Matematika 2

Grupa A

1. Ispitati i grafički predstaviti funkciju $y = \dfrac{x^3}{(1+x)^2}$.

2. Odrediti $\displaystyle\int \frac{7\,dx}{x^2(x+1)^2}$.

3. Izračunati $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \sqrt{3-x^2}\,dx$.

4. Naći ekstreme funkcije $z = x^2 - 2x^2y^2 + y^2$.

Grupa B

1. Ispitati i grafički predstaviti funkciju $y = \dfrac{(1+x)^2}{x}$.

2. Izračunati površinu površi omeđenu krugom $x^2 + y^2 = 3$ i poluosama $x \geq 0$, $y \geq 0$.

3. Dokazati da red $\dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{15} + ... + \dfrac{1}{n^2+2n} + ...$ konvergira i naći njegovu sumu.

4. Riješiti diferencijalnu jednačinu $xy' = -(x+y)$.

1. Ispitati i grafički predstaviti f-ju $y=\frac{x^3}{(1+x)^2}$.

Rj. $\mathcal{D}: x\in(-\infty,-1)\cup(-1,+\infty)$

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
Y	−	−	+

znak f-je

f-ja nije ni parna ni neparna

nije periodična

(0,0) je nula f-je i presjek sa Y-osom

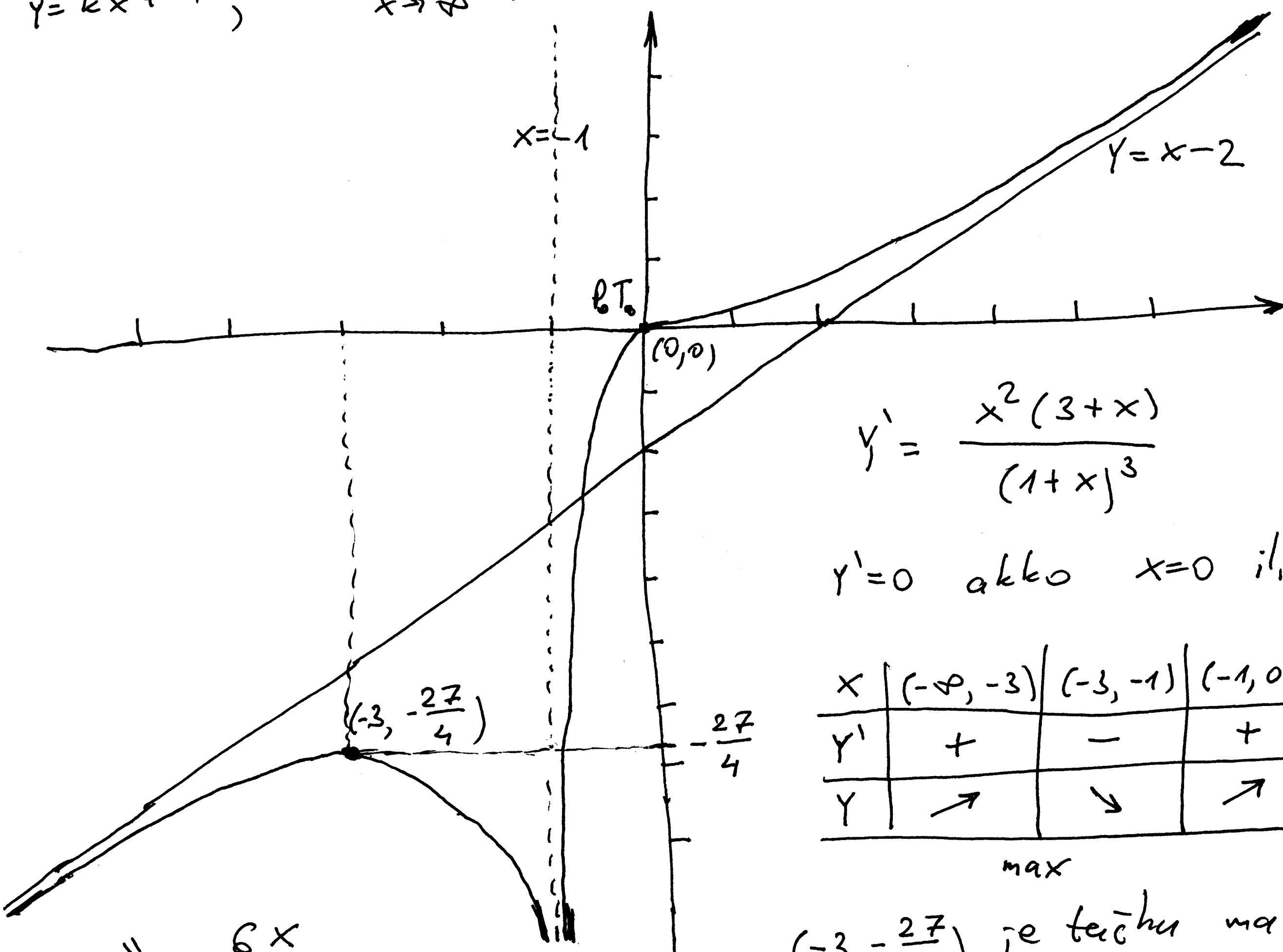
$\lim_{x\to -1-0} f(x) = -\infty \Rightarrow x=-1$ je vertikalna asimptota

