Univerzitet u Zenici
Fakultet za Metalurgiju i materijale
Odsjek za Hemiju

28.09.2009.

Pismeni ispit iz predmeta Matematika 1

1. Riješiti jednačinu $x^4 + \frac{9}{4} = 0$ i rješenja predstaviti u kompleksnoj ravni.

2. Riješiti jednačinu: $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0.$

3. Dati su vektori $\vec{a}\{\lambda, 3, 3\}$, $\vec{b}\{0, \lambda-1, \lambda+1\}$ i $\vec{c}\{\lambda, 3, 4\}$. Odrediti sve vrijednosti parametra λ tako da ovi vektori budu komplanarni pa za veću vrijednost parametr λ razložiti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

4. Napisati jednačinu ravni koja sadrži datu tačku $M(4, 5, 0)$ i datu pravu $\frac{x+3}{5} = \frac{y-4}{-3} = \frac{z-2}{2}$.

28.09.2009.

Pismeni ispit iz predmeta Matematika 2

1. Ispitati i grafički predstaviti funkciju: $y = \frac{1}{x} lnx$.

2. Odrediti: $\int x^2 sinx\, dx$.

3. Dokazati da red $\frac{1}{5 \cdot 14} + \frac{1}{14 \cdot 23} + ... + \frac{1}{(9n-4)\cdot(9n+5)} + ...$ konvergira i naći njegovu sumu.

4. Riješiti diferencijalnu jednačinu: $e^y\, dx + (xe^y - 2y)\, dy = 0$.

(Ova stranica je ostavljena prazna)

(#) Riješiti jednačinu $x^4+\frac{9}{4}=0$ i rješenja predstaviti u kompleksnoj ravni.

Rj. $x^4=-\frac{9}{4}$

$x=\sqrt[4]{-\frac{9}{4}}$

$x=\sqrt[4]{z}$

n-ti korijen kompleksnog broja tražimo po formuli:

$$z_k=\sqrt[n]{|z|}\left(\cos\frac{\omega+2k\pi}{n}+i\sin\frac{\omega+2k\pi}{n}\right),\quad k=1,2,\dots,n$$

$z=-\frac{9}{4}$ $\qquad |z|=\sqrt{\left(\frac{9}{4}\right)^2+0^2}=\frac{9}{4}$ $\qquad z=\frac{9}{4}(\cos\pi+i\sin\pi)$

$$\cos\omega=\frac{a}{|z|}=\frac{-\frac{9}{4}}{\frac{9}{4}}=-1$$

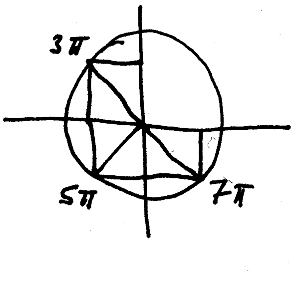
$$\Rightarrow \omega=\pi$$

$$\sin\omega=\frac{0}{|z|}=0$$

$$z_0=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt[4]{\left(\frac{3}{2}\right)^2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$$

$$z_0=\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

$$z_1=\sqrt{\frac{3}{2}}\left(\cos\frac{\pi+2\pi}{4}+i\sin\frac{\pi+2\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$



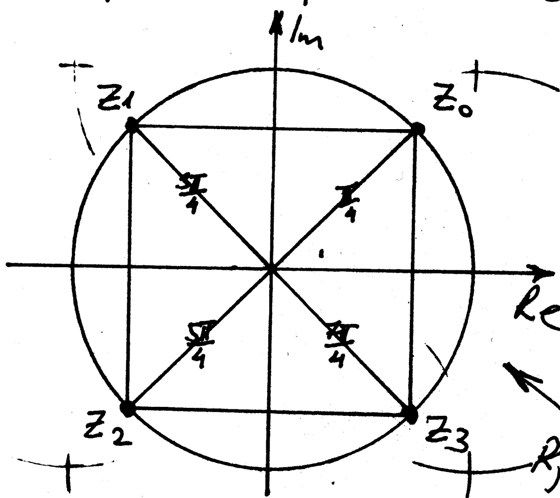
$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$$

$$z_2=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+4\pi}{4}+i\sin\frac{5\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$

$$z_3=\sqrt[4]{\frac{9}{4}}\left(\cos\frac{\pi+6\pi}{4}+i\sin\frac{7\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$$



Rješenja jednačine su:

$\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

i $\frac{\sqrt{3}}{2}-i\frac{\sqrt{3}}{2}$

Rješenja predstavljena u kompleksnoj ravni

Riješiti jednačinu: $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0$.

