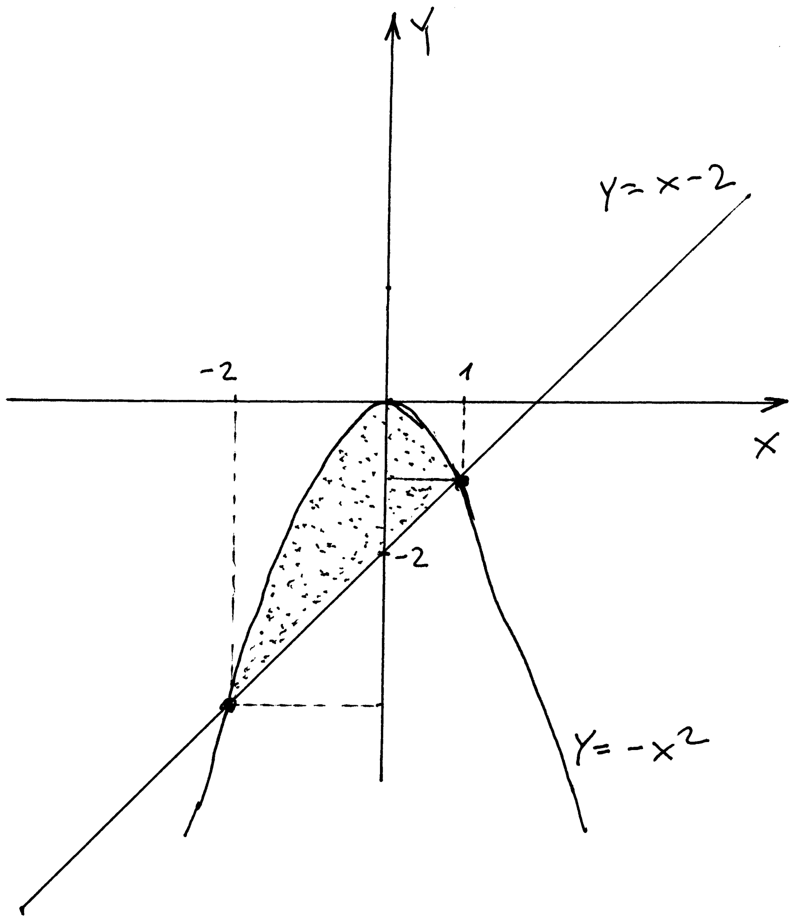
# Nađi površinu figure ograničene linijama $y = -x^2$, $x - y - 2 = 0$.

Rj. Nacrtajmo sliku




Pronađimo presječne tačke krive $y = -x^2$ i prave $x - y - 2 = 0$.

$$y = -x^2$$
$$x - y - 2 = 0$$
$$\overline{\quad\quad\quad\quad}$$
$$x + x^2 - 2 = 0$$
$$x^2 + x - 2 = 0$$
$$D = 1 + 8 = 9 \quad x_{1,2} = \frac{-1 \pm 3}{2}$$
$$x_1 = -2, \quad x_2 = 1$$
$$(x-1)(x+2) = 0$$
$$x = 1 \Rightarrow y = -1$$
$$x = -2 \Rightarrow y = -4$$

I način:

$$P = \int_{-2}^{1} \left(-x^2 - (x-2)\right) dx = \int_{-2}^{1} (-x^2 - x + 2)\, dx = -\frac{1}{3}x^3 \Big|_{-2}^{1} \overbrace{\,}^{1+8} - \frac{1}{2}x^2 \Big|_{-2}^{1} \overbrace{\,}^{1-4} + 2x \Big|_{-2}^{1} \overbrace{\,}^{1+2} =$$

$$= -\frac{1}{3} \cdot 9 + \frac{1}{2} \cdot 3 + 2 \cdot 3 = -3 + \frac{3}{2} + 6 = -3 - \frac{3}{2} = \frac{9}{2}$$

II način:

$$P = \iint_D dx\,dy \quad \text{gdje je} \quad D: \begin{cases} -2 \le x \le 1 \\ x - 2 \le y \le -x^2 \end{cases}$$

$$P = \iint_D dx\,dy = \int_{-2}^{1} dx \int_{x-2}^{-x^2} dy = \int_{-2}^{1} \left((-x^2) - (x-2)\right) dx = \ldots = \frac{9}{2}$$

(#) Naći ekstreme f-je $z = x^3 + 3xy^2 - 15x - 12y$.

Rj.

