

1. Nadi sve vrijednosti $\sqrt[3]{z}$ ako je $z = (\sqrt{3}-i)^2 (1+i\sqrt{3})^2$.

Rj. $z = z_1^2 \cdot z_2^2$ $\operatorname{tg} 30^\circ = \frac{\sqrt{3}}{3}$ $\varphi_1 = -\frac{\pi}{6}$ $z_1 = \sqrt{3}-i = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right)$

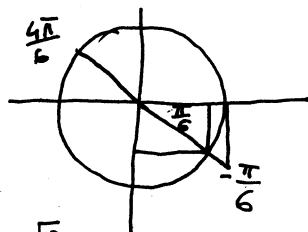
$z_1 = \sqrt{3}-i$

$|z_1| = 2$

$\sin \varphi_1 = \frac{-1}{2}$

$\cos \varphi_1 = \frac{\sqrt{3}}{2}$

$\operatorname{tg} \varphi_1 = -\frac{\sqrt{3}}{3}$



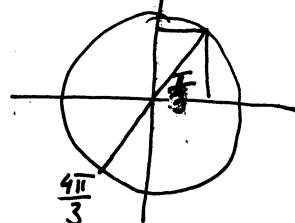
$z_2 = 1+i\sqrt{3}$

$|z_2| = 2$

$\cos \varphi_2 = \frac{1}{2}$

$\sin \varphi_2 = \frac{\sqrt{3}}{2}$

$\operatorname{tg} \varphi_2 = \sqrt{3}$



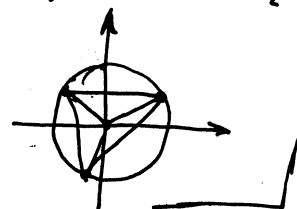
$z_2 = 1+i\sqrt{3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

$\varphi_2 = \frac{\pi}{3}$

$z_1^2 = 2^2 \left[\cos\left(-\frac{2\pi}{6}\right) + i \sin\left(-\frac{2\pi}{6}\right) \right]$

$z_2^2 = 2^2 \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right]$

$\frac{\pi}{3} = 20^\circ$



$z = z_1^2 \cdot z_2^2 = 4 \cdot 4 \left[\cos\left(\frac{2\pi}{3} - \frac{2\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} - \frac{2\pi}{6}\right) \right] = 16 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

$\sqrt[3]{z}$ ima vrijednosti $z_k = \sqrt[3]{|z|} \left(\cos \frac{\varphi + 2k\pi}{3} + i \sin \frac{\varphi + 2k\pi}{3} \right), k=0,1,2$

$z_0 = \sqrt[3]{16} \left(\cos \frac{\frac{\pi}{3}}{3} + i \sin \frac{\frac{\pi}{3}}{3} \right) = \sqrt[3]{16} \left(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9} \right) = 2\sqrt[3]{2} \left(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9} \right)$

$z_1 = \sqrt[3]{16} \left(\cos \frac{\frac{\pi}{3} + 2\pi}{3} + i \sin \frac{\frac{\pi}{3} + 2\pi}{3} \right) = 2\sqrt[3]{2} \left(\cos \frac{7\pi}{9} + i \sin \frac{7\pi}{9} \right)$

$z_2 = \sqrt[3]{16} \left(\cos \frac{\frac{\pi}{3} + 4\pi}{3} + i \sin \frac{\frac{\pi}{3} + 4\pi}{3} \right) = 2\sqrt[3]{2} \left(\cos \frac{13\pi}{9} + i \sin \frac{13\pi}{9} \right)$

2. Diskutovati ječicu sistema za razne vrijednosti parametra λ

$\lambda x + 2y + 4z = 5$

$3x + y + (\lambda + 2)z = 4$

$x + y + z = 2$

Rj.

