac{9}{2}-2,\ -2\right)=\left(\frac{1}{2},\ \frac{5}{2},\ -2\right)$ $\vec{c}=\left(3-\frac{9}{2},\ 2-\frac{9}{2},\ 1\right)=$
$=\left(-\frac{3}{2},\ -\frac{5}{2},\ 1\right)$

$\vec{a}=\mu\vec{b}+\eta\vec{c}$ – razlaganje vektora $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$

Pronađimo vrijednosti μ i η.

$$\left(\frac{15}{2},\ 1,\ \frac{13}{2}\right)=\mu\left(\frac{1}{2},\ \frac{5}{2},\ -2\right)+\eta\left(-\frac{3}{2},\ -\frac{5}{2},\ 1\right)$$

$$\begin{aligned}\frac{1}{2}\mu-\frac{3}{2}\eta&=\frac{15}{2}\\ \frac{5}{2}\mu-\frac{5}{2}\eta&=1\\ -2\mu+\eta&=\frac{13}{2}\end{aligned}$$

$\Rightarrow \mu=-\frac{69}{10},\ \eta=-\frac{73}{10}$ $\quad \vec{a}=\frac{-69\vec{b}-73\vec{c}}{10}$

sistem riješiti za vježbu

(#) Ispitati f-ju i nacrtati njen grafik $y = \frac{x^2+10}{x^2+4x+4}$.

Rj. $y = \frac{x^2+10}{x^2+4x+4} = \frac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2 \neq 0$

$x \neq -2$

$D: x \in (-\infty,-2) \cup (-2,+\infty)$

· parnost (neparnost), periodičnost

D nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom i znak f-je

$y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0 \ \forall x \in D$ to f-ja nema nule

$f(0) = \frac{0+10}{(0+2)^2} = \frac{10}{4} = \frac{5}{2}$

$(0, \frac{5}{2})$ je presjek sa Y-osom

$x^2+10>0 \ \forall x \in D$

$(x+2)^2>0 \ \forall x \in D$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim_{x\to -2-0} f(x) = \lim_{x\to -2-0} \frac{x^2+10}{(x+2)^2} = \frac{(-2-0)^2+10}{(-2-0+2)^2} = \frac{14+0}{+0} = +\infty \Rightarrow x=-2$ je V. A. (sa lijeve strane)

$\lim_{x\to -2+0} f(x) = \lim_{x\to -2+0} \frac{x^2+10}{(x+2)^2} = \frac{(-2+0)^2+10}{(-2+0+2)^2} = \frac{14-0}{+0} = +\infty \Rightarrow x=-2$ je V. A. (sa desne strane)

$\lim_{x\to \mp\infty} f(x) = \lim_{x\to \mp\infty} \frac{x^2+10 \ :x^2}{x^2+4x+4 \ :x^2} = \lim_{x\to \mp\infty} \frac{1+\frac{10}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}} = 1 \Rightarrow y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

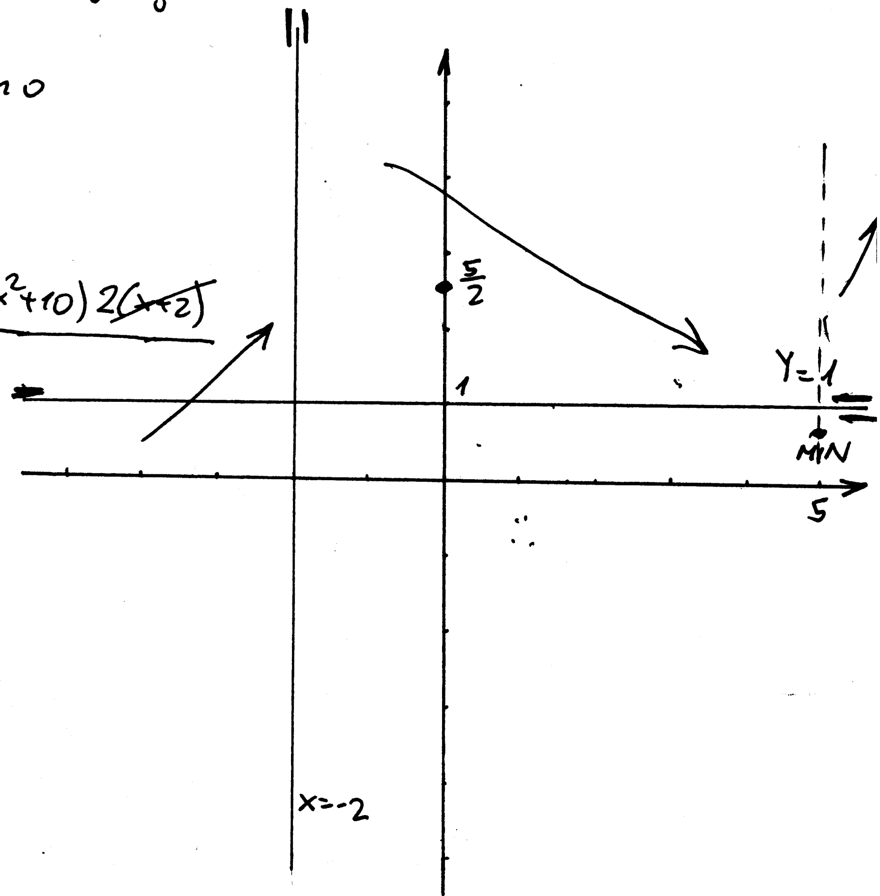
$y' = \left(\frac{x^2+10}{(x+2)^2}\right)' = \frac{2x\cdot(x+2)^2 - (x^2+10)2(x+2)}{(x+2)^4}$

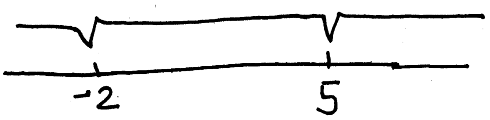
$y' = \frac{2x^2+4x-2x^2-20}{(x+2)^3}$

$y' = \frac{4x-20}{(x+2)^3} = 4\frac{x-5}{(x+2)^3}$

$y'=0$ akko $x-5=0$

$x=5$





x	$(-\infty,-2)$	$(-2,5)$	$(5,+\infty)$	
y'	+	—	+	
y	↗	↘	↗	rast i opadanje
			min	

ekstremi f-je

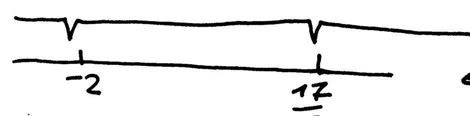
Stacionarna tačka je $x=5$.
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$$f(5)=\frac{25+10}{7^2}=\frac{35}{49}\approx 0{,}71 \qquad \left(5,\frac{35}{49}\right) \text{ je tačka minimuma}$$

prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(4\,\frac{x-5}{(x+2)^3}\right)'=4\,\frac{1\cdot(x+2)^3-(x-5)3(x+2)^2}{(x+2)^{6}}=4\,\frac{x+2-3x+15}{(x+2)^4}$$

