Univerzitet u Zenici
Fakultet za metalurgiju i materijale

Zadaci sa pismenog ispita rađenog 29.04.2009. iz predmeta MATEMATIKA 2

1. Ispitati i grafički predstaviti funkciju: $y = \dfrac{x^2-5}{3+x}$.

2. Izračunati: $\displaystyle\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$.

3. Odrediti ekstreme funkcije $z(x,y) = \dfrac{2}{3}x^3 - 5xy + \dfrac{5}{2}y^2 + 8x - 5y$.

4. Riješiti diferencijalnu jednačinu: $y' + y = cosx$.

1. Ispitati i grafički predstaviti f-ju $y = \frac{x^2 - 5}{3 + x}$

Rj. $\mathcal{D}: x \in \mathbb{R} \setminus \{-3\}$

$(-\sqrt{5}, 0)$ i $(\sqrt{5}, 0)$ su nule f-je

$(0, -\frac{5}{3})$ je tačka presjeka sa Y-osom

f-ja nije ni parna ni neparna

nije periodična

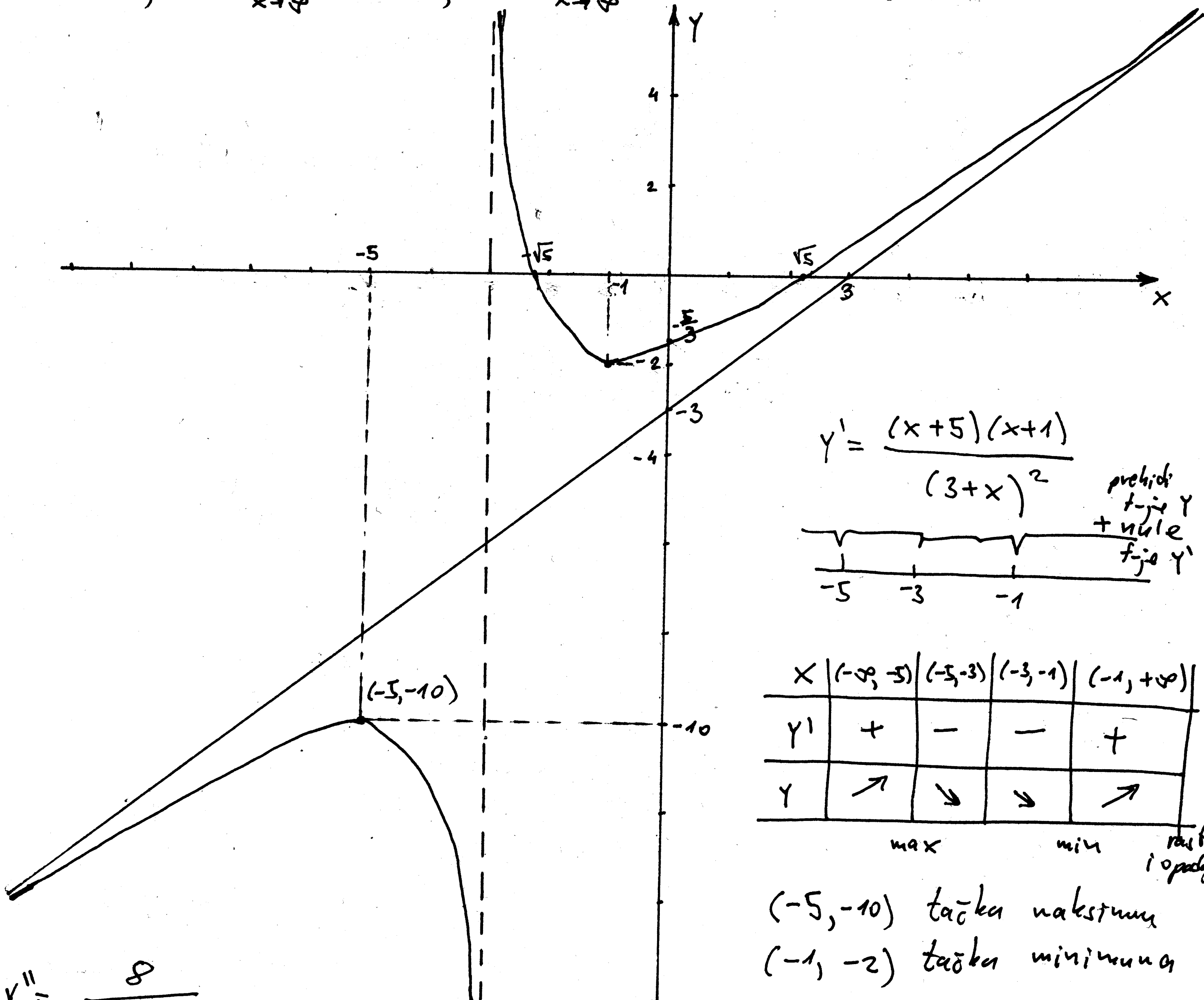
X	$(-\infty, -3)$	$(-3, -\sqrt{5})$	$(-\sqrt{5}, \sqrt{5})$	$(\sqrt{5}, +\infty)$	
Y	$-$	$+$	$-$	$+$	znak f-je