$$D = \begin{vmatrix} \lambda & 2 & 4 \\ 3 & 1 & \lambda+2 \\ 1 & 1 & 1 \end{vmatrix} \begin{matrix} I_k - II_k \\ III_k - II_k \end{matrix} \begin{vmatrix} \lambda-2 & 2 & 2 \\ 2 & 1 & \lambda+1 \\ 0 & 1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda+1 \end{vmatrix} = (-1) \cdot [(\lambda-2)(\lambda+1) - 4] = (-1) \cdot (\lambda^2 - \lambda - 2) = (-1) \cdot (\lambda+2)(\lambda-3)$$

$$D_x = \begin{vmatrix} 5 & 2 & 4 \\ 4 & 1 & \lambda+2 \\ 2 & 1 & 1 \end{vmatrix} = (-1)(\lambda-3)$$

$$D_y = \begin{vmatrix} \lambda & 5 & 4 \\ 3 & 4 & \lambda+2 \\ 1 & 2 & 1 \end{vmatrix} = -(2\lambda+1)(\lambda-3)$$

$$D_z = \begin{vmatrix} \lambda & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = (-2)(\lambda-3)$$

Diskusija:

$$1^\circ D \neq 0 \Rightarrow \lambda \neq 3 \quad ; \quad \lambda \neq -2$$

Sistem ima jedinstveno rješenje $x = \frac{D_x}{D} = \frac{(-1)(\lambda-3)}{(-1)(\lambda+2)(\lambda-3)} = \frac{1}{\lambda+2}$

$$y = \frac{D_y}{D} = \frac{(-1)(\lambda-3)(2\lambda+1)}{(-1)(\lambda+2)(\lambda-3)} = \frac{2\lambda+1}{\lambda+2}$$

$$z = \frac{D_z}{D} = \frac{(-2)(\lambda-3)}{(-1)(\lambda-3)(\lambda+2)} = \frac{1}{\lambda+2} \quad \text{rj.} \quad \left(\frac{1}{\lambda+2}, \frac{2\lambda+1}{\lambda+2}, \frac{1}{\lambda+2} \right), \quad \lambda \in \mathbb{R}$$

$$2^\circ \lambda = 3 \Rightarrow D = D_x = D_y = D_z = 0$$

sistem postaje

$$\begin{aligned} 3x + 2y + 4z &= 5 \\ 3x + y + 5z &= 4 \\ x + y + z &= 2 \quad | \cdot 3 \end{aligned}$$

$$\begin{aligned} 3x + 2y + 4z &= 5 \\ 3x + y + 5z &= 4 \\ 3x + 3y + 3z &= 6 \\ \hline -y + z &= -1 \\ -2y + 2z &= -2 \\ \hline y &= z + 1 \end{aligned}$$

$$x = 2 - z - y$$

$$x = 2 - z - z - 1$$

$$x = -2z + 1$$

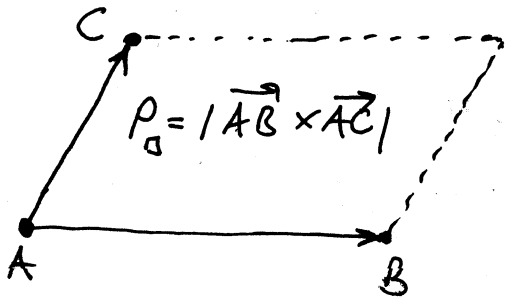
sistem ima ∞ mnogo rješenja oblika $(-2t+1, t+1, t)$

$$3^\circ \lambda = -2 \Rightarrow D = 0, D_x \neq 0 \quad \text{sistem nema rješenja}$$

3) Odrediti parameter λ tako da ΔABC ima površinu 14 ako su $A(1, 2, -1)$, $B(4, 3, -1)$, i $C(-3\lambda+4, 2, 2\lambda-3)$. Pored toga odrediti i visinu trougla koja odgovara stranici AB .

