Izračunati integrale $I_1=\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$, $I_2=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\sin x+\cos x}\,dx$.

Rj:

$$I_1=\int_0^4 \frac{dx}{1+\sqrt{2x+1}} = \left|\begin{array}{l} 2x+1=t^2 \\ x=0 \Rightarrow t=1 \\ x=4 \Rightarrow t=3 \\ 2\,dx=2t\,dt \end{array}\right| = \int_1^3 \frac{\overset{+1-1}{t}\,dt}{1+t} = \int_1^3 dt-\int_1^3\frac{dt}{1+t} =$$

$$= t\Big|_1^3-\ln|1+t|\Big|_1^3 = 2-(\ln 4-\ln 2)=2-\ln 2$$

$$I_2=\int_0^{\pi/4}\frac{\sin x \quad /:\cos x}{\sin x+\cos x \quad /:\cos x}\,dx = \int_0^{\frac{\pi}{4}}\frac{\operatorname{tg}x}{\operatorname{tg}x+1}\,dx = \left|\begin{array}{ll} \operatorname{tg}x=t & x=0\Rightarrow t=0 \\ x=\operatorname{arc\,tg} t & x=\frac{\pi}{4}\Rightarrow t=1 \\ dx=\frac{dt}{1+t^2} & \end{array}\right|$$

$$=\int_0^1\frac{t}{t+1}\cdot\frac{dt}{t^2+1}=\int_0^1\frac{t}{(t+1)(t^2+1)}\,dt$$

$$\frac{x}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1} \quad /\cdot(x+1)(x^2+1)$$

$$x=A(x^2+1)+B(x^2+x)+C(x+1)$$

$$\begin{array}{l} A+B=0 \\ B+C=1 \\ A+C=0 \\ \hline \end{array} \qquad \begin{array}{c} A=-B \\ B+C=1 \\ -B+C=0 \\ \hline A=-B \\ 2C=1 \\ C=\frac{1}{2} \\ B=C \end{array}$$

$$A=-\frac{1}{2},\quad B=\frac{1}{2},\quad C=\frac{1}{2}$$

$$I_2=\int_0^1\frac{-\frac{1}{2}}{x+1}\,dx+\frac{1}{2}\int_0^1\frac{x+1}{x^2+1}\,dx=-\frac{1}{2}\int_0^1\frac{dx}{x+1}+\frac{1}{2}\int_0^1\frac{x}{x^2+1}\,dx+$$

$$\left|\begin{array}{l} x^2+1=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \\ =\frac{1}{4}\ln|t| \end{array}\right|$$

$$+\frac{1}{2}\int_0^1\frac{dx}{x^2+1}=-\frac{1}{2}\ln|x+1|\Big|_0^1+\frac{1}{4}\ln|x^2+1|\Big|_0^1+$$

$$+\frac{1}{2}\operatorname{arc\,tg}x\Big|_0^1=-\frac{1}{2}(\ln 2-\ln 1)+\frac{1}{4}(\ln 2-\ln 1)$$

$$+\frac{1}{2}(\operatorname{arc\,tg}1-\operatorname{arc\,tg}0)=-\frac{1}{2}\ln 2+\frac{1}{4}\ln 2+\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}-\frac{1}{4}\ln 2$$

Izmeniti poredak integracije u integralu

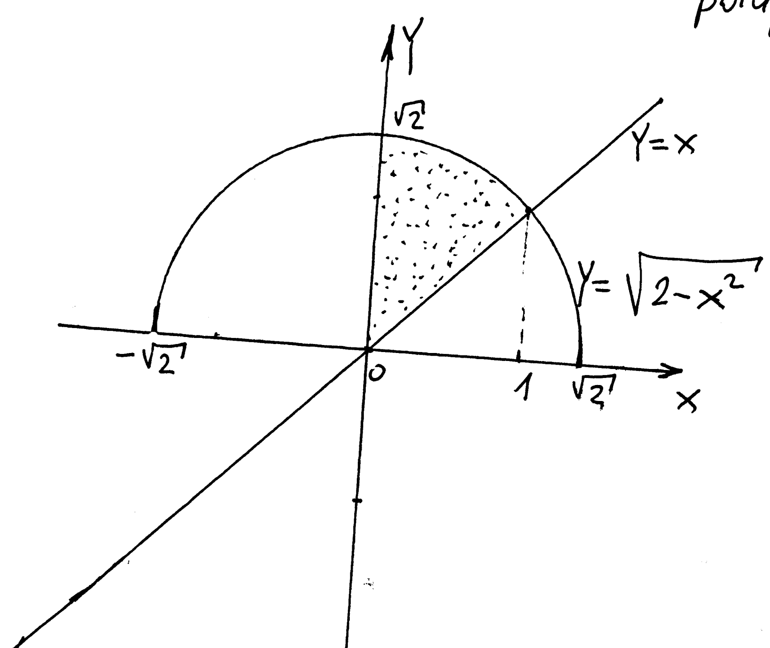
$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy\,.$$