$\lim_{x\to -1+0} f(x) = -\infty \Rightarrow x=-1$ je vertikalna asimptota

$\lim_{x\to \pm\infty} f(x) = \pm\infty \Rightarrow$ nema horizontalne asimptote

$y=kx+n$, $k=\lim_{x\to\infty}\frac{f(x)}{x}=1$, $n=\lim_{x\to\infty}(f(x)-kx)=-2$

$Y=x-2$ je kosa asimptota



$$y'=\frac{x^2(3+x)}{(1+x)^3}$$

$y'=0$ akko $x=0$ ili $x=-3$

x	$(-\infty,-3)$	$(-3,-1)$	$(-1,0)$	$(0,+\infty)$
Y'	+	−	+	+
Y	↗	↘	↗	↗

max

$(-3,-\frac{27}{4})$ je tačka maksimuma

$$y''=\frac{6x}{(1+x)^4}$$

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
Y''	−	−	+
Y	⌒	⌒	⌣

P.T.

(0,0) je prevojna tačka

(2.)[A] Odrediti $\int \frac{7\,dx}{x^2(x+1)^2}$.

Rj:

$$\frac{7}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \quad / \cdot x^2(x+1)^2$$

$$7 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$

$$7 = A(x^3+2x^2+x) + B(x^2+2x+1) + C(x^3+x^2) + Dx2$$

$$\begin{array}{l} A + C = 0 \\ 2A + B + C + D = 0 \\ A + 2B = 0 \\ B = 7 \end{array} \Rightarrow \begin{array}{ll} A = -14 & C = 14 \\ B = 7 & D = 7 \end{array}$$

$$\int \frac{7\,dx}{x^2(x+1)^2} = -14\int \frac{dx}{x} + 7\int \frac{dx}{x^2} + 14\int \frac{dx}{(x+1)} + 7\int \frac{dx}{(x+1)^2} =$$

$$= -14\ln|x| - \frac{7}{x} + 14\ln|x+1| - \frac{7}{x+1} + c$$

(3.)[A] Izračunati $\int_0^{\frac{\sqrt{3}}{2}} \sqrt{3-x^2}\,dx$.

Rj:

$$\int \sqrt{3-x^2}\,dx \cdot \frac{\sqrt{3-x^2}}{\sqrt{3-x^2}} = \int \frac{3-x^2}{\sqrt{3-x^2}}\,dx = (ax+b)\sqrt{3-x^2} + \lambda\int \frac{dx}{\sqrt{3-x^2}} \quad / \frac{d}{dx}$$

$$\frac{3-x^2}{\sqrt{3-x^2}} = a\sqrt{3-x^2} + (ax+b)\frac{-2x}{2\sqrt{3-x^2}} + \frac{\lambda}{\sqrt{3-x^2}} \quad / \cdot \sqrt{3-x^2}$$

$$3-x^2 = a(-x^2+3) + (-ax^2 - bx) + \lambda$$

$$\begin{array}{l} -2a = 1 \\ -b = 0 \\ 3a + \lambda = 3 \end{array} \Rightarrow a = \frac{1}{2},\ b = 0,\ \lambda = \frac{3}{2}$$

