Univerzitet u Zenici
Fakultet za metalurgiju i materijale
Osjek za hemiju

Zadaci sa pismenog ispita iz predmeta Matematika 2, obe grupe, rađen 16.02.2009.

1. Ispitati i grafički predstaviti funkciju $y = 3xe^{\frac{-3}{x}}$.

2. Ispitati i grafički predstaviti funkciju $y = 2 + ln\dfrac{x}{x-2}$.

3. Odrediti: $\displaystyle\int \frac{2x-5}{x^2+4x+11}\,dx$.

4. Odrediti: $\displaystyle\int \frac{e^x(12-4e^x)}{4e^{2x}-8e^x-12}\,dx$.

5. Izračunati površinu površi omeđenu elipsom $x^2 + \dfrac{y^2}{16} = 1$ i poluosama $x \geq 0$, $y \geq 0$.

6. Naći ekstreme funkcije $z = (x^2-1)^2 + 2y^2$.

7. Dokazati da red $\dfrac{1}{2\cdot 7} + \dfrac{1}{7\cdot 12} + ... + \dfrac{1}{(5n-3)\cdot(5n+2)} + ...$ konvergira i naći njegovu sumu.

8. Riješiti diferencijalnu jednačinu $2y'\sqrt{x^2-1} + 3y^2 = 3$.

1. Ispitati i grafički predstaviti f-ju $y = 3x e^{-\frac{3}{x}}$.

Rj. $\mathcal{D}: x \in (-\infty, 0) \cup (0, +\infty)$

f-ja nije ni parna ni neparna

f-ja ne siječe y-osu

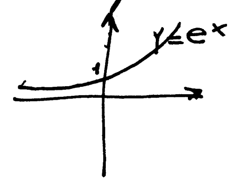
$3x e^{-\frac{3}{x}} \neq 0, \ \forall x \in \mathcal{D}$

f-ja nema nule

x	$(-\infty, 0)$	$(0, +\infty)$
y	$-$	$+$

znak f-je

$\lim\limits_{x \to 0-} f(x) = -\infty \Rightarrow x = 0$ je vertikalna asimptota



$\lim\limits_{x \to 0+} f(x) = 0$

$\lim\limits_{x \to \mp\infty} f(x) = \mp\infty \Rightarrow$ f-ja nema horizontalnu asimptotu