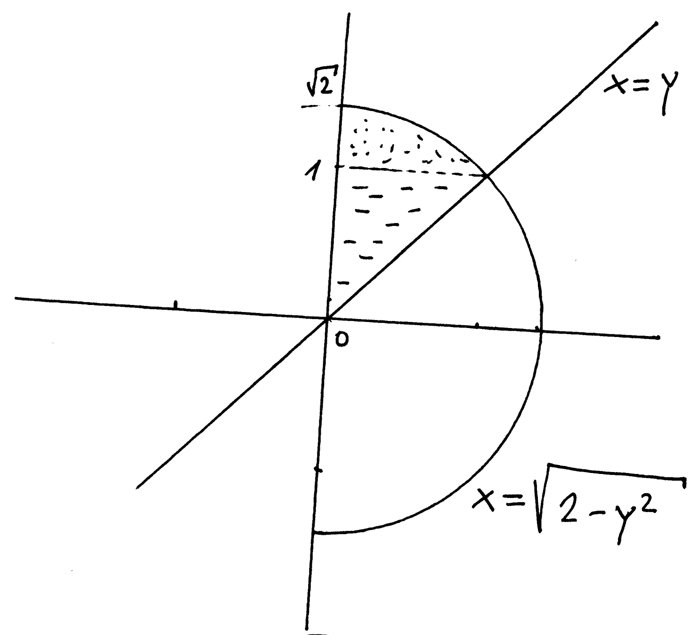
Rj.

$y=x$ prava $\qquad y^2=2-x^2$

$y=\sqrt{2-x^2}$ parabola $\qquad x^2+y^2=2$

krug sa centrom u tački $(0,0)$
poluprečnika $r=\sqrt{2}\approx 1{,}41$





$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy=\int_0^1 dy\int_0^{y} f(x,y)\,dx+\int_1^{\sqrt{2}} dy\int_0^{\sqrt{2-y^2}} f(x,y)\,dx$$

Izračunati površinski integral $\iint\limits_{(S)} \sqrt{-x^2+4}\, dS$, gdje je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$.

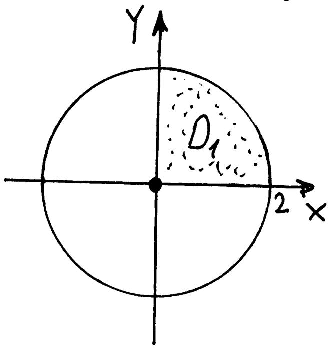
Rj. Skicirajmo površ $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$