Rj. $A(1, 2, -1)$ } $\vec{AB} = (3, 1, 0)$ $A(1, 2, -1)$ } $\vec{AC} = (-3\lambda+3, 0, 2\lambda-2)$
 $B(4, 3, -1)$ }

$$\rho_{\Delta ABC} = \frac{\rho_{\vec{d}}}{2} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$



$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ -3\lambda+3 & 0 & 2\lambda-2 \end{vmatrix} = (2\lambda-2)\vec{i} - 6(\lambda-1)\vec{j} + (-(-2\lambda+3))\vec{k} =$$

$$= (2(\lambda-1), -6(\lambda-1), 3(\lambda-1))$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{4(\lambda-1)^2 + 36(\lambda-1)^2 + 9(\lambda-1)^2}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{49(\lambda-1)^2}$$

$$|\vec{AB} \times \vec{AC}| = 7|\lambda-1|$$

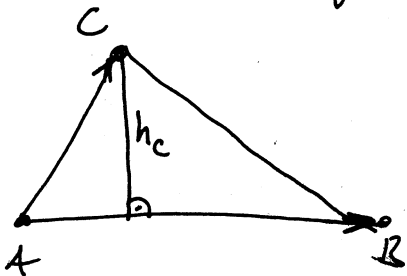
$$\rho_{\Delta ABC} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$

$$|\lambda-1| = 4$$

$$\lambda_1 = -3 \quad \lambda_2 = 5$$

$$14 = \frac{7|\lambda-1|}{2}$$

Za vrijednost parametra $\lambda = -3$ ili $\lambda = 5$ površina ΔABC je 14.



$$\rho_{\Delta ABC} = \frac{|\vec{AB}| \cdot h_c}{2}$$

$$14 = \frac{\sqrt{10} \cdot h_c}{2}$$

$$h_c = \frac{28\sqrt{10}}{10}$$

$$|\vec{AB}| = \sqrt{9+1} = \sqrt{10}$$

$$\sqrt{10} h_c = 28$$

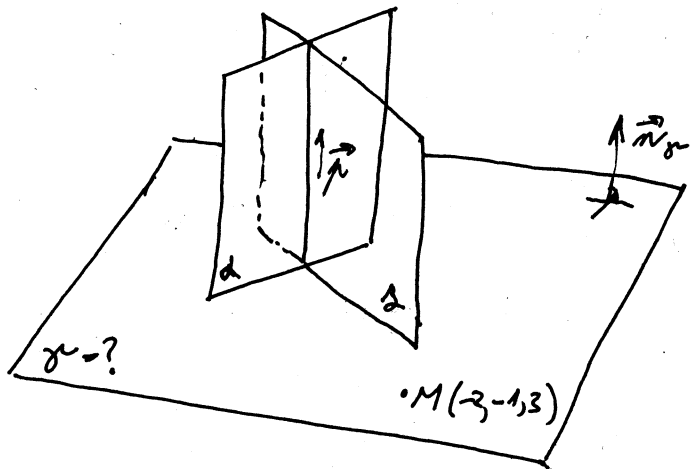
Visina trougla koja odgovara stranici AB iznosi $\frac{28\sqrt{10}}{10}$.

40) Napisati jednačinu ravni koja prolazi kroz tačku $M(-2, -1, 3)$ i normalna je na ravninama

$$A: -3x + 2y - 2z + 4 = 0$$

$$B: x - 2y + z - 5 = 0$$

Rj.



$$\vec{n}_A = (-3, 2, -2)$$

$$\vec{n}_B = (1, -2, 1)$$

$$\left. \begin{array}{l} \vec{n}_A \perp \vec{n} \\ \vec{n}_B \perp \vec{n} \end{array} \right\} \Rightarrow \vec{n} \parallel \vec{n}_A \times \vec{n}_B$$

$$\Downarrow$$

$$\vec{n} = k \cdot (\vec{n}_A \times \vec{n}_B)$$

$$\vec{n}_A \times \vec{n}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & -2 \\ 1 & -2 & 1 \end{vmatrix} = (2-4)\vec{i} - (-3+2)\vec{j} + (6-2)\vec{k} = (-2, 1, 4)$$