$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15$

$\frac{\partial z}{\partial y} = 6xy - 12$

$3x^2 + 3y^2 - 15 = 0 \quad /:3$

$6xy - 12 = 0 \quad /:6$

$x^2 + y^2 - 5 = 0 \quad /\cdot x^2$

$xy - 2 = 0 \Rightarrow xy = 2$

$x^4 + (xy)^2 - 5x^2 = 0$

$x^4 - 5x^2 + 4 = 0$

$x^2 = t \Rightarrow t^2 - 5t + 4 = 0$

$D = 25 - 16 = 9$

$t_{1,2} = \frac{5 \pm 3}{2}$

$t_1 = \frac{2}{2} = 1$

$t_2 = \frac{8}{2} = 4$

$t_1 = 1 \Rightarrow x^2 = 1$

$x_1 = -1 \Rightarrow -y = 2 \Rightarrow y_1 = -2$

$x_2 = 1 \Rightarrow y = 2 \Rightarrow y_2 = 2$

$t_2 = 4 \Rightarrow x^2 = 4$

$x_3 = -2 \Rightarrow -2y = 2 \Rightarrow y = -1$

$x_4 = 2 \Rightarrow 2y = 2 \Rightarrow y = 1$

Stacionarne tačke su $M_1(-1,-2)$, $M_2(1,2)$, $M_3(-2,-1)$ ; $M_4(2,1)$.

$\frac{\partial^2 z}{\partial x^2} = 6x$

$\frac{\partial^2 z}{\partial x \partial y} = 6y$

$\frac{\partial^2 z}{\partial y^2} = 6x$

$M_1(-1,-2)$: $A = -6$, $B = -12$, $C = -6$

$D = AC - B^2 = 36 - 12^2 = 36 - 144 < 0$

f-ja u tački $M_1$ nema ekstrema

$M_2(1,2)$: $A = 6$, $B = 12$, $C = 6$

$D = AC - B^2 = 36 - 144 < 0$ f-ja u tački $M_2$ nema ekstrema

$M_3(-2,-1)$: $A = -12$, $B = -6$, $C = -12$, $D = AC - B^2 = 12^2 - 6^2 > 0$

f-ja u tački $M_3$ ima ekstrem, $A < 0 \Rightarrow$ f-ja ima max

$z_{max}(-2,-1) = -8 - 6 + 30 + 12 = 42 - 14 = 28$

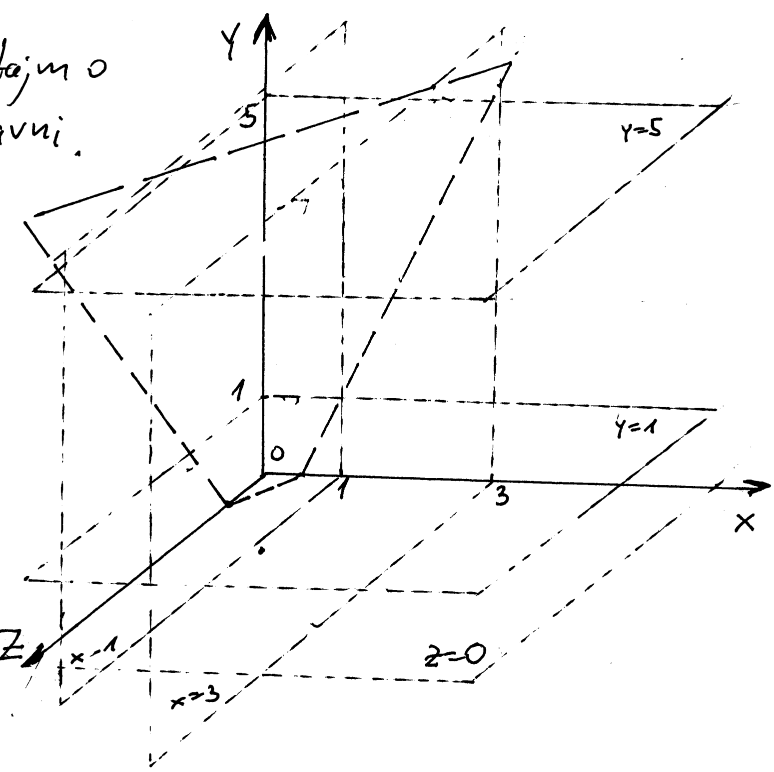
$M_4(2,1)$: $A = 12$, $B = 6$, $C = 12$, $D = AC - B^2 = 12^2 - 6^2 > 0$

f-ja u tački $M_3$ ima ekstrem, $A > 0 \Rightarrow$ f-ja ima min

$z_{min}(2,1) = 8 + 6 - 30 - 12 = 14 - 42 = -28$

(#) Naći zapreminu tijela ograničenog ravnima $x=1$, $x=3$, $y=1$, $y=5$, $2x-y+z-1=0$, $z=0$.