Rj: $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} \overset{I_k+II_k+III_k}{=\!=\!=} \begin{vmatrix} 3x-2 & x+2 & x-1 \\ 3x-2 & x-4 & x \\ 3x-2 & x+4 & x-5 \end{vmatrix} = (3x-2)\begin{vmatrix} 1 & x+2 & x-1 \\ 1 & x-4 & x \\ 1 & x+4 & x-5 \end{vmatrix}$

$\overset{I_k-II_k}{\underset{III_k-II_k}{=\!=\!=}} (3x-2)\begin{vmatrix} 0 & 6 & -1 \\ 1 & x-4 & x \\ 0 & 8 & -5 \end{vmatrix} = -(3x-2)\begin{vmatrix} 6 & -1 \\ 8 & -5 \end{vmatrix} = -(3x-2)(-30+8) =$

$= 22(3x-2)$

$$22(3x-2)=0$$

$$3x-2=0$$

$$3x=2$$

$x=\frac{2}{3}$ je rješenje jednačine

(#) Dati su vektori $\vec{a}\{\lambda, 3, 3\}$, $\vec{b}\{0, \lambda-1, \lambda+1\}$ i $\vec{c}\{\lambda, 3, 4\}$. Odrediti sve vrijednosti parametra λ tako da ovi vektori budu komplanarni, pa za veću vrijednost parametra λ razložiti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj.

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni akko $(\vec{a}\times\vec{b})\cdot\vec{c}=0$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} \lambda & 3 & 3 \\ 0 & \lambda-1 & \lambda+1 \\ \lambda & 3 & 4 \end{vmatrix} \overset{III_R - I_R}{=\!=} \begin{vmatrix} \lambda & 3 & 3 \\ 0 & \lambda-1 & \lambda+1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \lambda & 3 \\ 0 & \lambda-1 \end{vmatrix} =$$

$$= \lambda(\lambda-1)$$

$$\lambda(\lambda-1)=0$$

$$\lambda_1=0 \quad \lambda_2=1$$

Za vrijednost $\lambda=1$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

za $\lambda=1$ $\quad \vec{a}\{1,3,3\}$, $\vec{b}\{0,0,2\}$, $\vec{c}\{1,3,4\}$

$$\vec{a}=\alpha\vec{b}+\beta\vec{c}$$

$$\{1,3,3\} = \alpha\cdot\{0,0,2\}+\beta\{1,3,4\}$$

$$\begin{aligned} 0\cdot\alpha + \beta &= 1 \\ 0\cdot\alpha + 3\beta &= 3 \\ 2\alpha + 4\beta &= 3 \end{aligned}$$

$$\beta=1$$

$$2\alpha+4=3$$

$$2\alpha=-1$$

$$\alpha=-\frac{1}{2}$$

$\vec{a}=-\frac{1}{2}\vec{b}+\vec{c}$ $\quad$ vektor $\vec{a}$ razložen preko vektora $\vec{b}$ i $\vec{c}$

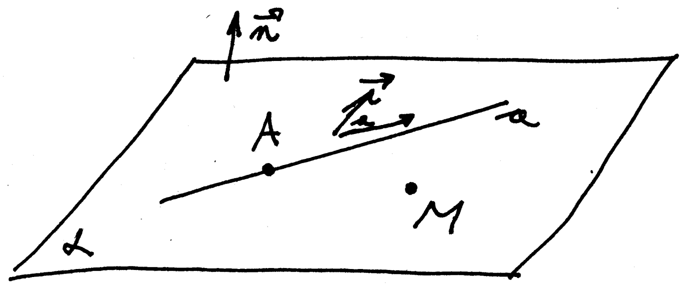
\# Napisati jednačinu ravni koja sadrži datu tačku $M(4,5,0)$ i datu pravu $\frac{x+3}{5} = \frac{y-4}{-3} = \frac{z-2}{2}$.