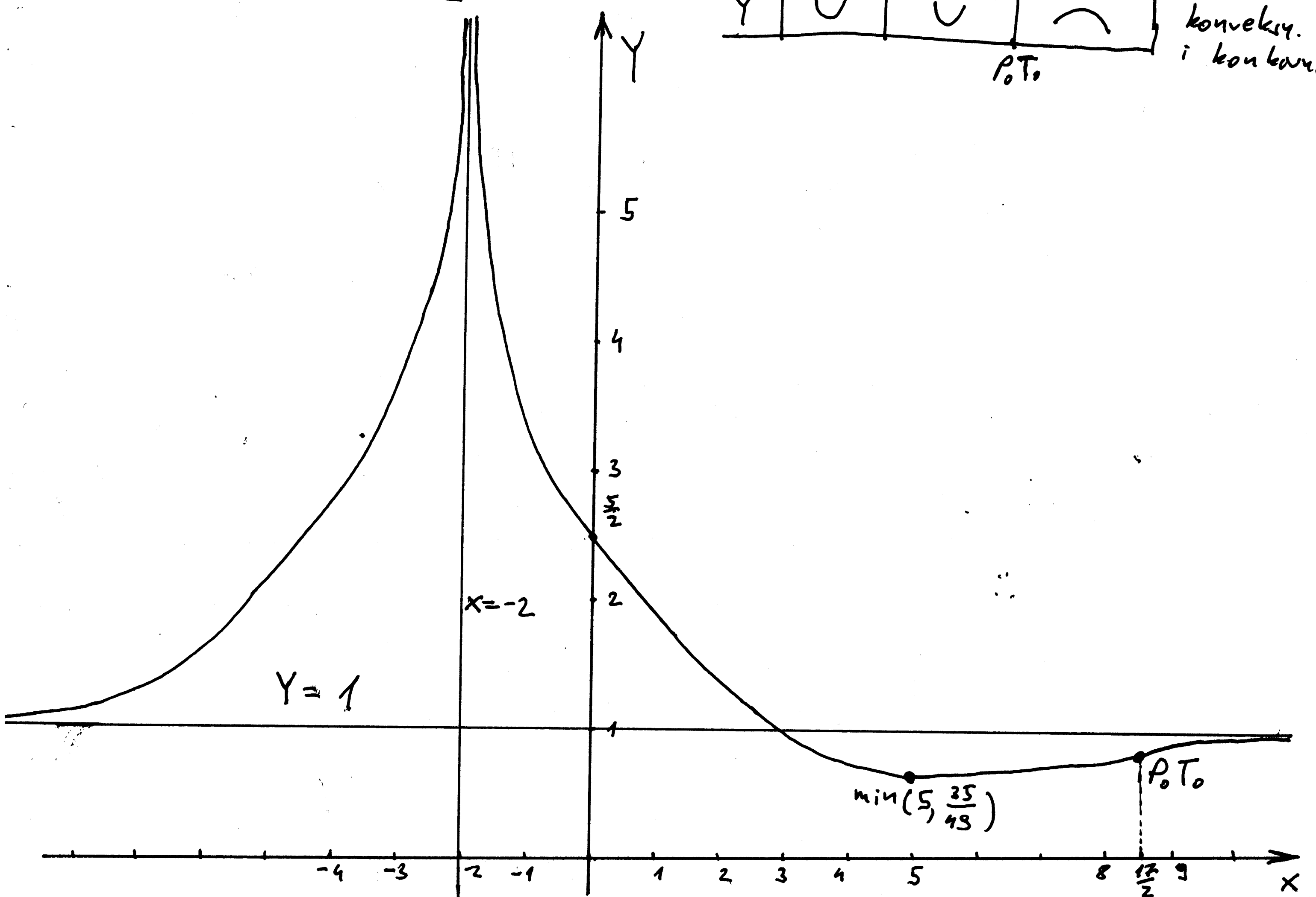
$$y''=4\,\frac{-2x+17}{(x+2)^4}=-4\,\frac{2x-17}{(x+2)^4}$$



$y''=0$ akko $2x-17=0$

$$x=\frac{17}{2}$$

x	$(-\infty,-2)$	$(-2,\frac{17}{2})$	$(\frac{17}{2},+\infty)$	
y''	+	+	—	
y	∪	∪	∩	intervali konveksn. i konkavn.
			PoTo	



Ispitati f-ju i nacrtati njen grafik: $y = \frac{x^3 - 2}{2x^2}$.

Rj. definiciono područje

$\mathcal{D}: x \neq 0$

parnost (neparnost), periodičnost

$f(-x) = \frac{(-x)^3 - 2}{2(-x)^2} = \frac{-x^3 - 2}{2x^2} \neq \pm f(x)$

· f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak

$y = 0$ akko $x^3 - 2 = 0$

$x = \sqrt[3]{2} \approx 1{,}26$

$(\sqrt[3]{2}, 0)$ je nula f-je

$f(0) =$ nije definisano

f-ja ne siječe y-osu

$2x^2 > 0 \quad \forall x \in \mathcal{D}$

$y > 0$ za $x > \sqrt[3]{2}$
$y < 0$ za $x < \sqrt[3]{2}$ } znak f-je.

ponašanje na krajevima intervala definisanosti i asimptote

za $x = 0$ f-ja ima prekid

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x^3 - 2}{2x^2} = \frac{(0-)^3 - 2}{2(0-)^2} = \frac{-2-0}{0+} = -\infty$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{(0+)^3 - 2}{2(0+)^2} = \frac{-2+0}{+0} = -\infty$

$\Rightarrow x = 0$ je V. A.

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^3 - 2 \ /:x^2}{2x^2 \ /:x^2} = \pm\infty$ f-ja nema H. A.

Tražimo kosu asimptotu u obliku $y = kx + n$.

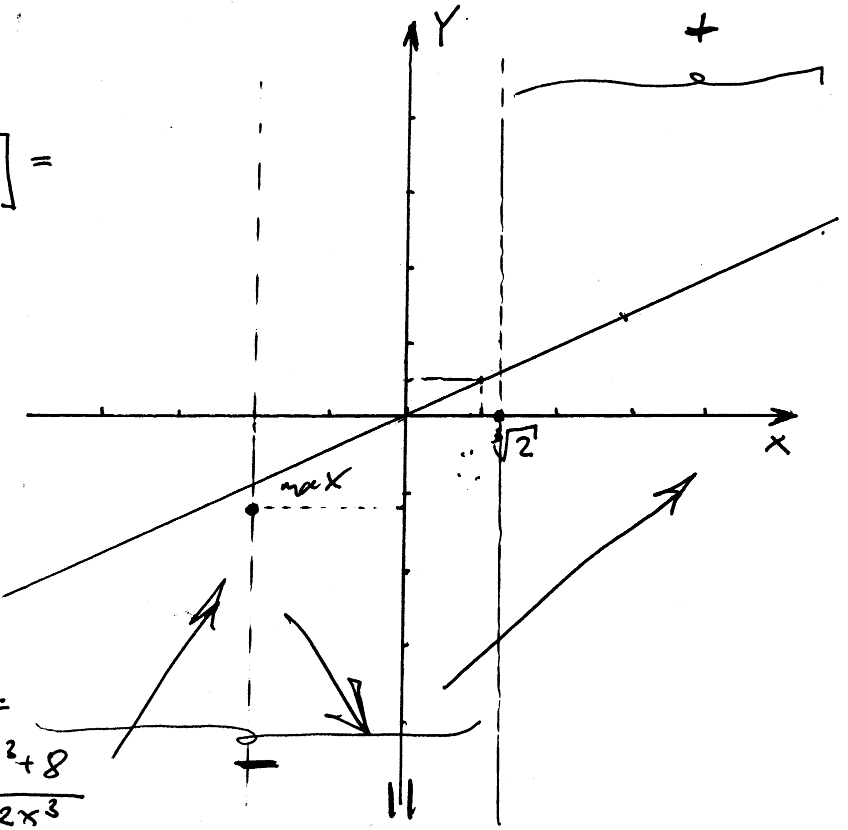
$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3 - 2 \ /:x^3}{2x^3 \ /:x^3} = \frac{1}{2}$

$n = \lim_{x \to \infty} [f(x) - kx] = \lim_{x \to \infty} \left[\frac{x^3 - 2}{2x^2} - \frac{1}{2}x\right] =$

$= \lim_{x \to \infty} \frac{x^3 - 2 - x^3}{2x^2} = \lim_{x \to \infty} \frac{-2}{2x^2} = 0$

kosa asimptota je $y = \frac{1}{2}x$

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$y' = \left(\frac{x^3 - 2}{2x^2}\right)' = \frac{3x^2 \cdot 2x^2 - (x^3 - 2)4x}{4x^4} =$

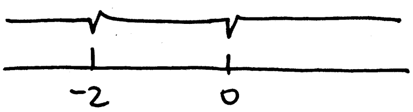
$= \frac{6x^4 - 4x^4 + 8x}{4x^4} = \frac{2x^4 + 8x}{4x^4} = \frac{x^3 + 8}{2x^3}$

$y' = \frac{x^3+8}{2x^3}$, $\quad y'=0$ akko $x^3+8=0$

$x^3=-8$

$x=-2$

prekidi y + nule y'



x	$(-\infty,-2)$	$(-2,0)$	$(0,+\infty)$
y'	+	−	+
y	↗	↘	↗

max N.D.