$\lim\limits_{x \to -3-0} f(x) = -\infty$

$\lim\limits_{x \to -3+0} f(x) = +\infty$

$\Rightarrow$ $x = -3$ je V.A., $\lim\limits_{x \to \pm\infty} f(x) = +\infty$ nema H.A.

$y = kx + n$, $k = \lim\limits_{x \to \infty} \frac{f(x)}{x} = 1$, $n = \lim\limits_{x \to \infty} [f(x) - k \cdot x] = -3$, $y = x - 3$ je K.A.



$y' = \frac{(x+5)(x+1)}{(3+x)^2}$

prekidi f-je y + nule f-je y'

-5 -3 -1

X	$(-\infty, -5)$	$(-5, -3)$	$(-3, -1)$	$(-1, +\infty)$
y'	$+$	$-$	$-$	$+$
Y	↗	↘	↘	↗

max min rast i opadanje

$(-5, -10)$ tačka maksimuma

$(-1, -2)$ tačka minimuma

$y'' = \frac{8}{(3+x)^3}$

X	$(-\infty, -3)$	$(-3, +\infty)$
y''	$-$	$+$
Y	$\frown$	$\smile$

konveksnost i konkavnost

f-ja nema prevojnih tački

2. Izračunati $\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$.

Rj.

$$\int \sqrt{4-x^2}\,dx = \int \frac{4-x^2}{\sqrt{4-x^2}}\,dx = (ax-b)\sqrt{4-x^2} + \lambda \int \frac{dx}{\sqrt{4-x^2}} \quad \Big/ \frac{d}{dx}$$

$$\frac{4-x^2}{\sqrt{4-x^2}} = a\sqrt{4-x^2} + (ax-b)\frac{-2x}{2\sqrt{4-x^2}} + \lambda \cdot \frac{1}{\sqrt{4-x^2}} \quad \Big/ \cdot \sqrt{4-x^2}$$

$$4-x^2 = a(4-x^2) + (-ax^2+bx) + \lambda$$

x^2: $-a-a=-1$, $-2a=-1$, $a=\frac{1}{2}$

x: $b=0$

x^0: $4a+\lambda=4$, $2+\lambda=4$, $\lambda=2$

$$\int \sqrt{4-x^2}\,dx = \frac{1}{2}x\sqrt{4-x^2} + 2\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C$$

$$\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx = \left(\frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\frac{x}{2}\right)\Big|_0^{\sqrt{3}} = \left(\frac{1}{2}\cdot\sqrt{3}\cdot\sqrt{4-3} + 2\arcsin\frac{\sqrt{3}}{2}\right)$$

$$-(0 + 2\arcsin 0) = \frac{\sqrt{3}}{2} + 2\cdot\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{2\pi}{3}$$

3. Odrediti ekstreme f-je $z(x,y) = \frac{2}{3}x^3 - 5xy + \frac{5}{2}y^2 + 8x - 5y$.

Rj.