Rj.

$a: \frac{x+3}{5} = \frac{y-4}{-3} = \frac{z-2}{2}$

$A \in a \quad A(-3,4,2)$

$\vec{p}_a\{5, -3, 2\}$



$\alpha = ?$ $\quad \alpha: A(x-x_1) + B(y-y_1) + C(z-z_1)$

$\vec{n}\{A,B,C\}$

$A(-3,4,2)$
$M(4,5,0)$ $\Rightarrow \overrightarrow{AM}\{7,1,-2\}$

$\left.\begin{array}{l}\vec{n} \perp \vec{p}_a \\ \vec{n} \perp \overrightarrow{AM}\end{array}\right\} \Rightarrow \vec{n} \parallel \vec{p}_a \times \overrightarrow{AM}$

$\Downarrow$

$\vec{n} = k(\vec{p}_a \times \overrightarrow{AM})$ $\quad k \in R$

$$\vec{p}_a \times \overrightarrow{AM} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -3 & 2 \\ 7 & 1 & -2 \end{vmatrix} = \vec{i}(6-2) - \vec{j}(-10-14) + \vec{k}(5+21)$$

$$= 4\vec{i} + 24\vec{j} + 26\vec{k} = \{4, 24, 26\}$$

$$= 2\{2, 12, 13\}$$

Pa mogu uzeti $\vec{n}\begin{array}{c} A \quad B \quad C \\ \{2, 12, 13\}\end{array}$

$\alpha: \ 2(x-4) + 12(y-5) + 13(z-0) = 0$

$2x + 12y + 13z - 8 - 60 = 0$

$2x + 12y + 13z - 68 = 0$

jednačina tražene ravni

Ispitati i grafički predstaviti f-ju $y=\frac{1}{x}\ln x$.

Rj. definiciono područje

$x\neq 0,\quad x>0$

$\mathcal{D}:\ x\in(0,+\infty)$

parnost neparnost periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

$\frac{1}{x}\ln x=0$

$\ln x=0$

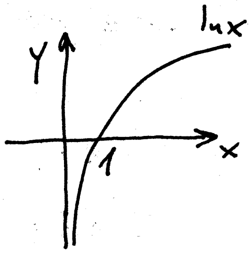
$x=e^0$

$x=1$

(1,0) je nula f-je

f(0) nije definisano

f-ja ne siječe y-osu



x	(0, 1)	(1, +∞)
ln x	−	+
y	−	+

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

$$\lim_{x\to+0} f(x)=\lim_{x\to+0}\frac{1}{x}\ln x\ \left(=\infty\cdot(-\infty)\right)=\frac{1}{+0}\ln(+0)=(+\infty)\cdot(-\infty)=-\infty$$