$$f(-2)=\frac{(-2)^3-2}{2(-2)^2}=\frac{-10}{8}=-\frac{5}{4}\approx -1{,}25$$

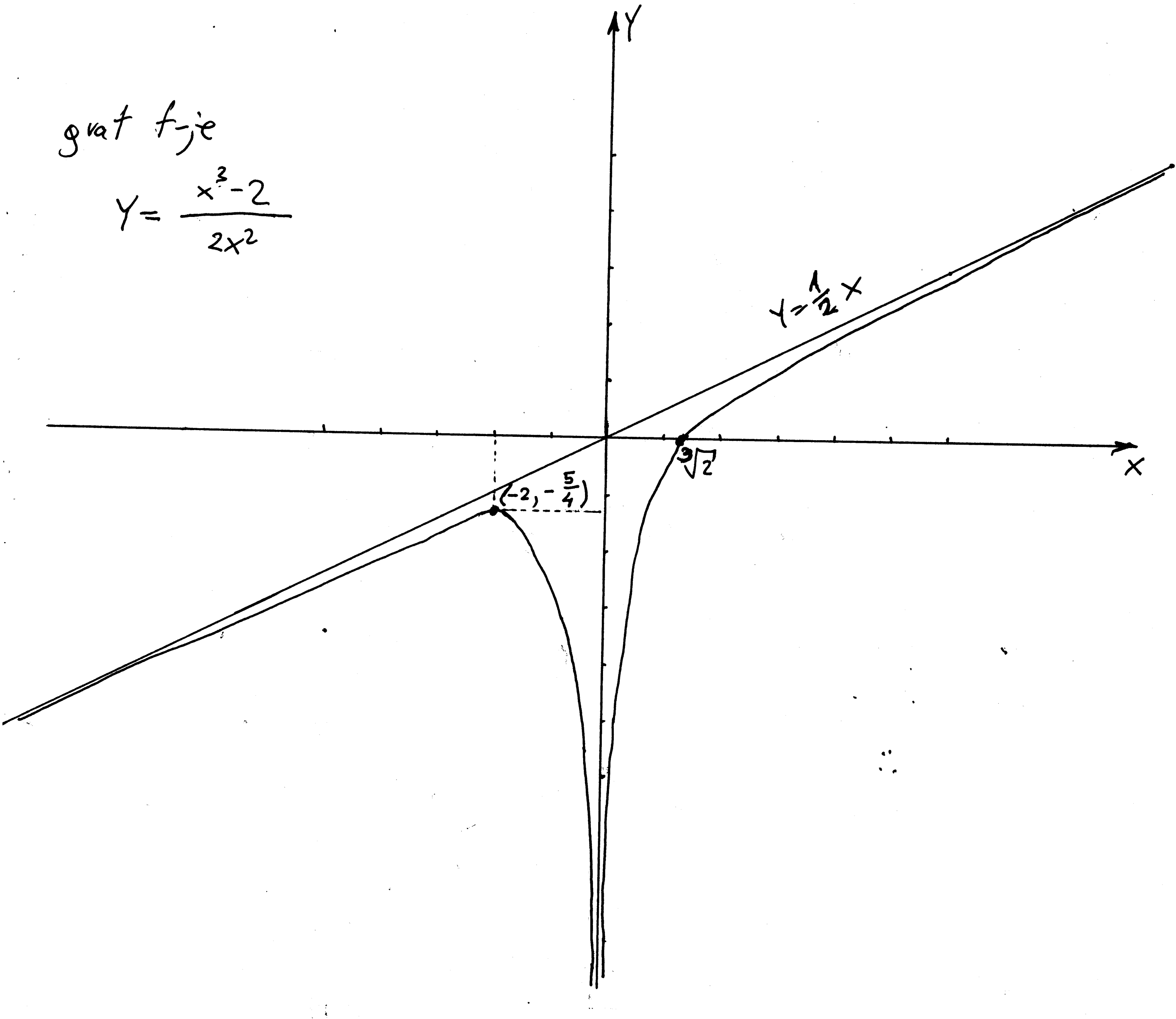
prevojne tačke i intervali konveksnosti i konkavnosti

$$y''=\left(\frac{x^3+8}{2x^3}\right)'=\frac{3x^2\cdot 2x^3-(x^3+8)\cdot 6x^2}{4x^6}=\frac{6x^5-6x^5-48}{4x^6}=\frac{-48}{4x^6}=-\frac{12}{x^6}<0$$

F-ja nema prevojnih tački i uvijek je negativna što znači uvijek je ∩ oblika.

graf f-je

$$y=\frac{x^3-2}{2x^2}$$



(#) Ispitati f-ju i nacrtati njen grafik $y = e^{\frac{x}{1-x}} - 1$.

Rj. definiciono područje

$1-x \neq 0$

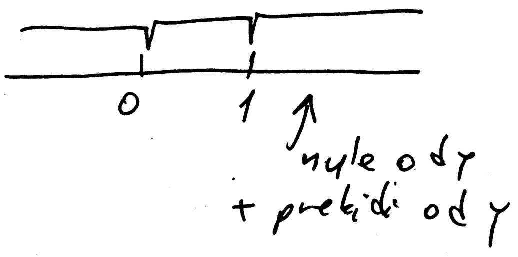
$x \neq 1$ $\quad \mathcal{D}: x \in (-\infty, 1) \cup (1, +\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična



nule, presjek sa y-osom, znak f-je

$y=0$ ako $e^{\frac{x}{1-x}} = 1$

tj. $\frac{x}{1-x} = 0 \Rightarrow x = 0$

(0,0) je nula f-je i presjek sa y-osom

$y > 0 \Leftrightarrow e^{\frac{x}{1-x}} - 1 > 0$

	$(-\infty, 0)$	$(0,1)$	$(1,+\infty)$
x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
y	$-$	$+$	$-$

znak f-je

$e^{\frac{x}{1-x}} > 1$

$e^{\frac{x}{1-x}} > e^0$

$\frac{x}{1-x} > 0$

ponašanje na krajevima intervala definisanosti i asimptote

za $x=1$ f-ja ima prekid

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1-0}{1-1+0}} - 1 = e^{\frac{1-0}{+0}} - 1 = e^{\infty} - 1 = \infty$$

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}\left(e^{\frac{x}{1-x}} - 1\right) = e^{\frac{1+0}{1-1-0}} - 1 = e^{\frac{1+0}{-0}} - 1 = e^{-\infty} - 1 = \frac{1}{e^{\infty}} - 1 = -1$$

$x=1$ je vertikalna asimptota (sa lijeve strane)

$$\lim_{x\to\mp\infty} f(x) = \lim_{x\to\mp\infty}\left(e^{\frac{x}{1-x}\,\frac{:x}{:x}} - 1\right) =$$

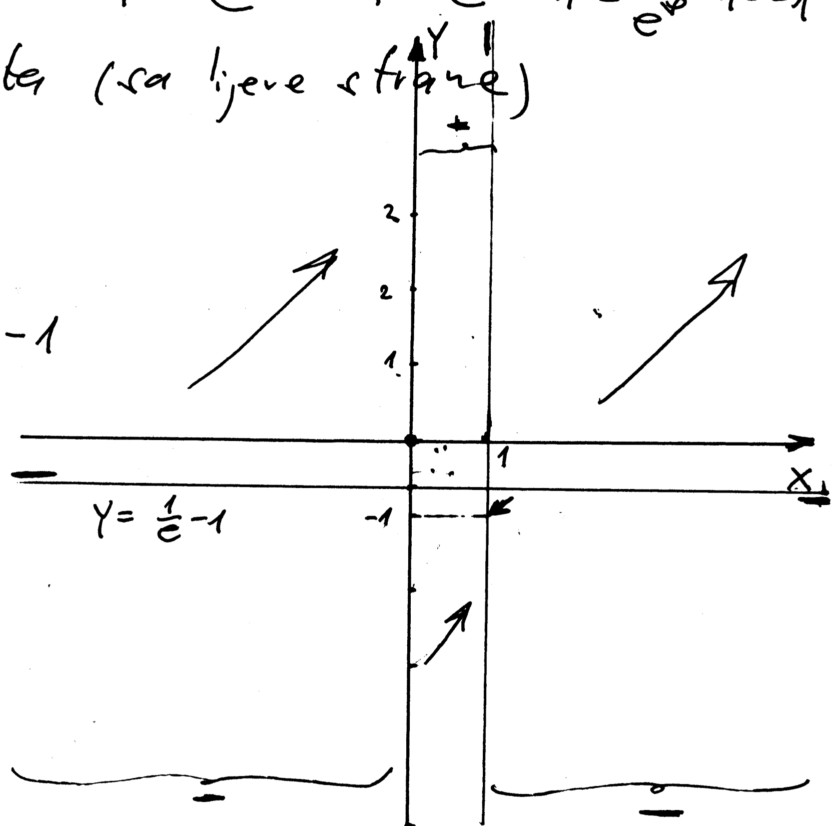
$$= \lim_{x\to\mp\infty}\left(e^{\frac{1}{\frac{1}{x} - 1}} - 1\right) = e^{-1} - 1 = \frac{1}{e} - 1$$