$y = kx + n$

$k = \lim\limits_{x \to \infty} \frac{f(x)}{x} = 3$

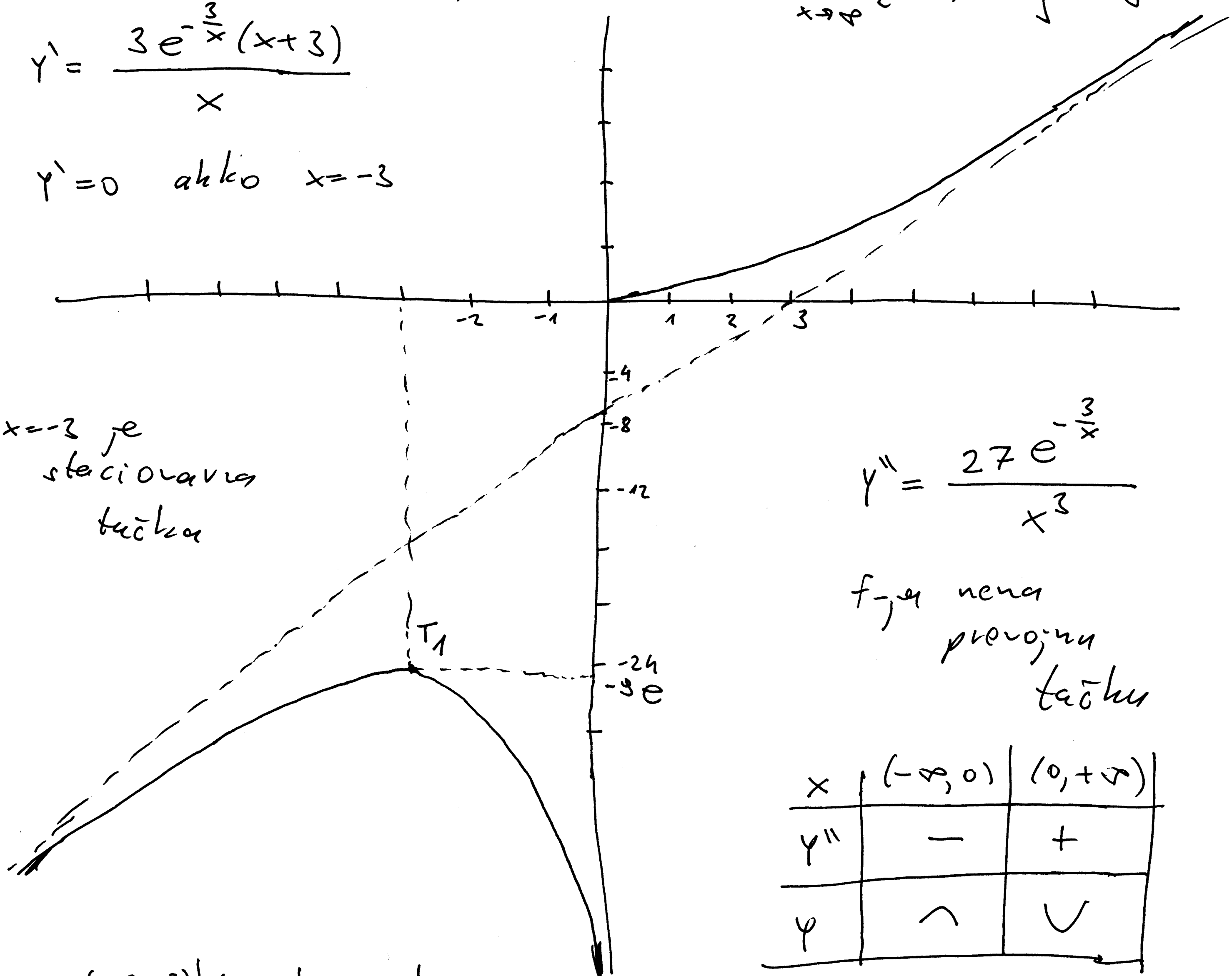
$n = \lim\limits_{x \to \infty} [f(x) - 3x] = -9$

$y = 3x - 9$ kosa asimptota

$y' = \frac{3e^{-\frac{3}{x}}(x+3)}{x}$

$y' = 0$ ako $x = -3$

$x = -3$ je stacionarna tačka



$y'' = \frac{27 e^{-\frac{3}{x}}}{x^3}$

f-ja nema prevojnu tačku

x	$(-\infty, 0)$	$(0, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

x	$(-\infty, -3)$	$(-3, 0)$	$(0, +\infty)$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$	$\nearrow$
	max		