Za vektor normale tražene ravni; mogu uzeti $\vec{n}_p = (-2, 1, 4)$

$$\vec{n}_p = (A, B, C)$$

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$$

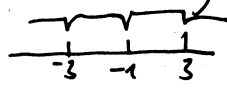
$$-2(x - (-2)) + 1(y - (-1)) + 4(z - 3) = 0$$

$$-2(x+2) + (y+1) + 4(z-3) = 0$$

$$-2x + y + 4z - 4 + 1 - 12 = 0$$

$$-2x + y + 4z - 15 = 0 \quad \text{jednačina tražene ravni;}$$

1) Ispitati i grafički predstaviti f-ju $y = \frac{x^2 - 9}{1+x}$



x	$(-\infty, -3)$	$(-3, -1)$	$(-1, 3)$	$(3, +\infty)$
$(x-3)$	-	-	-	+
$x+3$	-	+	+	+
$1+x$	-	-	+	+
Y	-	+	-	+

zak f-je

D: $x \in (-\infty, -1) \cup (-1, +\infty)$

nije ni parna ni neparna
nije periodična

$(0, -9)$ presjek sa y-osom

$(-3, 0)$ i $(3, 0)$ nule f-je

-1 je tačka preloma f-je

$\lim_{x \rightarrow -1-0} f(x) = +\infty$

$\lim_{x \rightarrow -1+0} f(x) = -\infty$

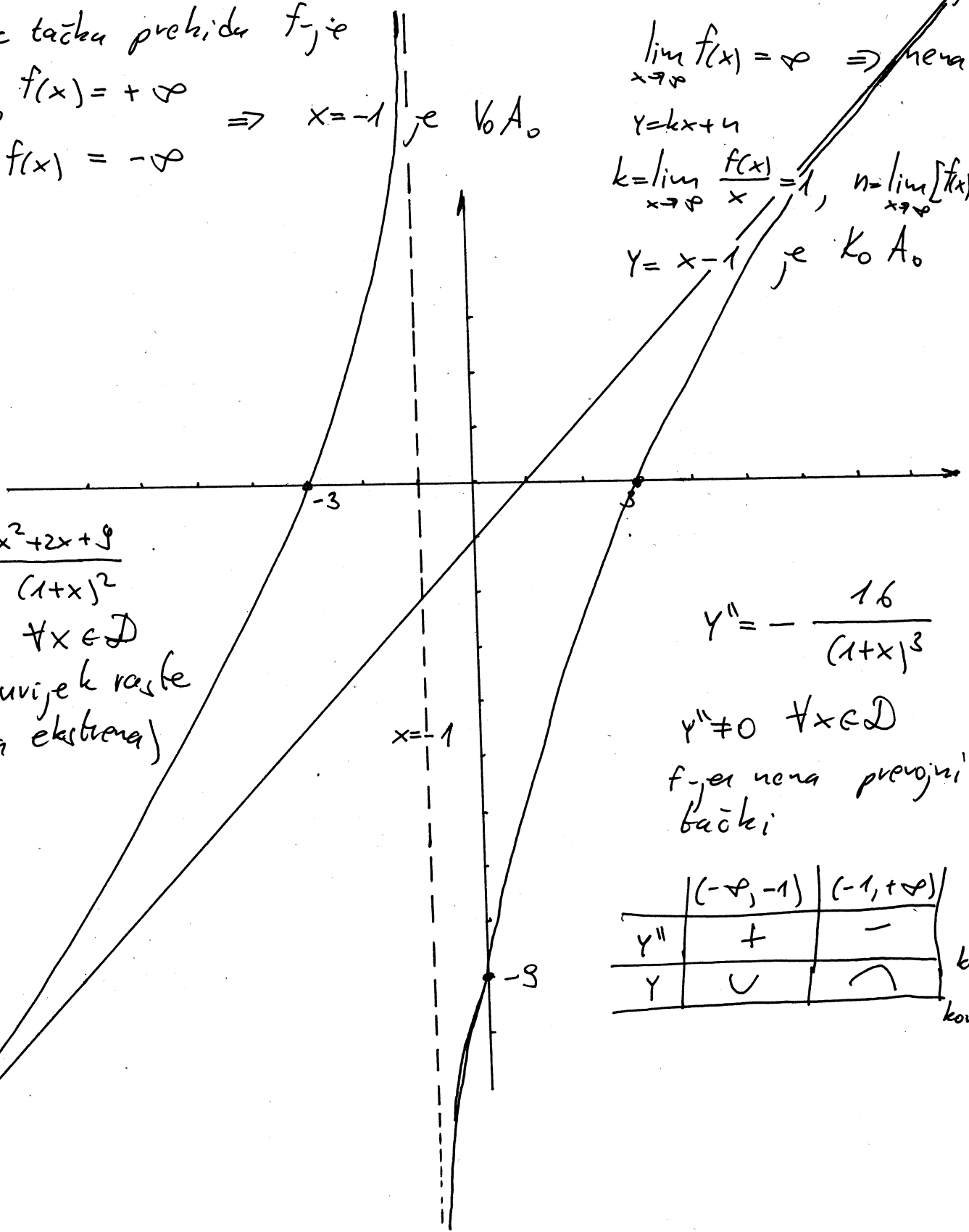
$\Rightarrow x = -1$ je $V_0 A_0$

$\lim_{x \rightarrow \infty} f(x) = \infty \Rightarrow$ nema $H_0 A_0$

$y = kx + n$

$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 1, n = \lim_{x \rightarrow \infty} [f(x) - kx] = -1$

$y = x - 1$ je $K_0 A_0$



$y' = \frac{x^2 + 2x + 9}{(1+x)^2}$

$y' > 0 \forall x \in D$

f-ja uvijek raste
(nema ekstrema)

$x = -1$

$y'' = -\frac{16}{(1+x)^3}$

$y'' \neq 0 \forall x \in D$

f-ja nema prevojnih
tački

	$(-\infty, -1)$	$(-1, +\infty)$
y''	+	-
Y	∪	∩

konveksna
i
konkavna

2) Odrediti: $\int \frac{2x+3}{\sqrt{x^2-x+1}} dx$

R; I način: Metoda Ostrogradskog

II način: $\int \frac{2x+3}{\sqrt{x^2-x+1}} dx = \int \frac{2x-1+4}{\sqrt{x^2-x+1}} dx = \int \frac{2x-1}{\sqrt{x^2-x+1}} dx + 4 \int \frac{dx}{\sqrt{x^2-x+1}}$