$y = \frac{1}{e} - 1 \approx -0{,}63$

je H. A.

kose asimptote nema

Poslije ovog koraka počinjemo sa skiciranjem grafa f-je



rast i opadanje

$$y' = \left(e^{\frac{x}{1-x}} - 1\right)' = e^{\frac{x}{1-x}} \cdot \left(\frac{x}{1-x}\right)' = \frac{1\cdot(1-x) - x\cdot(-1)}{(1-x)^2}\, e^{\frac{x}{1-x}} = \frac{e^{\frac{x}{1-x}}}{(1-x)^2}$$

$$y' = \frac{1}{(1-x)^2}\, e^{\frac{x}{1-x}}$$ $y' > 0$ za $\forall x \in D$, f-ja $\nearrow$ za $\forall x$

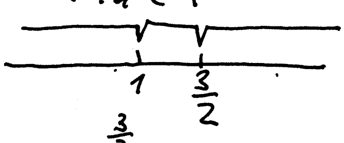
ekstremi f-je

$y' \neq 0$ $\forall x$ f-ja nema ekstrema

$$y'' = \left(\frac{1}{(1-x)^2}\, e^{\frac{x}{1-x}}\right)' = (-2)(1-x)^{-3}(-1)e^{\frac{x}{1-x}} + \frac{1}{(1-x)^2} \cdot \frac{1}{(1-x)^2}\, e^{\frac{x}{1-x}}$$

$$y'' = \frac{2\cdot(1-x) + 1}{(1-x)^4}\, e^{\frac{x}{1-x}} = \frac{-2x+3}{(1-x)^4}\, e^{\frac{x}{1-x}}$$ $y'' = 0$ akko $x = \frac{3}{2}$

prelazi ose i nule y''



$$f\left(\tfrac{3}{2}\right) = e^{\frac{\frac{3}{2}}{1-\frac{3}{2}}} - 1 = e^{\frac{\frac{3}{2}}{-\frac{1}{2}}} - 1$$

$$f\left(\tfrac{3}{2}\right) = e^{-3} - 1 \approx -0{,}95$$

x	$(-\infty, 1)$	$(1, \frac{3}{2})$	$(\frac{3}{2}, +\infty)$
y''	+	+	—
y	∪	∪	∩

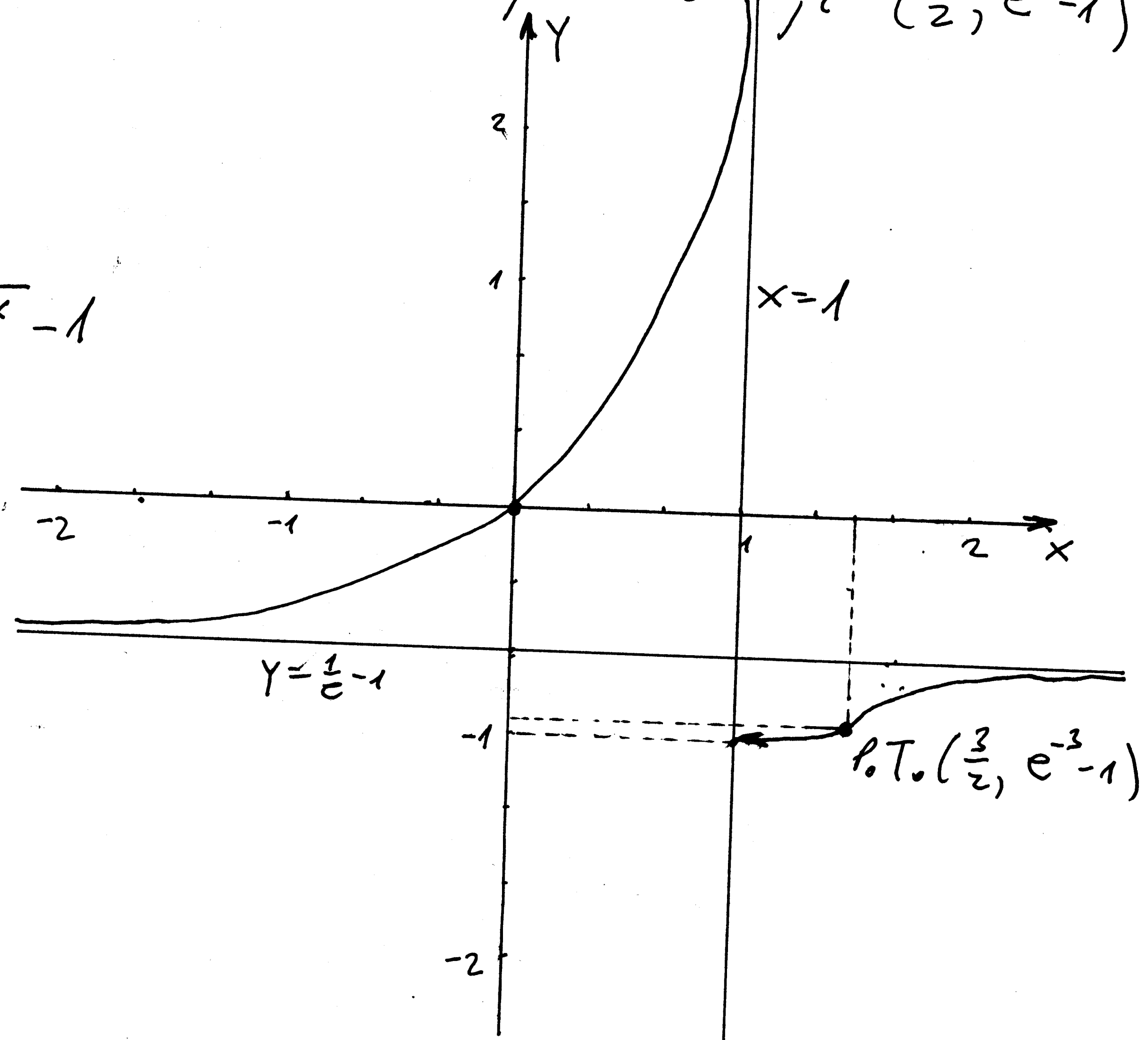
konveksnost i konkavnost

P.T.

Prevojna tačka je $\left(\frac{3}{2}, e^{-3} - 1\right)$

graf f-je

$$y = e^{\frac{x}{1-x}} - 1$$



(#) Ispitati f-ju i nacrtati njen grafik: $y = \frac{\ln^2 x + 1}{x^2}$.