u xOy ravni

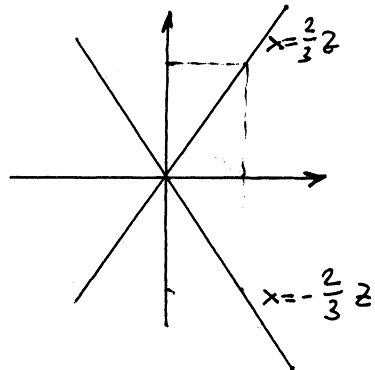
$$\frac{x^2}{4}+\frac{y^2}{4}=0$$

za $z=0$, $x^2+y^2=0$

tačka $(0,0)$



u xOz ravni



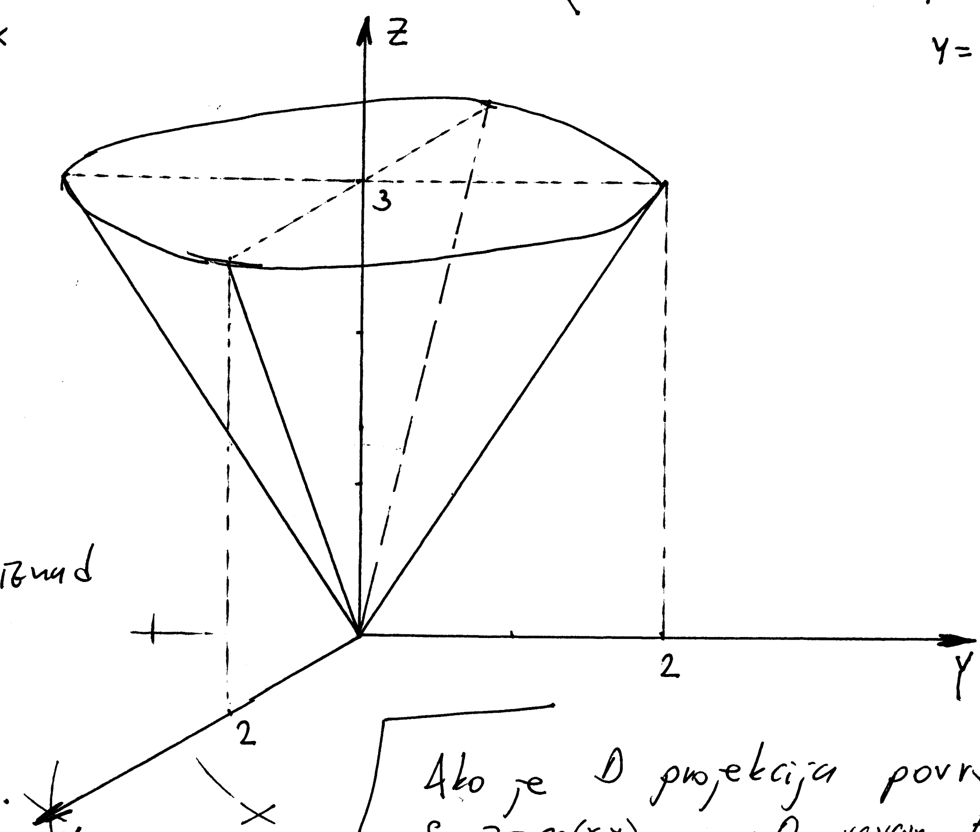
$$\frac{x^2}{4}=\frac{z^2}{9}$$

$$x^2=\frac{4}{9}z^2$$

$$x=\pm\frac{2}{3}z$$

yOz ravan

$$y=\pm\frac{2}{3}z$$



Za $z=3$ $x^2+y^2=4$

$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$$

$$z^2=\frac{9}{4}(x^2+y^2)$$

Kako je data površ iznad xOy ravni

$$z=\frac{3}{2}\sqrt{x^2+y^2}$$

$$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{3x}{2\sqrt{x^2+y^2}} \qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint\limits_{S} f(x,y,z)\,dS=\iint\limits_{D} f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)}=\frac{13x^2+13y^2}{4(x^2+y^2)}=\frac{13}{4}$$

Primjetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = \frac{\sqrt{13}}{2} \iint\limits_{D} \sqrt{-x^2+4}\, dx\,dy = 4 \cdot \frac{\sqrt{13}}{2} \iint\limits_{D_1} \sqrt{4-x^2}\, dx\,dy$$

gde je $D_1 : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$

$$\iint\limits_{(S)} \sqrt{-x^2+4}\, dS = 2\sqrt{13} \int_0^2 \sqrt{4-x^2}\, dx \int_0^{\sqrt{4-x^2}} dy = 2\sqrt{13} \int_0^2 (4-x^2)\, dx =$$

$$= 2\sqrt{13} \left(4x \Big|_0^2 - \frac{1}{3} x^3 \Big|_0^2 \right) = 2\sqrt{13} \left(8 - \frac{8}{3} \right) = 2\sqrt{13} \cdot \frac{\overset{16}{24-8}}{3}$$

$$= \frac{32}{3} \sqrt{13}$$ traženo rešenje

(#) Izračunati pomoću diferenciranja po parametru integral

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx, \quad \alpha>0.$$

Rj. Ako je data f-ja dvije promjenjive $f(x,\alpha)$, ako su $f(x,\alpha)$ i $f'_\alpha(x,\alpha)$ neprekidne f-je tada za integral $I(\alpha)=\int_a^b f(x,\alpha)\,dx$ vrijedi $I'_\alpha(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$.