$$\int \frac{2x-1}{\sqrt{x^2-x+1}} dx = \left| \begin{array}{l} x^2-x+1=t^2 \\ (2x-1)dx=2t dt \\ t=\sqrt{x^2-x+1} \end{array} \right| = \int \frac{2t dt}{\sqrt{t^2}} = 2t + C_1 = 2\sqrt{x^2-x+1} + C_1$$

$$\int \frac{dx}{\sqrt{x^2-x+1}} = \left| x^2-x+1 = x^2 - 2 \cdot \frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \right| =$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}} = \left| \begin{array}{l} x - \frac{1}{2} = \frac{\sqrt{3}}{2} t \\ dx = \frac{\sqrt{3}}{2} dt \\ t = \frac{2x-1}{\sqrt{3}} \end{array} \right| = \int \frac{\frac{\sqrt{3}}{2} dt}{\sqrt{\frac{3}{4}t^2 - \frac{3}{4}}} =$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2-1}} = \ln|t + \sqrt{t^2-1}| + C_2$$

$$= \ln \left| \frac{2x-1}{\sqrt{3}} + \sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2 - 1} \right| + C_2$$

$$\int \frac{2x+3}{\sqrt{x^2-x+1}} dx = 2\sqrt{x^2-x+1} + \ln \left| \frac{\sqrt{3}}{3} (2x-1) + \sqrt{\frac{1}{3}(2x-1)^2 - 1} \right| + C$$

3. Odrediti područje konvergencije reda $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

Rj. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (x-1)^n$ ovo je stepeni red $(\sum_{n=1}^{\infty} a_n (x-a)^n)$,
ovaj red za $|x-a| < R$ KV

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}}} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \left| \sqrt{1 + \frac{1}{n}} \right| = 1$$

$$R = 1$$

$$|x-1| < 1$$

$$-1 < x-1 < 1$$

$$x \in (0, 2)$$

Ispitajmo još konvergenciju reda u graničnim tačkama.