R:
definiciono područje
$x \neq 0$ i $x > 0$
$\mathcal{D}: x \in (0, +\infty)$

nule, presjek sa Y-osom, znak f-je
$y = 0$ akko $\ln^2 x + 1 = 0$
$(\ln x)^2 = -1$
f-ja nema nulu
$f(0)$ nije definisano
f-ja ne siječe Y-osu
$\ln^2 x + 1 > 0 \ \forall x \in \mathcal{D}$
$x^2 > 0 \ \forall x \in \mathcal{D}$
f-ja je uvijek pozitivna

parnost (neparnost), periodičnost
$\mathcal{D}$ nije simetrično
$\Rightarrow$ f-ja nije ni parna ni neparna
f-ja nije periodična

ponašanje na krajevima intervala definisanosti i asimptote
za $x \leq 0$ f-ja nije definisana

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln^2 x + 1}{x^2} = \frac{+\infty}{0+} = +\infty \Rightarrow x = 0 \text{ je vertikalna asimptota}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln^2 x + 1}{x^2} \left(=\frac{\infty}{\infty}\right) \overset{L.P.}{=} \lim_{x\to+\infty} \frac{2\ln x \cdot \frac{1}{x}}{2x} = \lim_{x\to+\infty} \frac{\ln x}{x^2} \left(\frac{\infty}{\infty}\right) \overset{L.P.}{=}$$

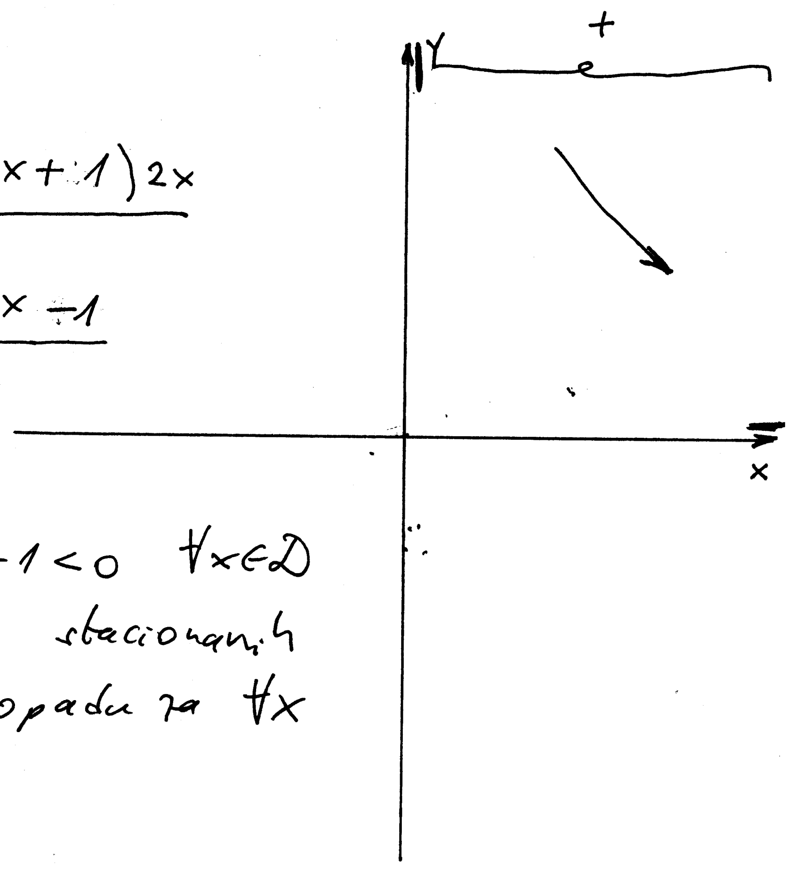
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = 0 \Rightarrow y = 0 \text{ je horizontalna asimptota}$$

f-ja nema kosu asimptotu
počinjemo skicirati grafik

rast i opadanje

$$y' = \left(\frac{\ln^2 x + 1}{x^2}\right)' = \frac{2\ln x \cdot \frac{1}{x} \cdot x^2 - (\ln^2 x + 1)2x}{x^4}$$

$$= \frac{2x(\ln x - \ln^2 x - 1)}{x^4} = 2\,\frac{\ln x - \ln^2 x - 1}{x^3}$$



$y' = 0$ akko $-\ln^2 x + \ln x - 1 = 0$

$\ln x = t$
$-t^2 + t - 1 = 0$
$t^2 - t + 1 = 0$
$D = 1 - 4 < 0$

$-\ln^2 x + \ln x - 1 < 0 \ \forall x \in \mathcal{D}$
f-ja nema stacionarnih tačaka i opada za $\forall x$

ekstremi f-je

f-ja nema stacionarnih tački ⇒ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = 2\left(\frac{\ln x - \ln^2 x - 1}{x^3}\right)' = 2\,\frac{\left(\frac{1}{x} - 2\ln x \cdot \frac{1}{x}\right)x^3 - (\ln x - \ln^2 x - 1)\cdot 3x^2}{x^6} =$$

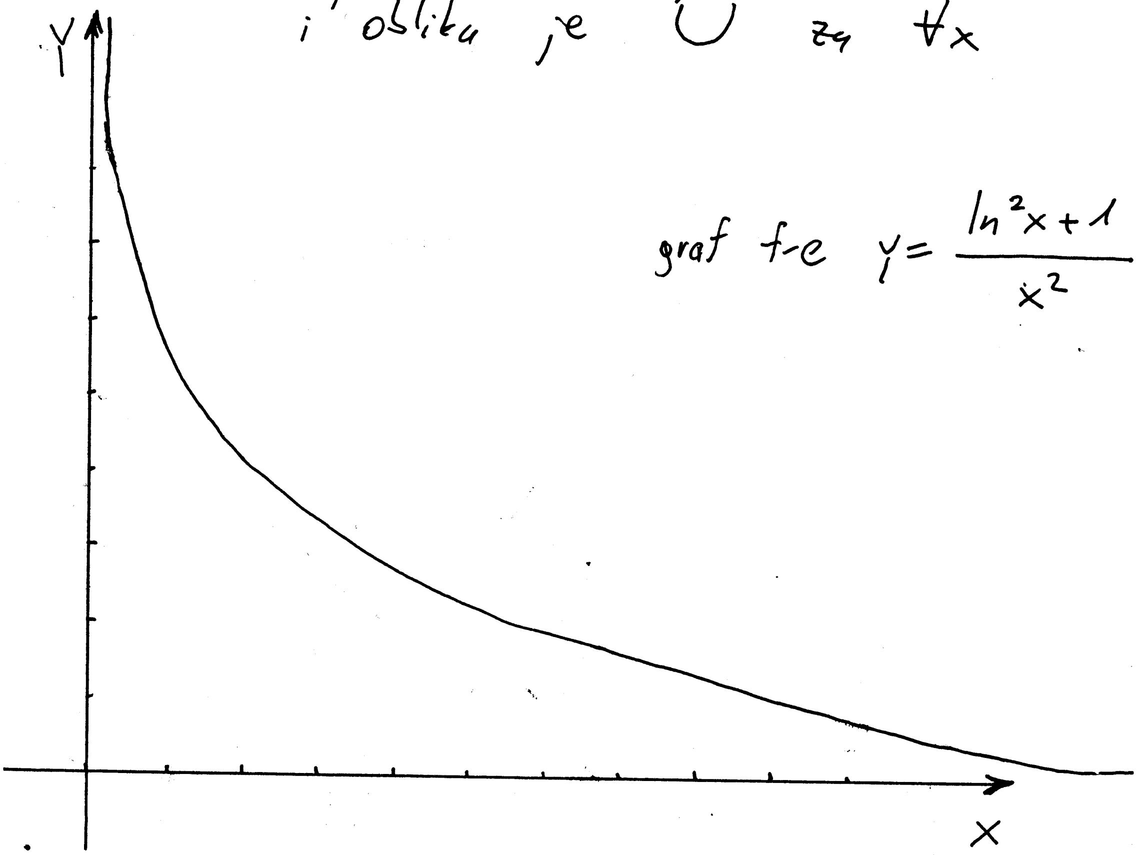
$$= 2\,\frac{1 - 2\ln x - 3\ln x + 3\ln^2 x + 3}{x^4} = 2\,\frac{3\ln^2 x - 5\ln x + 4}{x^4}$$

$3\ln^2 x - 5\ln x + 4 = 0$

$\ln x = t \qquad 3t^2 - 5t + 4 = 0 \qquad \Rightarrow \quad 3\ln^2 x - 5\ln x + 4 > 0 \quad \forall x$

$D = 25 - 48 < 0 \qquad\qquad x^4 > 0 \quad \forall x$

$y'' > 0 \quad \forall x \in D \quad \Rightarrow$ f-ja nema prevojnih tački i oblika je $\cup$ za $\forall x$



graf f-e $y = \dfrac{\ln^2 x + 1}{x^2}$

(#) Izračunati integral $I = \int \frac{dx}{x^5 - x^2}$.