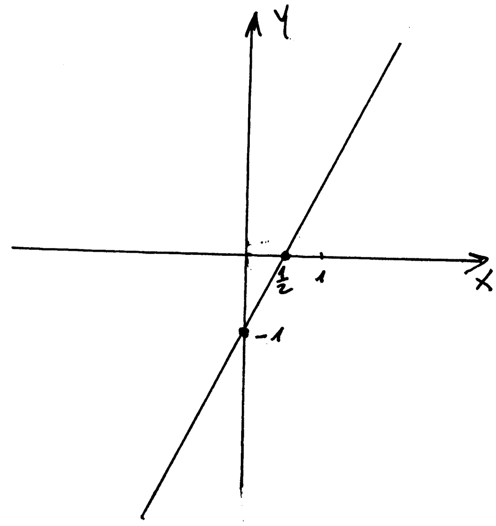
Rj. Nacrtajmo ove ravni.




$$2x-y+z-1=0$$

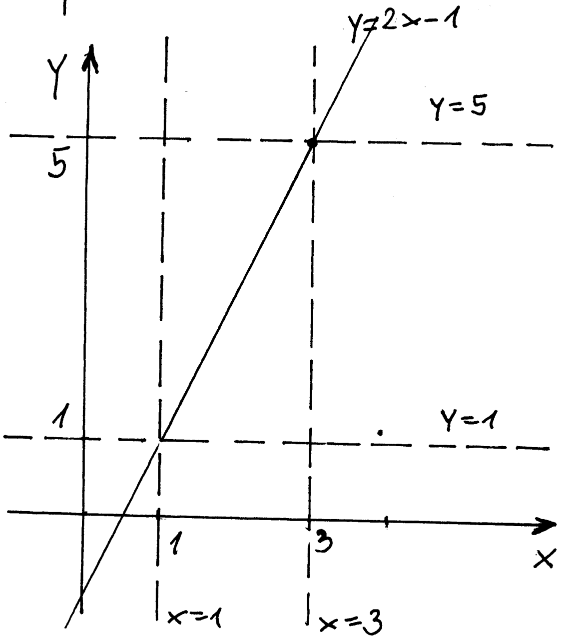
$$z=-2x+y+1$$

projekcija ove ravni na $xOy$ ravan




$$y=2x-1$$

Slika u prostoru je komplikovana i sa nje ne možemo pročitati granice. Nacrtajmo projekcije ovih ravni na $xOy$ ravan.



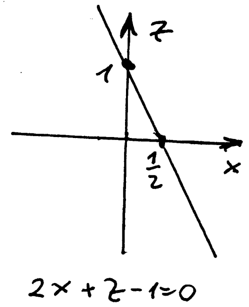

$$2x-y-1=0$$

$$y=2x-1$$

$$x=3 \Rightarrow y=5$$

$$x=1 \Rightarrow y=1$$

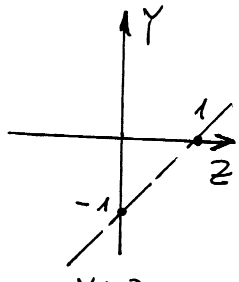
na $xOz$ ravan




$$2x+z-1=0$$

$$z=-2x+1$$

na $yOz$ ravan



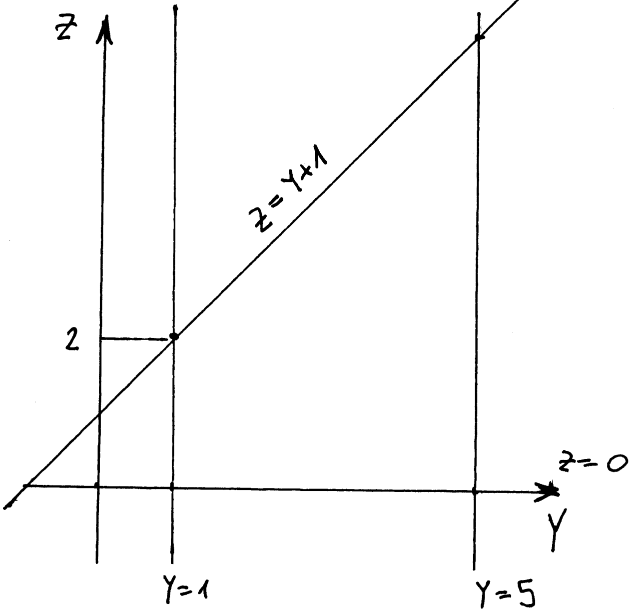

$$-y+z-1=0$$

$$y=z-1$$

Sad na osnovu slike u prostoru i projekcija na ravni možemo pročitati granice za tijelo

$$\Omega: \begin{cases} 1 \le x \le 3 \\ 2x-1 \le y \le 5 \\ 0 \le z \le -2x+y+1 \end{cases}$$

Da su napisane granice ispravne provjerimo projekcijom ravni na $yOz$ ravan.