Za $x=0$ imamo red $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Ovo je naizmjenični red.

Teorema: Neka je dat naizmjenični red $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$.
Ako je $\{a_n\}$ opadajući niz koji konvergira nuli (0) tad naizmjenični red KV.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$: Kako je $\left\{ \frac{1}{\sqrt{n}} \right\}$ opadajući niz koji KV 0
 $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ KV.

Za $x=2$ red izgleda $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Ovo je red sa pozitivnim članovima. Možemo primjeniti kriterij poredenja
 $n > \sqrt{n} \Rightarrow \frac{1}{n} < \frac{1}{\sqrt{n}}$, kako $\sum_{n=1}^{\infty} \frac{1}{n}$ DV to i $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DV.

Područje KV reda je $[0, 2)$.

4) Riješiti diferencijalnu jednačinu $Y' + \frac{2}{x} Y = \frac{2\sqrt{Y}}{\cos^2 x}$.

Rj: $Y' + \frac{2}{x} Y = \frac{2}{\cos^2 x} \sqrt{Y}$ ovo je Bernulijeva difer. jedn. uvodimo smjenu $Y = u \cdot v$, $Y' = u'v + u \cdot v'$

$$u' \cdot v + u \cdot v' + \frac{2}{x} uv = \frac{2}{\cos^2 x} \sqrt{Y}$$

$$u' \cdot v = \frac{2}{\cos^2 x} \sqrt{u \cdot v}$$

$$u' \cdot v + u \left(v' + \frac{2}{x} v \right) = \frac{2}{\cos^2 x} \sqrt{Y}$$

$\underbrace{\hspace{10em}}_{=0}$

$$u' \cdot \frac{1}{x^2} = \frac{2\sqrt{u \cdot \frac{1}{x^2}}}{\cos^2 x}$$

$$v' + \frac{2}{x} v = 0$$

$$\int \frac{dv}{v} = -2 \int \frac{dx}{x}$$

$$u' \cdot \frac{1}{x^2} = \frac{2}{\cos^2 x} \sqrt{\frac{u}{x^2}}$$

$$v' = -\frac{2}{x} v$$

$$\ln|v| = -2 \ln|x|$$

$$\frac{dv}{v} = -\frac{2}{x} v$$

$$v = x^{-2}$$

$$u' \cdot \frac{1}{x^2} = \frac{2\sqrt{u}}{x \cos^2 x} \quad | \cdot x^2$$

$$\frac{dv}{v} = -2 \frac{dx}{x} \quad \int \int$$

$$v = \frac{1}{x^2}$$

$$u' = \frac{2x\sqrt{u}}{\cos^2 x}$$

$$\frac{du}{dx} = \frac{2x\sqrt{u}}{\cos^2 x}$$

$$\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{u}$$

$$\frac{du}{\sqrt{u}} = \frac{2x}{\cos^2 x} dx \quad \int \int$$

$$\int \frac{x}{\cos^2 x} dx = \int \left. \begin{array}{l} u=x \quad dv = \frac{dx}{\cos^2 x} \\ du=dx \quad v = \tan x \end{array} \right| = x \tan x - \int \tan x dx$$

$$\int \frac{du}{\sqrt{u}} = 2 \int \frac{x}{\cos^2 x} dx$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \left. \begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array} \right| = -\int \frac{dt}{t} = -\ln|t| + C$$

$$\int \frac{x}{\cos^2 x} dx = x \tan x + \ln|\cos x| + C$$

$$= -\ln|\cos x| + C$$

$$2\sqrt{u} = 2(x \tan x + \ln|\cos x| + C)$$

$$u = (x \tan x + \ln|\cos x| + C)^2$$

$$Y = u \cdot v = \frac{(x \tan x + \ln|\cos x| + C)^2}{x^2}$$

opšte rješenje
diferencijalne
jednačine