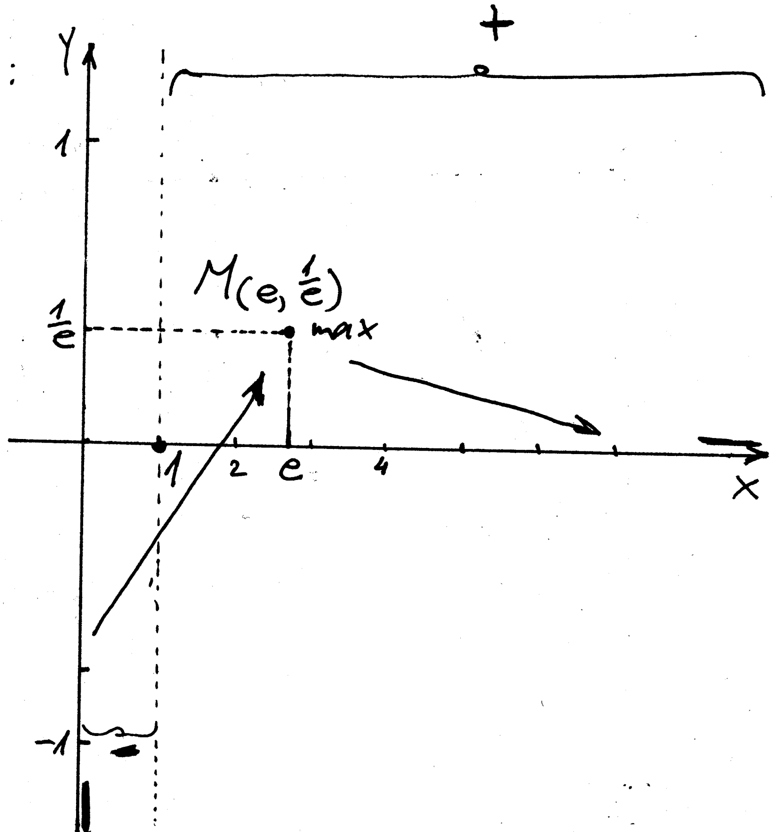
$\Rightarrow\ x=0$ je V. A. (sa desne strane)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\ln x}{x}\ \left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=\!=}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0\ \Rightarrow$$

$\Rightarrow\ y=0$ je H. A.

f-ja nema kosu asimptotu

počinjemo sa skiciranjem grafa:



rast i opadanje

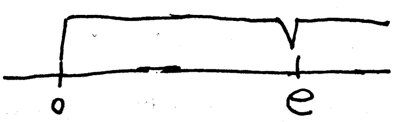
$$y'=\left(\frac{1}{x}\ln x\right)'=\left(\frac{\ln x}{x}\right)'=$$

$$=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$y'=0$ akko $1-\ln x=0$

$\ln x=1$

$x=e\approx 2{,}7183$



← nule y'
+ prekidi y

x	$(0,e)$	$(e,+\infty)$
y'	$+$	$-$
y	$\nearrow$	$\searrow$

max

rast i opadanje

$$f(e)=\frac{1}{e}\ln e=\frac{1}{e}\approx 0{,}3679$$

ekstremi f-je

Na osnovu tabele rasta i opadanja, f-ja ima maksimum u tački $M(e,\frac{1}{e})$.

prevojne tačke i intervali konveksnosti i konkavnosti.

$$y''=\left(\frac{1-\ln x}{x^2}\right)'=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)2x}{x^4}=\frac{-\cancel{x}-(1-\ln x)\cdot 2\cancel{x}}{x^{\cancel{4}\,3}}=\frac{-1-2+2\ln x}{x^3}$$

$$y''=\frac{2\ln x-3}{x^3},\qquad y''=0 \text{ akko } 2\ln x-3=0$$

$$2\ln x=3$$

$$\ln x=\frac{3}{2}$$

$$x=e^{\frac{3}{2}}=\sqrt{e^3}\approx 4{,}4817$$

$$f(e^{\frac{3}{2}})=\frac{1}{\sqrt{e^3}}\cdot\frac{3}{2}=\frac{3}{2\sqrt{e^3}}\approx 0{,}3347.$$