$$-y+z-1=0$$
$$z=y+1$$

$$V=\iiint_{\Omega} dx\,dy\,dz =$$

$$=\int_1^3 dx \int_{2x-1}^{5} dy \underbrace{\int_0^{-2x+y+1} dz}_{z\Big|_0^{-2x+y+1}} =$$

$$=\int_1^3 dx \int_{2x-1}^{5} (-2x+y+1)\,dy = \int_1^3 \left( (-2x)\cdot y\Big|_{2x-1}^{5} + \frac{1}{2}y^2\Big|_{2x-1}^{5} + y\Big|_{2x-1}^{5} \right) dx =$$

$$=\int_1^3 \left( (-2x)(5-(2x-1)) + \frac{1}{2}\left(5^2-(2x-1)^2\right) + 5-(2x-1) \right) dx =$$

$$=\int_1^3 \left( (-2x)(6-2x) + \frac{1}{2}\left(25-(4x^2-4x+1)\right) + 6-2x \right) dx =$$

$$=\int_1^3 \left( -12x + 4x^2 + \overbrace{\frac{1}{2}(-4x^2+4x+24)}^{-2x^2+2x+12} + 6 - 2x \right) dx = \int_1^3 (2x^2-12x+18)\,dx$$

$$=\frac{2}{3}x^3\Big|_1^3 - \frac{12}{2}x^2\Big|_1^3 + 18x\Big|_1^3 = \frac{2}{3}\cdot 26 - \overset{48}{6\cdot 8} + \overset{36}{18\cdot 2} = \frac{52}{3} - 12 = \frac{16}{3}$$

Zapremina tijela ograničenog spomenutim ravnima iznosi $\frac{16}{3}$.

(#) Izračunati krivoliniski integral $I = \int_c z\sqrt{x^2+y^2+2z^2}\,dS$

ako je c kriva $x = \frac{r\sqrt{2}}{2}\cos t$,

$y = \frac{r\sqrt{2}}{2}\cos t$, $z = r\sin t$, $t \in [0, \pi]$.

Rj. Ako je c kriva opisana parametarskim jednačinama

$$c: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ z = \psi(t) \end{cases}, \quad t_1 \leq t \leq t_2 \quad \text{tada}$$

$$\int_c f(x,y,z)\,dS = \int_{t_1}^{t_2} f(\mu(t), \eta(t), \psi(t)) \underbrace{\sqrt{(\mu'(t))^2 + (\eta'(t))^2 + (\psi'(t))^2}\,dt}_{dS}$$

$$x^2+y^2+2z^2 = \frac{2r^2}{4}\cos^2 t + \frac{2r^2}{4}\cos^2 t + 2r^2\sin^2 t = r^2\cos^2 t + 2r^2\sin^2 t$$

$$x'_t = -\frac{\sqrt{2}}{2} r\sin t, \quad y'_t = -\frac{\sqrt{2}}{2} r\sin t, \quad z'_t = r\cos t$$

$$(x'_t)^2 + (y'_t)^2 + (z'_t)^2 = \frac{1}{2}r^2\sin^2 t + \frac{1}{2}r^2\sin^2 t + r^2\cos^2 t = r^2\sin^2 t + r^2\cos^2 t$$
$$= r^2(\sin^2 t + \cos^2 t) = r^2$$

$$\sqrt{(\mu'(t))^2 + (\eta'(t))^2 + (\psi'(t))^2} = \sqrt{r^2} = r$$

$$I = \int_c z\sqrt{x^2+y^2+2z^2}\,dS = \int_0^{\pi} r\sin t \sqrt{\underbrace{r^2\cos^2 t + 2r^2\sin^2 t}_{r^2(\cos^2 t + 2\sin^2 t)}}\; r\,dt =$$

$$= r^3 \int_0^{\pi} \sin t \sqrt{\cos^2 t + 2\underbrace{\sin^2 t}_{1-\cos^2 t}}\,dt = r^3 \int_0^{\pi} \sin t \sqrt{2-\cos^2 t}\,dt =$$

$$= \left| \begin{matrix} \cos t = u \\ -\sin t\,dt = du \\ \sin t\,dt = -du \end{matrix} \quad t\Big|_0^{\pi} \Rightarrow u\Big|_1^{-1} \right| = r^3 \int_{-1}^{1} \sqrt{2-t^2}\,dt = r^3 \int_{-1}^{1} \frac{2-t^2}{\sqrt{2-t^2}}\,dt$$

ZA VJEŽBU

$$\ldots = r^3 \cdot \frac{1}{2} t\sqrt{2-t^2}\Big|_{-1}^{1} + r^3 \int_{-1}^{1} \frac{dt}{2-t^2} = \ldots = \left(1 + \frac{\pi}{2}\right) r^3$$