f'_α – predstavlja izvod f-je f po promjenjivoj α

$$I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)\,dx$$

$$f(x,\alpha)=\ln(\sin^2 x+\alpha^2\cos^2 x)$$

$$f'_\alpha=\frac{1}{\sin^2 x+\alpha^2\cos^2 x}\cdot 2\alpha\cos^2 x=\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}$$

$$I'_\alpha(\alpha)=\int_0^{\frac{\pi}{2}} f'_\alpha\,dx=\int_0^{\frac{\pi}{2}}\frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}\,dx \begin{matrix} /:\cos^2 x \\ /:\cos^2 x\end{matrix} =2\alpha\int_0^{\pi/2}\frac{dx}{tg^2 x+\alpha^2}$$

$$=\left|\begin{matrix} tg\,x=t & x=arctg\,t \\ x=0\Rightarrow t=0 & dx=\frac{dt}{1+t^2} \\ x=\frac{\pi}{2}\Rightarrow t=\infty & \end{matrix}\right| = 2\alpha\int_0^{\infty}\frac{dt}{(t^2+\alpha^2)(t^2+1)}$$

$$\frac{1}{(x^2+\alpha^2)(x^2+1)}=\frac{Ax+B}{x^2+\alpha^2}+\frac{Cx+D}{x^2+1} \quad /(x^2+\alpha^2)(x^2+1)$$

$$1=A(x^3+x)+B(x^2+1)+C(x^3+\alpha^2x)+D(x^2+\alpha^2)$$

$A+C=0$ (1)

$B+D=0$ (2)

$A+\alpha^2C=0$ (3)

$B+\alpha^2D=1$ (4)

(1)-(4): $C-\alpha^2C=0 \Rightarrow C=0 \Rightarrow A=0$

(2)-(4): $D-\alpha^2D=-1$ $\quad (\alpha^2-1)D=1$

$\alpha^2D-D=1$ $\quad D=\frac{1}{\alpha^2-1} \Rightarrow B=\frac{-1}{\alpha^2-1}$

$$I'_\alpha(\alpha) = 2\alpha \int_0^\infty \frac{dx}{(x^2+\alpha^2)(x^2+1)} = \frac{-2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+\alpha^2} + \frac{2\alpha}{\alpha^2-1}\int_0^\infty \frac{dx}{x^2+1} =$$

$$- \frac{2\alpha}{\alpha^2-1} \cdot \frac{1}{\alpha} \operatorname{arctg} \frac{x}{\alpha}\Big|_0^\infty + \frac{2\alpha}{\alpha^2-1} \operatorname{arctg} x \Big|_0^\infty =$$

$$= - \frac{2}{\alpha^2-1}\left(\frac{\pi}{2} - 0\right) + \frac{2\alpha}{\alpha^2-1}\left(\frac{\pi}{2} - 0\right) =$$

$$= - \frac{\pi}{\alpha^2-1} + \frac{\pi}{\cancel{2}} \cdot \frac{\cancel{2}\alpha}{\alpha^2-1} = \frac{\pi(\alpha-1)}{\underbrace{\alpha^2-1}_{(\alpha-1)(\alpha+1)}} = \frac{\pi}{\alpha+1}$$

$$I'_\alpha(\alpha) = \frac{\pi}{\alpha+1} \Rightarrow \quad I(\alpha) = \pi \ln|\alpha+1| + C = \Big| \text{kako je } \alpha > 0 \Big|$$

$$= \pi \ln(\alpha+1) + C \quad \ldots (*)$$

$$I(\alpha) = \int_0^{\pi/2} \ln(\sin^2 x + \alpha^2 \cos^2 x)\, dx \Rightarrow I(1) = \int_0^{\pi/2} \ln(1)\, dx = 0 \quad \ldots (**)$$

$$I(1) \overset{(*)}{=} \pi \ln 2 + C \overset{(**)}{=} 0 \quad \Rightarrow \quad C = -\pi \ln 2$$

$$\int_0^{\pi/2} \ln(\sin^2 x + \alpha^2 \cos^2 x)\, dx = \pi \ln(\alpha+1) - \pi \ln 2 = \pi \ln \frac{\alpha+1}{2}$$

traženo rješenje