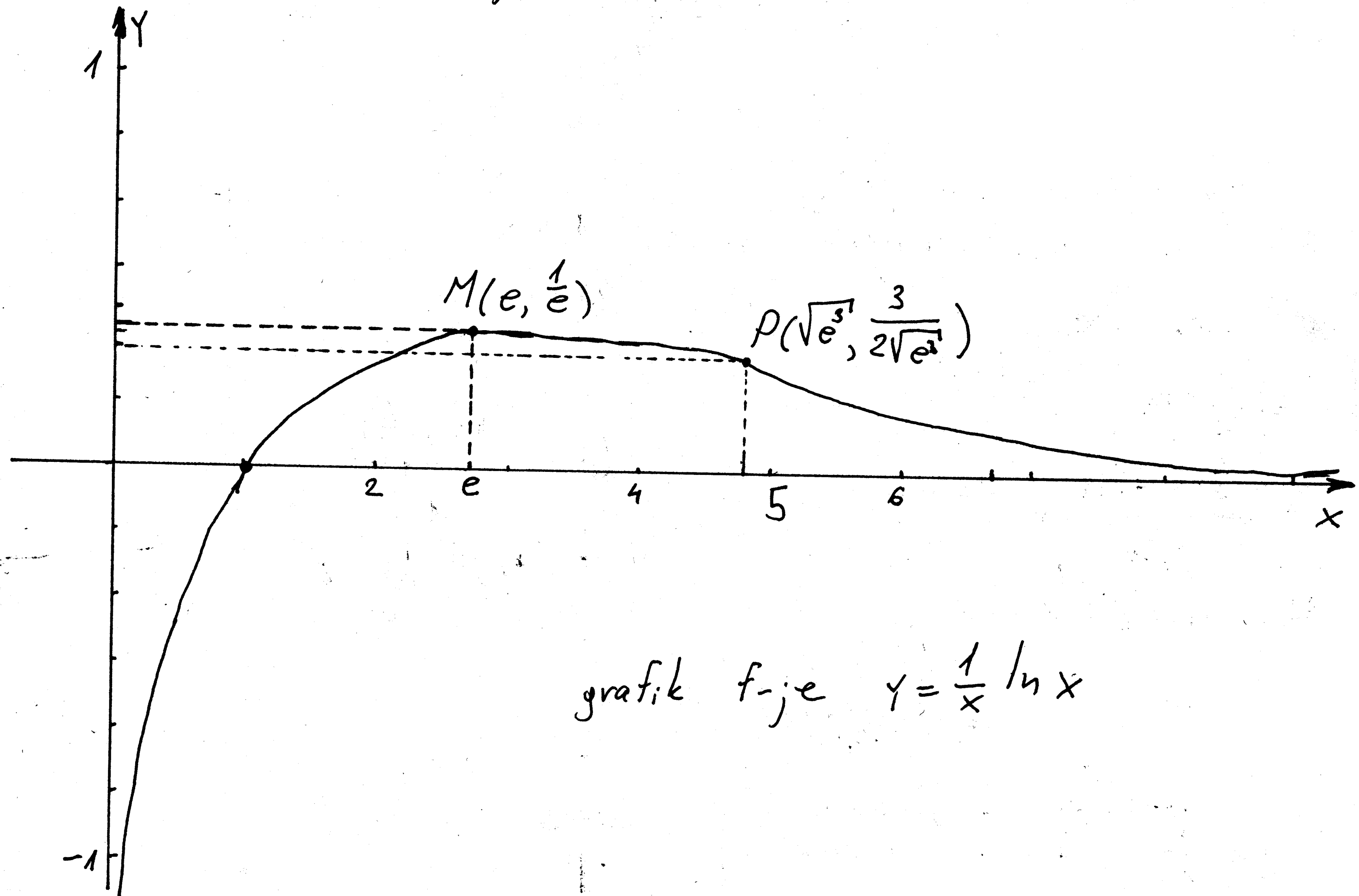
nule y'' + prekidi y

0 $\quad\sqrt{e^3}$

x	$(0,\sqrt{e^3})$	$(\sqrt{e^3},+\infty)$
y''	$-$	$+$
y	$\frown$	$\cup$

P_0T_0

$P(\sqrt{e^3},\frac{3}{2\sqrt{e^3}})$ je prevojna tačka



grafik f-je $y=\frac{1}{x}\ln x$

(#) Odrediti $I=\int x^2 \sin x \, dx$.

Rj.

$$I=\int x^2 \sin x \, dx = \left| \begin{array}{ll} u=x^2 & dv=\sin x \, dx \\ du=2x\,dx & v=-\cos x \end{array} \right| =$$

$$= -x^2 \cos x - \int (-\cos x)\cdot 2x \, dx = -x^2 + 2\int x \cos x \, dx$$

$$\int x \cos x \, dx = \left| \begin{array}{ll} u=x & dv=\cos x \, dx \\ du=dx & v=\sin x \end{array} \right| = x \sin x - \int \sin x \, dx$$

$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C_1$$

$$I = -x^2 \cos x + 2(x \sin x + \cos x + C_1) = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

$$= 2x \sin x - (x^2-2)\cos x + C$$

Dokazati da red $\frac{1}{5\cdot 14}+\frac{1}{14\cdot 23}+\dots+\frac{1}{(9n-4)(9n+5)}+\dots$ konvergira i naći njegovu sumu.