$$\frac{\partial z}{\partial x} = 2x^2 - 5y + 8$$

$$\frac{\partial z}{\partial y} = -5x + 5y - 5$$

$$2x^2 - 5y + 8 = 0$$
$$-5x + 5y - 5 = 0 \quad /:5$$

$$2x^2 - 5y + 8 = 0$$
$$-x + y - 1 = 0$$

$$2x^2 - 5y + 8 = 0$$
$$y = x + 1$$

$$2x^2 - 5x - 5 + 8 = 0$$
$$2x^2 - 5x + 3 = 0$$
$$D = 25 - 24 = 1$$
$$x_{1,2} = \frac{5 \pm 1}{4}$$
$$x_1 = 1, \quad x_2 = \frac{6}{4} = \frac{3}{2}$$

za $x_1 = 1 \Rightarrow y_1 = 2$

za $x_2 = \frac{3}{2} \Rightarrow y_2 = \frac{5}{2}$

Tačke $M_1(1,2)$ i $M_2(\frac{3}{2}, \frac{5}{2})$ su stacionarne tačke.

$$\frac{\partial^2 z}{\partial x^2} = 4x \qquad M_1(1,2)$$

$A = 4 \qquad D = AC - B^2$

$B = -5 \qquad D = 20 - 25 < 0$

$$\frac{\partial^2 z}{\partial x \partial y} = -5 \qquad C = 5$$

u tački M_1 f-ja nema ekstrem

$$\frac{\partial^2 z}{\partial y^2} = 5 \qquad M_2\left(\frac{3}{2}, \frac{5}{2}\right)$$

$A = 6 \qquad D = 30 - 25 > 0$

$B = -5 \qquad$ f-ja u tački M_2 ima ekstrem

$C = 5 \qquad A > 0$ f-ja ima minimum

$$z_{min}\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{2}{3} \cdot \frac{27}{8} - 5 \cdot \frac{3}{2} \cdot \frac{5}{2} + \frac{5}{2} \cdot \frac{25}{4} + 8 \cdot \frac{3}{2} - 5 \cdot \frac{5}{2}$$

$$z_{min} = -1{,}375$$

4. Riješiti diferencijalnu jednačinu $y' + y = \cos x$.

Rj. $y' + y = \cos x$

ovo je linearna diferencijalna jednačina

uvodimo smjenu $y = uv$

$y' = u'v + u \cdot v'$

$u'v + u \cdot v' + uv = \cos x$

$u'v + \underbrace{(v' + v)}u = \cos x$

ovaj izraz izjednačavamo sa 0

$v' + v = 0$

$v' = -v$

$\frac{dv}{dx} = -v$

$\frac{dv}{v} = -dx \quad //$

$\int \frac{dv}{v} = -\int dx$

$\ln|v| = -x$

$v = e^{-x}$

$u'v = \cos x$

$u' \cdot e^{-x} = \cos x$

$u' = e^{x} \cos x$

$\frac{du}{dx} = e^{x} \cos x$

$du = e^{x} \cos x \, dx \quad //$

$\int du = \int e^{x} \cos x \, dx$

$$I = \int e^{x} \cos x \, dx = \left| \begin{matrix} u = e^{x} & dv = \cos x\,dx \\ du = e^{x}dx & v = \sin x \end{matrix} \right|$$

$= e^{x} \sin x - \int e^{x} \sin x \, dx$

$$\int e^{x} \sin x \, dx = \left| \begin{matrix} u = e^{x} & dv = \sin x\,dx \\ du = e^{x}dx & v = -\cos x \end{matrix} \right|$$

$= -e^{x} \cos x + \int \cos x \, e^{x} dx$

$I = e^{x} \sin x + e^{x} \cos x - I$

$2I = e^{x}(\cos x + \sin x)$

$$I = \frac{e^{x}(\cos x + \sin x)}{2} + C$$

$u = \frac{1}{2} e^{x} (\cos x + \sin x) + C$

$y = uv = C e^{-x} + \frac{1}{2} (\cos x + \sin x)$ rješenje diferencijalne jednačine