R:

$$\frac{1}{x^5 - x^2} = \frac{1}{x^2(x^3-1)} = \frac{1}{x^2(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1} \quad / \cdot (x^5 - x^2)$$

$$1 = A(x^4 - x) + B(x^3 - 1) + C(x^4 + x^3 + x^2) + (Dx + E)(x^3 - x^2)$$

x^4: $A + C + D = 0$

x^3: $B + C - D + E = 0$

x^2: $C - E = 0$

x: $-A = 0$

1: $-B = 1$

$A = 0$

$B = -1$

$C = E$

$C + D = 0$

$2C - D = 1$

$3C = 1$

$C = \frac{1}{3} \quad D = -\frac{1}{3}$

$E = \frac{1}{3}$

$\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{x^{-1}}{-1} + C$

$$I = \int \frac{dx}{x^5 - x^2} = -\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1} dx$$

$$\int \frac{x-1}{x^2+x+1} dx = \frac{1}{2}\int \frac{2x-2 \overset{+1-1}{}}{x^2+x+1} dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1} dx - \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$x^2 + x + 1 = x^2 + 2 \cdot x \cdot \frac{1}{2} + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} = \left| \begin{matrix} x + \frac{1}{2} = \frac{\sqrt{3}}{2} t \\ dx = \frac{\sqrt{3}}{2} dt \end{matrix} \right| = \frac{\sqrt{3}}{2} \int \frac{dt}{\frac{3}{4}t^2 + \frac{3}{4}} = \frac{\sqrt{3}}{2} \cdot \frac{4}{3} \int \frac{dt}{t^2+1}$$

$$= \frac{2\sqrt{3}}{3} \operatorname{arctg} t + C = \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C$$

$$I = \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{3}\left(\frac{1}{2}\ln|x^2+x+1| - \frac{3}{2} \cdot \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{2x+1}{\sqrt{3}}\right) + C =$$

$$= \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{\sqrt{3}}{3} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C$$

(#) Izračunati integral $I=\int \frac{x^4}{x^4+x^2-6}\,dx$.

Rj.

$$\int \frac{x^4}{x^4+x^2-6}\,dx = \int \frac{x^4+x^2-6-x^2+6}{x^4+x^2-6}\,dx$$

$$= \int dx - \int \frac{x^2-6}{x^4+x^2-6}\,dx$$

$x^4+x^2-6=0$

$x^2=t$

$t^2+t-6=0$

$(t+3)(t-2)=0$

$$\frac{x^2-6}{x^4+x^2-6} = \frac{x^2-6}{(x^2+3)(x^2-2)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2-2}$$

$$x^2-6 = (Ax+B)(x^2-2) + (Cx+D)(x^2+3)$$

$$x^2-6 = A(x^3-2x) + B(x^2-2) + C(x^3+3x) + D(x^2+3)$$

$$\begin{aligned} A + C &= 0 \quad \ldots(I)\\ B + D &= 1 \quad \ldots(II)\\ -2A + 3C &= 0 \quad \ldots(III)\\ -2B + 3D &= -6 \quad \ldots(IV) \end{aligned}$$

$2\cdot(I)+(III)$: $5C=0$

$C=0 \Rightarrow A=0$

$2\cdot(II)+(IV)$: $5D=-4$

$D=-\frac{4}{5}$ $\quad B=1-D=1+\frac{4}{5}$

$B=\frac{9}{5}$

$$\int \frac{x^2-6}{x^4+x^2-6}\,dx = \frac{9}{5}\int \frac{dx}{x^2+3} - \frac{4}{5}\int \frac{dx}{x^2-2}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}+c \qquad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$$

$$I = x - \frac{9}{5}\cdot\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}} + \frac{4}{5}\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C$$

$$= x - \frac{3\sqrt{3}}{5}\operatorname{arctg}\frac{x\sqrt{3}}{3} + \frac{\sqrt{2}}{5}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C$$

traženo rješenje

(#) Izračunati površinu figure određene linijama

$y = 8x - 2x^2, \quad 3x + y = 0, \quad 3x - y - 12 = 0.$

Rj.

Prisjetimo se da krive oblika $y = ax^2 + bx + c$ izgledaju $\cup$ ili $\cap$ u zavisnosti od parametra $a > 0$ ili $a < 0$

Nađimo presjek krive i datih pravih

$$\begin{aligned} y &= 8x - 2x^2 \\ 3x + y &= 0 \end{aligned}$$

$$\begin{aligned} y &= 8x - 2x^2 \\ - \quad y &= -3x \end{aligned}$$

$11x - 2x^2 = 0$

$x(11 - 2x) = 0$

$x_1 = 0 \quad x_2 = \frac{11}{2}$

$x_1 = 0 \Rightarrow y_1 = 0$

$x_2 = \frac{11}{2} \Rightarrow y_2 = -3 \cdot \frac{11}{2} = \frac{-33}{2}$

Presječne tačke su

$A(0,0)$ i

$B\left(\frac{11}{2}, -\frac{33}{2}\right)$

$$\begin{aligned} y &= 8x - 2x^2 \\ 3x - y - 12 &= 0 \end{aligned}$$

$$\begin{aligned} y &= 8x - 2x^2 \\ - \quad y &= 3x - 12 \end{aligned}$$

$5x - 2x^2 + 12 = 0 \quad / \cdot (-1)$

$2x^2 - 5x - 12 = 0$

$D = 25 + 96 = 121$

$x_{1,2} = \frac{5 \pm 11}{4}$

$x_1 = \frac{-6}{4} = -\frac{3}{2}$

$x_2 = \frac{16}{4} = 4$

$y = 3x - 12$

$x_1 = -\frac{3}{2} \Rightarrow y = -\frac{9}{2} - 12 = \frac{-33}{2}$

$x_2 = 4 \Rightarrow y = 12 - 12 = 0$

Presječne tačke su

$C\left(-\frac{3}{2}, -\frac{33}{2}\right)$ i $D(4,0)$

Nađimo presječne tačke pravih

$$\begin{aligned} 3x + y &= 0 \\ - \quad 3x - y - 12 &= 0 \end{aligned}$$

$2y + 12 = 0$

$y = -6$

$3x + y = 0$

$3x - 6 = 0$

$3x = 6$

$x = 2$

Presječna tačka pravih je $E(2,-6)$

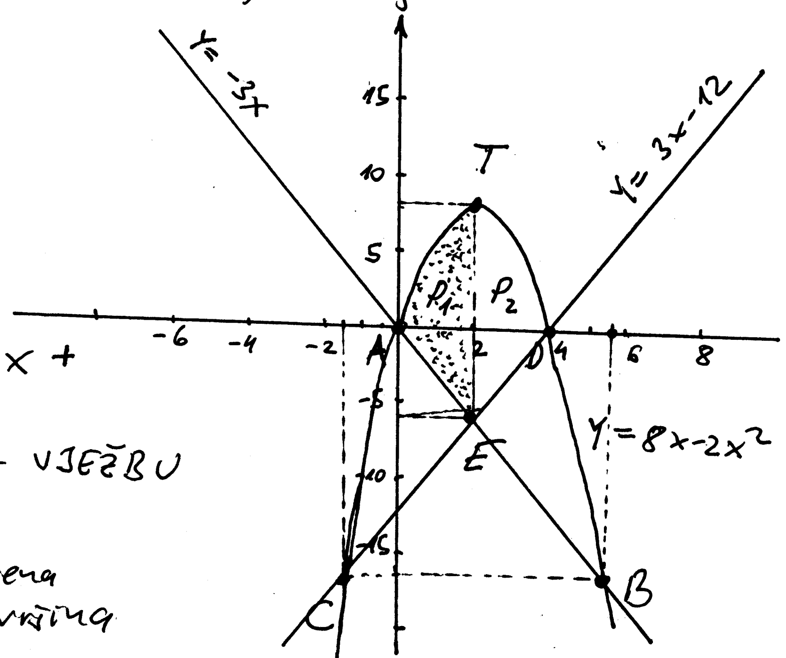
Nađimo tjeme krive $y = -2x^2 + 8x$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a} = \frac{8}{4} = 2$. $T(2,8)$

$D = 64 \quad -\frac{D}{4a} = \frac{64}{8} = 8$

Skiciramo grafik



$$P = P_1 + P_2 = \int_0^2 \left[(8x - 2x^2) - (-3x)\right] dx + \int_2^4 \left[(8x - 2x^2) - (3x - 12)\right] dx = \ldots$$

ZA VJEŽBU

$\ldots = \frac{50}{3} + \frac{50}{3} = \frac{100}{3}$ tražena površina

Ispitati površinu figure omeđene linijama

$y = -\frac{1}{2}x + 2, \quad y = \sqrt{x-1}, \quad y = 0$

Rj.