f-ja ima maksimum u $T_1(-3, -9e)$

2. Ispitati i grafički predstaviti f-ju $y = 2 + \ln\frac{x}{x-2}$.

Rj. $\mathcal{D}: x \in (-\infty, 0) \cup (2, +\infty)$

x	$(-\infty, 0)$	$(2, +\infty)$
y	$-$	$+$

znak f-je

f-ja nije ni parna ni neparna

nije periodična

$\left(\frac{-2}{e^2-1}, 0\right)$ nula f-je

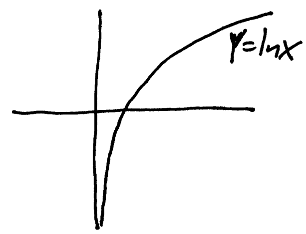
f-ja ne siječe y osu

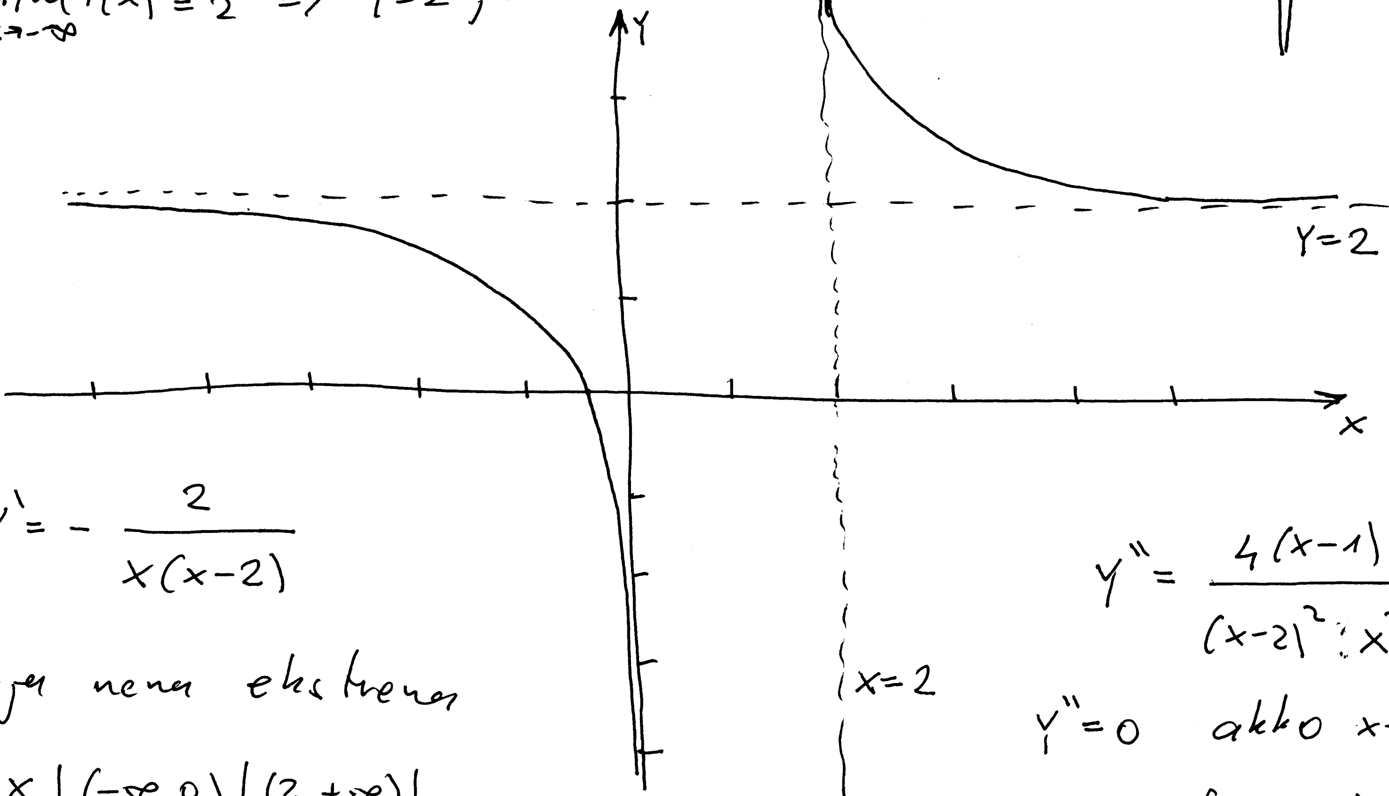
$\lim\limits_{x\to 0-} f(x) = -\infty \Rightarrow x=0$ je vertikalna asimptota