$$\int \sqrt{3-x^2}\,dx = \frac{1}{2}x\sqrt{3-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{3-x^2}}$$

$$\int \frac{dx}{\sqrt{3-x^2}}\,dx = \left| \begin{array}{l} x = \sqrt{3}\,t \\ dx = \sqrt{3}\,dt \end{array} \right| = \int \frac{\sqrt{3}\,dt}{\sqrt{3-3t^2}} = \frac{\sqrt{3}}{\sqrt{3}}\int \frac{dt}{1-t^2} = \arcsin\frac{x}{\sqrt{3}} + c$$

$$\int_0^{\frac{\sqrt{3}}{2}} \sqrt{3-x^2}\,dx = \left(\frac{1}{2}x\sqrt{3-x^2} + \frac{3}{2}\arcsin\frac{x}{\sqrt{3}}\right)\Bigg|_0^{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{4}\cdot\sqrt{\frac{9}{4}} + \frac{3}{2}\cdot\frac{\pi}{6} = \frac{3\sqrt{3}}{8} + \frac{\pi}{4}.$$

traženo rješenje

4.[A] Naći ekstreme f-je $z = x^2 - 2x^2y^2 + y^2$.

Rj.

$$\frac{\partial z}{\partial x} = 2x - 4xy^2$$

$$\frac{\partial z}{\partial y} = -4x^2y + 2y$$

$$2x - 4xy^2 = 0$$
$$-4x^2y + 2y = 0$$

$$2x(1-2y^2) = 0$$
$$-2y(2x^2-1) = 0$$

$x = 0 \lor 2y^2 = 1$

$y = 0 \lor 2x^2 = 1$

stacionarne tačke su

$M_1(0,0)$

$M_2(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$M_3(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$M_4(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$M_5(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\frac{\partial^2 z}{\partial x^2} = 2 - 4y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = -8xy$$

$$\frac{\partial^2 z}{\partial y^2} = -4x^2 + 2$$

$M_1(0,0)$

$A = 2$ $\quad D = AC - B^2$

$B = 0$ $\quad D = 4 > 0$ f-ja ima ekstrem

$C = 2$ $\quad A > 0$ f-ja ima minimum

$z_{min}(0,0) = 0$

$M_2(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$A = 0$ $\quad D = -16 < 0$

$B = -4$ $\quad$ u tački M_2

$C = 0$ $\quad$ f-ja nema ekstrem

$M_3(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$A = 0$ $\quad D = -16 < 0$

$B = 4$ $\quad$ u tački M_3

$C = 0$ $\quad$ f-ja nema ekstrem

$M_4(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$A = 0$ $\quad D = -16 < 0$

$B = 4$ $\quad$ u tački M_4

$C = 0$ $\quad$ f-ja nema ekstrem

$M_5(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$A = 0$ $\quad D = -16 < 0$

$B = 4$ $\quad$ u tački M_5

$C = 0$ $\quad$ f-ja nema ekstrem

1.[B] Ispitati i grafički predstaviti f-ju $y=\frac{(1+x)^2}{x}$.