$y = -\frac{1}{2}x + 2$ i $y = 0$ su prave koje nije teško nacrtati.

Kako izgleda f-ja $y = \sqrt{x-1}$? Ispitajmo ukratko ovu f-ju

$y = \sqrt{x-1}$

$D: x > 1$

nije ni parna ni neparna

$(1,0)$ je nula f-je

f-ja ne siječe y-osu

$\lim_{x \to 1+0} \sqrt{x-1} = \sqrt{1+0-1} = 0$

$\lim_{x \to \infty} \sqrt{x-1} = \infty$

$\Rightarrow$ f-ja nema H.A.

$y = kx + n$

$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\sqrt{x-1}}{x} = 0$

f-ja nema K.A.

$y' = \frac{1}{2\sqrt{x-1}}$

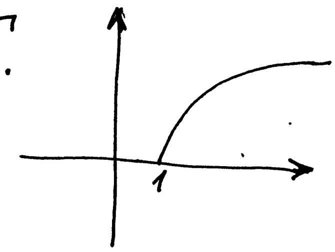
$y' > 0 \ \forall x \in D$ ↗ za $\forall x \in D$

f-ja nema ekstrema

$y'' = \left(\frac{1}{2}(x-1)^{-\frac{1}{2}}\right)' = -\frac{1}{4}(x-1)^{-\frac{3}{2}}$

$= -\frac{1}{4} \cdot \frac{1}{\sqrt{(x-1)^3}} < 0 \ \forall x \in D$

f-ja je uvijek ∩ i nema prevojnih tačaka



Nađimo presječne tačke prave $y = -\frac{1}{2}x + 2$ i f-je $y = \sqrt{x-1}$.

$y = -\frac{1}{2}x + 2 \quad / \cdot 2$

$y = \sqrt{x-1} \quad /^2$

$2y = -x + 4$

$+ \ y^2 = x - 1$

$y^2 + 2y - 3 = 0$

$D = 4 + 12 = 16$

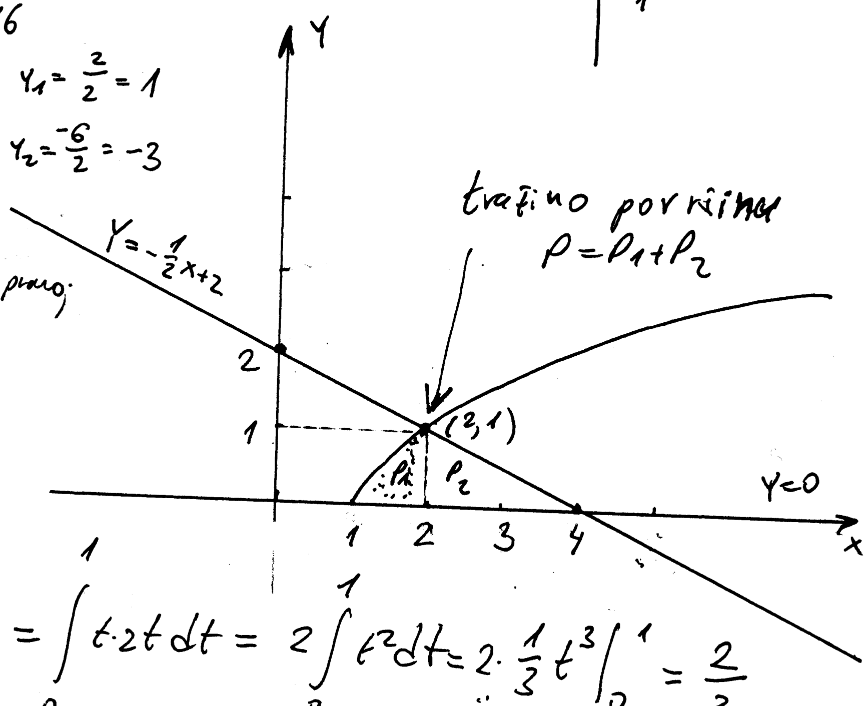
$y_{1,2} = \frac{-2 \pm 4}{2} \qquad y_1 = \frac{2}{2} = 1 \qquad y_2 = \frac{-6}{2} = -3$

$y_1 = 1 \Rightarrow x = 2$

$y_2 = -3 \Rightarrow x = 10$

Tačka $(-3, 10)$ nije na pravoj

Presječna tačka je $(2, 1)$.



$P = P_1 + P_2$

$P_1 = \int_1^2 \sqrt{x-1}\,dx = \left| \begin{array}{l} x - 1 = t^2 \\ dx = 2t\,dt \\ x = 1 \Rightarrow t = 0 \\ x = 2 \Rightarrow t = 1 \end{array} \right| = \int_0^1 t \cdot 2t\,dt = 2\int_0^1 t^2\,dt = 2 \cdot \frac{1}{3} t^3 \Big|_0^1 = \frac{2}{3}$

$P_2 = \int_2^4 \left(-\frac{1}{2}x + 2\right) dx = -\frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_2^4 + 2x \Big|_2^4 = -\frac{1}{4} \cdot (16 - 4) + 2(4 - 2) = -3 + 4 = 1$

$P = P_1 + P_2 = \frac{2}{3} + 1 = \frac{5}{3}$ tražena površina

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj. Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{1}{x^2+y^2+1}\cdot 2x = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{1}{x^2+y^2+1}\cdot 2y = 1 - \frac{3y}{x^2+y^2+1}$$

Nađimo stacionarne tačke

$$1 - \frac{3x}{x^2+y^2+1} = 0$$

$$1 - \frac{3y}{x^2+y^2+1} = 0$$

$$3x = 3y \Rightarrow x = y$$

$$1 - \frac{3x}{2x^2+1} = 0 \quad / \cdot 2x^2+1$$

$$2x^2+1-3x=0$$

$$2x^2-3x+1=0$$

$$D = 9-8 = 1$$

$$x_{1,2} = \frac{3\pm 1}{4}$$

$$x_1 = \frac{2}{4} = \frac{1}{2}$$

$$x_2 = 1$$

Stacionarne tačke su $M_1(\frac{1}{2}, \frac{1}{2})$ i $M_2(1,1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 0 - 3\cdot\frac{1\cdot(x^2+y^2+1) - x\cdot 2x}{(x^2+y^2+1)^2} = -3\cdot\frac{-x^2+y^2+1}{(x^2+y^2+1)^2} = 3\,\frac{x^2-y^2-1}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = 0 - 3x\cdot(-1)(x^2+y^2+1)^{-2}\cdot 2y = 6\,\frac{xy}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 - 3\,\frac{1\cdot(x^2+y^2+1) - y\cdot 2y}{(x^2+y^2+1)^2} = -3\,\frac{x^2-y^2+1}{(x^2+y^2+1)^2}$$

Za $M_1(\frac{1}{2}, \frac{1}{2})$, $A = 3\cdot\frac{-1}{(\frac{1}{2}+1)^2} = \frac{-3}{\frac{9}{4}} = \frac{-12}{9} = -\frac{4}{3}$, $B = \frac{2}{3}$, $C = -\frac{4}{3}$

$D = AC - B^2 = \frac{16}{9} - \frac{4}{9} > 0$ f-ja ima ekstrem u tački M_1

$A<0$ f-ja ima minimum $z_{min}(\frac{1}{2}, \frac{1}{2}) = 1 - \frac{3}{2}\ln\frac{3}{2}$

Za $M_2(1,1)$, $A = -\frac{1}{3}$, $B = \frac{2}{3}$, $C = -\frac{1}{3}$

$D = AC - B^2 = \frac{1}{9} - \frac{4}{9} < 0$ f-ja u tački M_2 nema ekstrem

(#) Naći uslovne ekstreme f-je $z=(x-3)^2+(y-4)^2$ uz uslov $x^2+y^2=\frac{25}{4}$.