Rj. Dat je red $\sum_{n=1}^{\infty}\frac{1}{(9n-4)(9n+5)}$

Izraz $\frac{1}{(9n-4)(9n+5)}$ ćemo napisati u obliku zbira dva razlomka.

$$\frac{1}{(9n-4)(9n+5)}=\frac{A}{9n-4}+\frac{B}{9n+5}\quad /(9n-4)(9n+5)$$

$$1=A(9n+5)+B(9n-4)$$

$$1=(9A+9B)n+(5A-4B)$$

$$\begin{aligned}9A+9B&=0\\5A-4B&=1\\\hline A+B&=0\\5A-4B&=1\\\hline\end{aligned}\qquad\begin{aligned}A&=-B\\5A-4B&=1\\\hline -5B-4B&=1\\-9B&=1\end{aligned}$$

$B=-\frac{1}{9}\qquad A=\frac{1}{9}$

Red možemo napisati u obliku

$$\sum_{n=1}^{\infty}\left(\frac{\frac{1}{9}}{9n-4}+\frac{-\frac{1}{9}}{9n+5}\right)$$

tj. $\frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{1}{9n-4}-\frac{1}{9n+5}\right)$

Za red $\sum_{n=1}^{\infty}a_n$ kažemo da KV ako niz $\{S_n\}$ njegovih parcijalnih suma $(S_n=a_1+a_2+\dots+a_n)$ KV.

$$S_1=\frac{1}{9}\left(\frac{1}{5}-\frac{1}{14}\right)$$

$$S_2=\frac{1}{9}\left(\frac{1}{5}-\frac{1}{14}+\frac{1}{14}-\frac{1}{23}\right)=\frac{1}{9}\left(\frac{1}{5}-\frac{1}{23}\right)$$

$\vdots$

$$S_n=\frac{1}{9}\left(\frac{1}{5}-\underline{\frac{1}{14}}+\underline{\frac{1}{14}}-\underline{\underline{\frac{1}{23}}}+\dots+\underline{\underline{\frac{1}{9n-4}}}-\frac{1}{9n+5}\right)=\frac{1}{9}\left(\frac{1}{5}-\frac{1}{9n+5}\right)$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{9}\left(\frac{1}{5}-\underbrace{\left(\frac{1}{9n+5}\right)}_{\to 0}\right)=\frac{1}{9}\cdot\frac{1}{5}=\frac{1}{45}$$

Prema tome red KV i njegova suma iznosi $\frac{1}{45}$.

(#) Riješiti diferencijalnu jednačinu

$$e^y dx + (xe^y - 2y)\,dy = 0.$$

Rj.

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$P(x,y) = e^y \qquad \frac{\partial P}{\partial y} = e^y \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$Q(x,y) = xe^y - 2y \qquad \frac{\partial Q}{\partial x} = e^y$$

ovo je egzaktna diferencijalna jednačina

Diferencijalnu jednačinu možemo pisati u obliku

$$du(x,y) = 0$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x} = e^y \qquad \frac{\partial u}{\partial y} = xe^y - 2y$$

$$\partial u = e^y\,\partial x \qquad \frac{\partial u}{\partial y} = xe^y + \varphi'(y)$$

$$u = \int e^y dx = xe^y + \varphi(y) \qquad \varphi'(y) = -2y$$

$$\varphi(y) = -y^2 + C$$

$$u = xe^y - y^2 + C$$

$$d(xe^y - y^2 + C) = e^y dx + (xe^y - 2y)\,dy = 0$$

prema tome $xe^y - y^2 = C$ je opće rješenje diferencijalne jednačine