Rj: $D: x \in (-\infty, 0) \cup (0, +\infty)$

f-ja nije ni parna ni neparna

f-ja nije periodična

x	$(-\infty, 0)$	$(0, +\infty)$
y	$-$	$+$

znak f-je

$f(0)$ nije definisano

– f-ja ne siječe y-osu

$x=-1 \Rightarrow y=0$

$(-1,0)$ je nula f-je

$\lim_{x\to 0^-} f(x) = -\infty \Rightarrow x=0$ je vertikalna asimptota

$\lim_{x\to 0^+} f(x) = +\infty \Rightarrow x=0$ je vertikal. asimptota

$\lim_{x\to\pm\infty} f(x) = \pm\infty$ f-ja nema horizontalnu asimptotu

$y = kx + n$

$k = \lim_{x\to\infty} \frac{f(x)}{x} = 1$

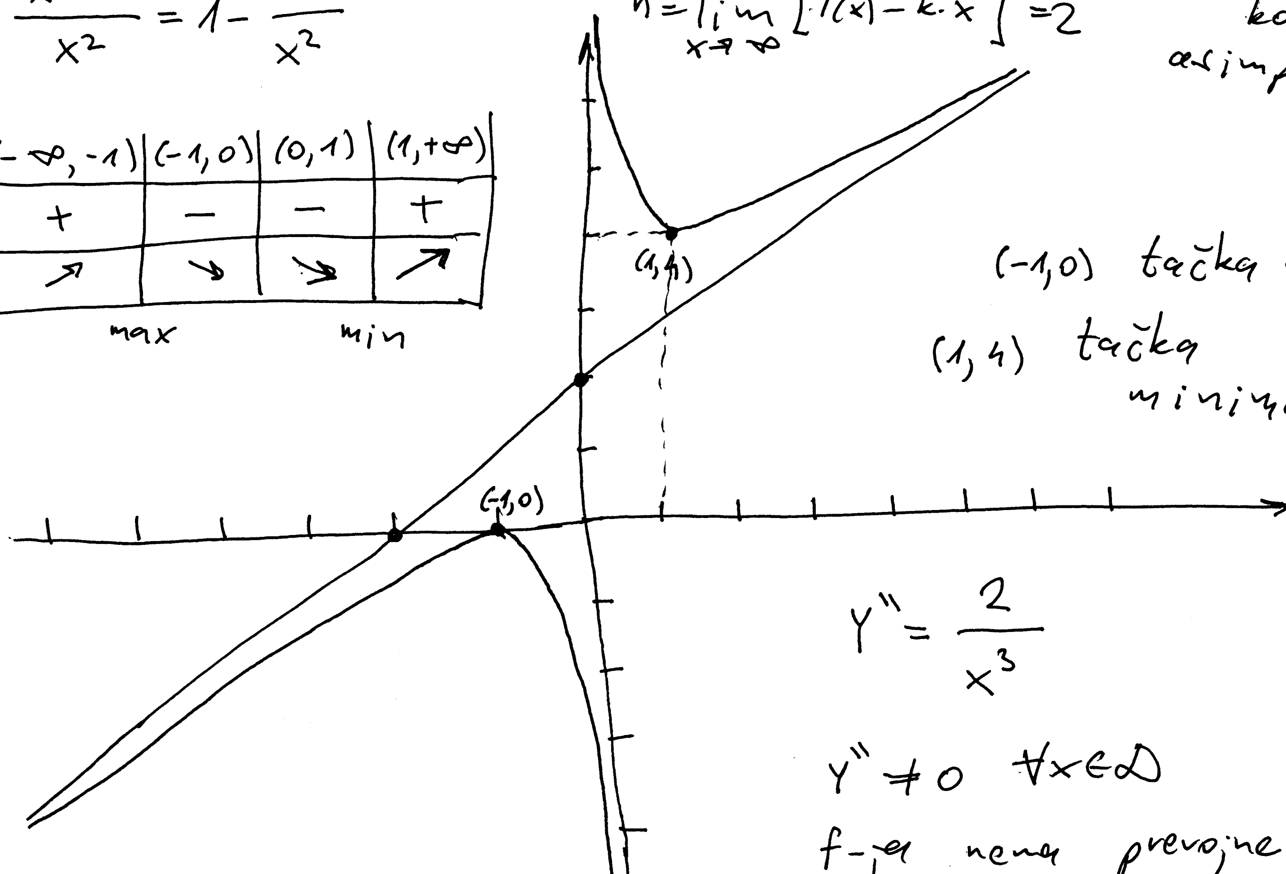
$n = \lim_{x\to\infty} [f(x) - k\cdot x] = 2$

$y = x+2$ je kosa asimptota

$$y' = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$$

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,+\infty)$
y'	$+$	$-$	$-$	$+$
y	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

max min



$(-1,0)$ tačka max

$(1,4)$ tačka minimuma

$$y'' = \frac{2}{x^3}$$

$y'' \neq 0 \ \forall x \in D$

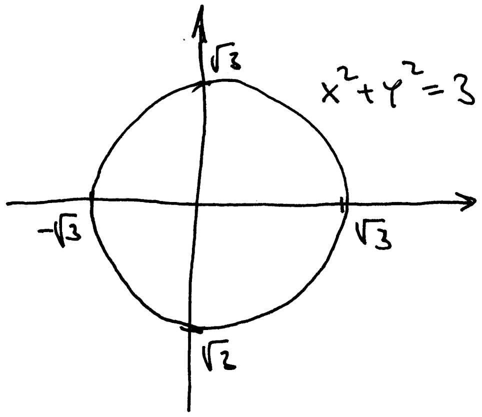
f-ja nema prevojne tačke

y	$(-\infty, 0)$	$(0, +\infty)$
y''	$-$	$+$
y	$\cap$	$\cup$

(2.)[B] Izračunati površinu površi omeđenu krugom $x^2+y^2=3$ i poluosama $x \geq 0$, $y \geq 0$.