$\lim\limits_{x\to 2+0} f(x) = \infty \Rightarrow x=2$ je vertikalna asimptota

$\lim\limits_{x\to\infty} f(x) = 2 \Rightarrow y=2$ je horizontalna asimptota

$\lim\limits_{x\to-\infty} f(x) = 2 \Rightarrow y=2$ je horizontalna asimptota





$$y' = -\frac{2}{x(x-2)}$$

f-ja nema ekstrema

x	$(-\infty, 0)$	$(2, +\infty)$
y'	$-$	$-$
y	$\searrow$	$\searrow$

$$y'' = \frac{4(x-1)}{(x-2)^2 \cdot x^2}$$

$y'' = 0$ akko $x = 1$

u $x=1$ f-ja nije definisana

f-ja nema prevojnih tački

x	$(-\infty, 0)$	$(2, +\infty)$
y''	$-$	$+$
y	$\frown$	$\smile$

3. Odrediti $\int \frac{2x-5}{x^2+4x+11}\,dx$.

Rj. $(x^2+4x+11)' = 2x+4$, $\quad x^2+4x+11 = x^2+2\cdot 2x+4+7 = (x+2)^2+7$

$$\int \frac{2x-5}{x^2+4x+11}\,dx = \int \frac{2x+4-9}{x^2+4x+11}\,dx = \int \frac{2x+4}{x^2+4x+11}\,dx - 9\int \frac{dx}{x^2+4x+11}$$

$$\int \frac{dx}{x^2+4x+11} = \int \frac{dx}{(x+2)^2+7} = \left| \begin{array}{l} x+2=\sqrt{7}\,t \\ dx=\sqrt{7}\,dt \end{array} \right| = \int \frac{\sqrt{7}\,dx}{7t^2+7} = \frac{\sqrt{7}}{7}\int \frac{dx}{t^2+1}$$

$$= \frac{\sqrt{7}}{7}\,\mathrm{arctg}\,t + c = \frac{\sqrt{7}}{7}\,\mathrm{arctg}\left(\frac{x+2}{\sqrt{7}}\right) + c$$

$$\int \frac{2x-5}{x^2+4x+11}\,dx = \ln|x^2+4x+11| - \frac{9\sqrt{7}}{7}\,\mathrm{arctg}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

4. Odrediti $\int \frac{e^x(12-4e^x)}{4e^{2x}-8e^x-12}\,dx$.

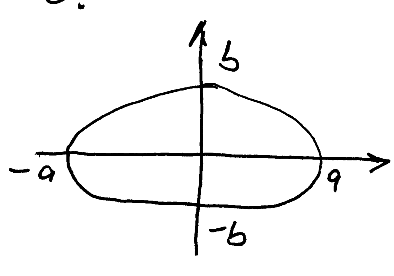
Rj.
$$\int \frac{e^x(12-4e^x)}{4e^{2x}-8e^x-12}\,dx = \left| \begin{array}{l} e^x=t \\ e^x dx=dt \\ e^{2x}=t^2 \end{array} \right| = \int \frac{12-4t}{4t^2-8t-12}\,dt \begin{array}{l} :4 \\ :4 \end{array} =$$

$$= \int \frac{3-t}{t^2-2t-3}\,dt = \int \frac{-\cancel{(t-3)}}{(t+1)\cancel{(t-3)}}\,dt = -\int \frac{dt}{t+1} = -\ln|t+1| + c = -\ln(e^x+1) + c$$

5. Izračunati površinu površi omeđenu elipsom $x^2+\frac{y^2}{16}=1$ i poluosama $x \geq 0$, $y \geq 0$.

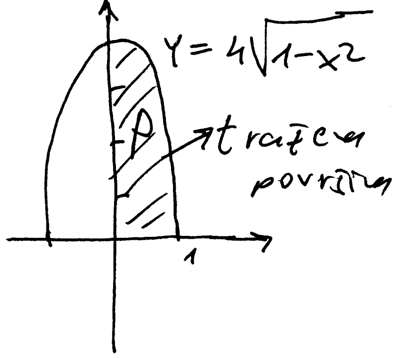
Rj.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ je jednačina elipse



$$16x^2+y^2=16 \Rightarrow y^2=16-16x^2 \Rightarrow y=\sqrt{16-16x^2}$$

$Y = 4\sqrt{1-x^2}$



$$P = \int_0^1 4\sqrt{1-x^2}\,dx$$

$$\int \sqrt{1-x^2}\,d\cancel{x} = \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx =$$

$$= (ax+b)\sqrt{1-x^2} + \lambda \int \frac{dx}{\sqrt{1-x^2}} \Rightarrow \quad a = \frac{1}{2} \quad b = 0 \quad \lambda = \frac{1}{2}$$

$$4\int \sqrt{1-x^2}\,dx = 2x\sqrt{1-x^2} + 2\arcsin x + C$$

$$P = 2\left(x\sqrt{1-x^2}\right)\Big|_0^1 + 2\left(\arcsin x\right)\Big|_0^1 = 2\cdot\frac{\pi}{2} = \pi$$ tražena površina