Rj. Formiramo f-ju $F(x,y,\lambda)=(x-3)^2+(y-4)^2+\lambda(x^2+y^2-\frac{25}{4})$

$F'_x = 2(x-3)+2\lambda x$

$F'_y = 2(y-4)+2\lambda y$

$F'_\lambda = x^2+y^2-\frac{25}{4}$

$2x-6+2\lambda x=0 \quad /:2 \Rightarrow \lambda=\frac{3-x}{x}$

$2y-8+2\lambda y=0 \quad /:2 \Rightarrow \lambda=\frac{4-y}{y}$

$x^2+y^2=\frac{25}{4}$

$\frac{3-x}{x}=\frac{4-y}{y} \Rightarrow \frac{3}{x}=\frac{4}{y}$

$x=\frac{3}{4}y$

$(\frac{3}{4}y)^2+y^2=\frac{25}{4}$

$\frac{25y^2}{16}=\frac{25}{4} \Rightarrow y^2=4 \Rightarrow y=\pm 2$

$y=\pm 2 \Rightarrow x=\pm\frac{3}{2} \Rightarrow \lambda=\pm 2-1 \Rightarrow \lambda_1=1,\ \lambda_2=-3$

Stacionarne tačke su $M_1(\frac{3}{2},2)$ za $\lambda_1=1$ i $M_2(-\frac{3}{2},-2)$ za $\lambda_2=-3$

Pronađimo druge parcijalne izvode

$F''_{xx}=2+2\lambda$

$F''_{xy}=0$

$F''_{yy}=2+2\lambda$

Za $M_1(\frac{3}{2},2)$, $\lambda_1=1$, $D=AC-B^2$

$A=C=4$, $B=0 \Rightarrow D=16>0 \Rightarrow$ f-ja ima ekstrem, pa kako je $A<0$ f-ja ima minimum u tački M_1

$z_{min}(\frac{3}{2},2)=(\frac{3}{2}-3)^2+(2-4)^2=(-\frac{3}{2})^2+(-2)^2=\frac{9}{4}+4=\frac{25}{4}$

Za $M_2(-\frac{3}{2},-2)$, $\lambda_2=-3$, $D=AC-B^2$

$A=C=-4$, $B=0$, $D=16>0$ f-ja ima ekstrem

$A<0 \Rightarrow$ f-ja ima maksimum

$z_{max}(-\frac{3}{2},-2)=(-\frac{3}{2}-3)^2+(-2-4)^2=(-\frac{9}{2})^2+(-6)^2=\frac{225}{4}$

(4) Riješiti diferencijalnu jednačinu

$$y^3y' + 3xy^2 + 2x^3 = 0.$$

Rj.

$$y^3y' + 3xy^2 + 2x^3 = 0$$

$$y^3y' = -3xy^2 - 2x^3 \quad / : y^3$$

$$y' = \frac{-3xy^2 - 2x^3}{y^3} \quad \begin{matrix} :x^3 \\ :x^3 \end{matrix}$$

$$y' = \frac{-3\left(\frac{y}{x}\right)^2 - 2}{\left(\frac{y}{x}\right)^3}$$

ovo je homogena diferencijalna jednačina

uvodimo smjenu $\frac{y}{x} = u$

tj. $y = ux$

$y' = u'x + u$

$$u'x + u = \frac{-3u^2 - 2}{u^3}$$

$$u'x = \frac{-3u^2 - 2}{u^3} - u$$

$$u'x = \frac{-3u^2 - 2 - u^4}{u^3}$$

$$\frac{du}{dx}x = \frac{-u^4 - 3u^2 - 2}{u^3}$$

$$\frac{u^3}{-u^4 - 3u^2 - 2}\,du = \frac{dx}{x}$$

$$\frac{u^3}{u^4 + 3u^2 + 2}\,du = -\frac{dx}{x}$$

$$u^4 + 3u^2 + 2 = 0$$

$$u^2 = t, \quad t^2 + 3t + 2 = 0 \qquad t_{1,2} = \frac{-3 \pm 1}{2}$$

$$D = 9 - 8 = 1 \qquad t_1 = \frac{-4}{2} = -2$$

$$(u^2+2)(u^2+1) = 0 \qquad t_2 = \frac{-2}{2} = -1$$

$$\frac{u^3}{u^4 + 3u^2 + 2} = \frac{Au + B}{u^2 + 2} + \frac{Cu + D}{u^2 + 1} \quad /(u^2+2)(u^2+1)$$

$$u^3 = A(u^3 + u) + B(u^2 + 1) + C(u^3 + 2u) + D(u^2 + 2)$$

$$\begin{aligned} A + C &= 1 \\ B + D &= 0 \\ A + 2C &= 0 \\ B + 2D &= 0 \end{aligned}$$

$$B = D = 0$$

$$\begin{aligned} A + C &= 1 \\ A + 2C &= 0 \quad / \cdot (-1) \end{aligned}$$

$$\begin{aligned} A + C &= 1 \\ -A - 2C &= 0 \end{aligned}$$

$$-C = 1$$

$$C = -1$$

$$\therefore A = 2$$

$$\frac{u^3}{u^4+3u^2+2} = \frac{2u}{u^2+2} + \frac{-u}{u^2+1}$$

$$\frac{u^3}{u^4+3u^2+2}\,du = -\frac{dx}{x} \quad /\int$$

$$\ln|u^2+2| - \frac{1}{2}\ln|u^2+1| = -\ln|x| + \ln c$$

$$\ln\frac{|u^2+2|}{\sqrt{u^2+1}} = \ln\frac{c}{x}$$

$$\frac{u^2+2}{\sqrt{u^2+1}} = \frac{c}{x}$$

$$\frac{\left(\frac{y}{x}\right)^2+2}{\sqrt{\left(\frac{y}{x}\right)^2+1}} = \frac{c}{x}$$

rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu

$$y' + \frac{y}{4x} + y^3 e^{\sqrt{x}} = 0 \quad \text{ako je} \quad y(1) = 1.$$

Rj.

$$y' + \frac{1}{4x} y = -e^{\sqrt{x}} y^3$$ ovo je Bernulijeva diferencijalna jednačina.

uvodimo smjenu $y = uv$

$y' = u'v + uv'$

$$u'v + uv' + \frac{1}{4x} uv = -e^{\sqrt{x}} u^3 v^3$$

$$u'v + u\underbrace{\left(v' + \frac{1}{4x} v\right)}_{=0} = -e^{\sqrt{x}} u^3 v^3$$

a) $v' + \frac{1}{4x} v = 0$

$$\frac{dv}{dx} = \frac{-v}{4x}$$

$$\frac{dv}{v} = \frac{-dx}{4x}$$

$$\frac{dv}{v} = -\frac{1}{4} \cdot \frac{dx}{x} \quad / \int$$

$$\ln v = -\frac{1}{4} \ln|x|$$

$$\ln v = \ln|x|^{-\frac{1}{4}}$$

$$v = \frac{1}{\sqrt[4]{x}}$$

b) $u'v = -e^{\sqrt{x}} u^3 v^3$

$$u' \cdot \frac{1}{\sqrt[4]{x}} = -e^{\sqrt{x}} u^3 \frac{1}{\sqrt[4]{x^3}} \quad / \cdot \sqrt[4]{x}$$

$$\frac{du}{dx} = -e^{\sqrt{x}} \frac{u^3}{\sqrt[4]{x^2}}$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt[4]{x^2}} dx$$

$$\frac{du}{u^3} = -\frac{e^{\sqrt{x}}}{\sqrt{x}} dx \quad / \int$$

$$\left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$\frac{u^{-2}}{-2} = -2e^{\sqrt{x}} + c_1 \quad / \cdot (-2)$$

$$\frac{1}{u^2} = 4e^{\sqrt{x}} + c.$$

$$u^2 = \frac{1}{4e^{\sqrt{x}} + c} \Rightarrow u = \frac{1}{\sqrt{4e^{\sqrt{x}} + c}}$$

$$y = uv = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + c}}$$ opšte rješenje diferenc. jedn.

$$y(1) = 1 \Rightarrow \frac{1}{\sqrt{4e + c}} = 1$$

$$\sqrt{4e + c} = 1$$

$$4e + c = 1 \Rightarrow c = 1 - 4e$$

$$y = \frac{1}{\sqrt[4]{x}\sqrt{4e^{\sqrt{x}} + 1 - 4e}}$$ partikularno rješenje diferencijalne jednačine