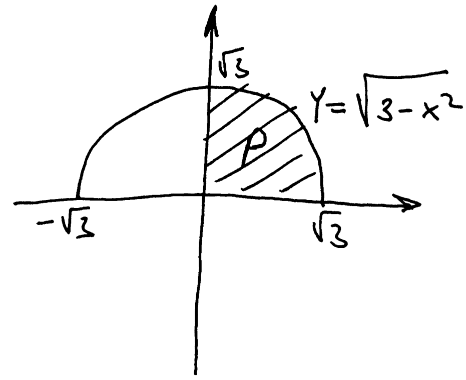
Rj. $(x-p)^2+(y-q)^2=r^2$

je jednačina kruga poluprečnika r sa centrom u tački (p,q)



$x^2+y^2=3$ ima centar u $(0,0)$ poluprečnik $\sqrt{3}$

$$P=\int_0^{\sqrt{3}} \sqrt{3-x^2}$$



$$\int \sqrt{3-x^2} = \frac{1}{2}x\sqrt{3-x^2}+\frac{3}{2}\arcsin\frac{x}{\sqrt{3}}+C$$

ovaj rezultat smo dobili u 3. zadatku grupe A.

$$\int_0^{\sqrt{3}} \sqrt{3-x^2} = \left(\frac{1}{2}x\sqrt{3-x^2}+\frac{3}{2}\arcsin\frac{x}{\sqrt{3}}\right)\Big|_0^{\sqrt{3}} = \frac{3}{2}\cdot\arcsin 1 = \frac{3}{4}\pi$$

tražena površina

(3.)[B] Dokazati da red $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots+\frac{1}{n^2+2n}+\ldots$ konvergira i naći njegovu sumu.

Rj.: $$\frac{1}{n^2+2n}=\frac{1}{n(n+2)}=\frac{A}{n}+\frac{B}{n+2} \quad \Big/\cdot(n+2)\cdot n$$

$$1=A(n+2)+Bn$$

$$\begin{aligned} A+B&=0\\ 2A&=1\end{aligned} \Rightarrow \begin{aligned} A&=\frac{1}{2}\\ B&=-\frac{1}{2}\end{aligned}$$

$$\frac{1}{n^2+2n}=\frac{\frac{1}{2}}{n}+\frac{-\frac{1}{2}}{n+2}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\ldots+\frac{1}{n^2+2n}=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\ldots+\frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n+2}\right)=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}\right)$$

Imamo: $\sum_{k=1}^{n} \frac{1}{k^2+2k} = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right)$

$$S_n = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right)$$

$S_n \to \frac{3}{4}$ kad $n \to \infty$

Vidimo da red konvergira i njegova suma iznosi $\frac{3}{4}$.

4.[B] Riješiti diferencijalnu jednačinu $xy' = -(x+y)$.

Rj. $y' = -\frac{x+y}{x}$

$y' = -1 + \frac{y}{x}$ ovo je homogena diferencijalna jednačina $y' = f\left(\frac{y}{x}\right)$

smjena: $u = \frac{y}{x}$, $y = ux$, $y' = u'x + u$

$u'x + u = -1 - u$

$\frac{du}{dx} x = -1 - 2u$

$\frac{du}{1+2u} = -\frac{dx}{x}$ $\quad / \int$

$\frac{1}{2}\ln(1+2u) = \ln\frac{c}{x}$

$1 + 2u = \left(\frac{c}{x}\right)^2$

$2u = \frac{c}{x^2} - 1$

$2\frac{y}{x} = \frac{c}{x^2} - 1$

$2y = \frac{c}{x^2} - x$

$y = \frac{c}{x^2} - \frac{x}{2}$

rješenje diferencijalne jednačine