6. Naći ekstreme f-je $z = (x^2-1)^2 + 2y^2$.

Rj.

$$\frac{\partial z}{\partial x} = 2(x^2-1)\cdot 2x = 4x^3 - 4x$$

$$\frac{\partial z}{\partial y} = 4y$$

$4(x^2-1)\cdot x = 0 \Rightarrow x = -1$ ili $x = 0$ ili $x = 1$

$4y = 0 \Rightarrow y = 0$

$M_1(-1,0)$
$M_2(0,0)$ stacionarne tačke
$M_3(1,0)$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = 4$$

$M_1(-1,0)$

$A = 8$

$B = 0$

$C = 4$

$D = AC - B^2$

$D = 32 > 0 \Rightarrow$ f-ja ima ekstrem

$A > 0$ f-ja ima minimum

$z_{min}(-1,0) = 0$

$M_2(0,0)$

$A = -4$

$B = 0$

$C = 4$

$D = -16 < 0$ f-ja nema ekstrem u M_2

$M_3(1,0)$

$A = 8$

$B = 0$

$C = 4$

$D = 32 > 0$ f-ja ima ekstrem u M_3

$A > 0$ f-ja ima minimum

$z_{min}(1,0) = 0$

7. Dokazati da red $\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\ldots+\frac{1}{(5n-3)\cdot(5n+2)}+\ldots$ konvergira i naći njegovu sumu.

Rj. Posmatraćemo niz parcijalnih suma $S_n=\sum_{i=1}^{n}\frac{1}{(5i-3)(5i+2)}$

Ako S_n teži konačnom broju kad $n\to\infty$ tad niz konvergira.

$$\frac{1}{(5n-3)(5n+2)}=\frac{A}{5n-3}+\frac{B}{5n+2}\quad\Rightarrow\quad A=\frac{1}{5},\ B=-\frac{1}{5}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{5}\left(\frac{1}{2}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}-\cancel{\frac{1}{12}}+\ldots+\cancel{\frac{1}{5n-3}}-\frac{1}{5n+2}\right)=$$

$$=\frac{1}{5}\lim_{n\to\infty}\left(\frac{1}{2}-\underbrace{\frac{1}{5n+2}}_{\to 0}\right)=\frac{1}{10}$$

Red konvergira i njegova suma iznosi $\frac{1}{10}$.

8. Riješiti diferencijalnu jednačinu $2y'\sqrt{x^2-1}+3y^2=3$.

Rj.

$$2y'\sqrt{x^2-1}+3y^2=3$$

$$2y'\sqrt{x^2-1}=3-3y^2$$

$$2y'=\frac{1}{\sqrt{x^2-1}}\cdot 3(1-y^2)\quad /:2$$

$$y'=\frac{3}{2}\cdot\frac{1}{\sqrt{x^2-1}}\cdot(1-y^2)$$

ovo je dif. jedn. sa razdvojenim promjenjivim (oblika $y'=f(x)\cdot g(x)$)

$$\frac{dy}{dx}=\frac{3}{2}\cdot\frac{1}{\sqrt{x^2-1}}(1-y^2)$$

$$\frac{2\,dy}{1-y^2}=\frac{3\,dx}{\sqrt{x^2-1}}\quad /\int$$

$$2\int\frac{dy}{1-y^2}=3\int\frac{dx}{\sqrt{x^2-1}}$$

$$2\cdot\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|=3\ln\left|x+\sqrt{x^2-1}\right|+\ln C$$

$$\frac{1+y}{1-y}=\left(x+\sqrt{x^2-1}\right)^3\cdot C$$

rješenje diferencijalne jednačine