Univerzitet u Zenici
Mašinski fakultet
Odsjek: Opšte mašinstvo
Zenica, 02.09.2010.

Pismeni ispit iz predmeta Matematika 1

1. Riješiti matričnu jednačinu $AXB = BA$ ako je $A = \begin{bmatrix} -2 & 6 \\ 4 & -10 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -4 & 8 \end{bmatrix}$.

2. Date su tačke $A(1, -6\lambda, 2)$, $B(4, 0, 1)$, $C(5, 2, 3)$ i $D(6, 4, 4)$ gdje je λ realan broj. Odrediti zapreminu tetraedra ABCD i visinu tetraedra koja odgovara osnovici BCD. Kolika je zapremina tetraedra ako stavimo $\lambda = 8$.

3. Kroz tačku $M(1, 2, 3)$ postaviti pravu koja siječe pravu $\dfrac{x-1}{2} = \dfrac{y+1}{6} = \dfrac{z+3}{3}$ pod uglom od $90°$.

4. Ispitati i grafički predstaviti funkciju $y = \dfrac{2x+1}{(x+1)^2}$.

(Rješenja skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Riješiti matričnu jednačinu $AXB = BA$ ako je

$$A = \begin{bmatrix} -2 & 6 \\ 4 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -4 & 8 \end{bmatrix}.$$

Rj:

$AXB = BA \quad / \cdot A^{-1}$ sa lijeve str.

$A^{-1}AXB = A^{-1}BA \quad / \cdot B^{-1}$ sa desne strane

$X = A^{-1}BA \cdot B^{-1}$

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^{T}$$

$$\det A = \begin{vmatrix} -2 & 6 \\ 4 & -10 \end{vmatrix} = 20 - 24 = -4$$

$$\det B = \begin{vmatrix} 2 & -2 \\ -4 & 8 \end{vmatrix} = 16 - 8 = 8$$

$A_{11} = (-1)^2 \cdot (-10) = -10$

$A_{12} = (-1)^3 \cdot 4 = -4$

$A_{21} = (-1)^3 \cdot 6 = -6$

$A_{22} = (-1)^4 \cdot (-2) = -2$

$$A_{kof} = \begin{bmatrix} -10 & -4 \\ -6 & -2 \end{bmatrix}, \quad A_{kof}^{T} = \begin{bmatrix} -10 & -6 \\ -4 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 10 & 6 \\ 4 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{kof}^{T}$$

$B_{11} = (-1)^2 \cdot 8 = 8$

$B_{12} = (-1)^3 \cdot (-4) = 4$

$B_{21} = (-1)^3 \cdot (-2) = 2$

$B_{22} = (-1)^4 \cdot 2 = 2$

$$B_{kof} = \begin{bmatrix} 8 & 4 \\ 2 & 2 \end{bmatrix}$$

$$B_{kof}^{T} = \begin{bmatrix} 8 & 2 \\ 4 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 2 \\ 4 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 2 & -2 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} -2 & 6 \\ 4 & -10 \end{bmatrix} = \begin{bmatrix} -12 & 32 \\ 40 & -104 \end{bmatrix}$$

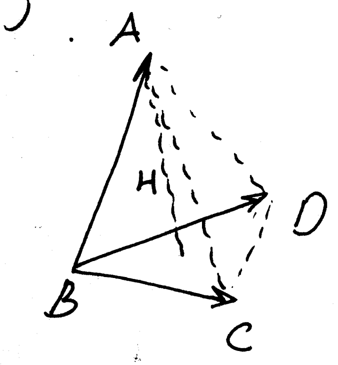
$$A^{-1} \cdot B \cdot A = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -12 & 32 \\ 40 & -104 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 60 & -152 \\ 16 & -40 \end{bmatrix} = \begin{bmatrix} 30 & -76 \\ 8 & -20 \end{bmatrix}$$

$$X = A^{-1} \cdot B \cdot A \cdot B^{-1} = \begin{bmatrix} 30 & -76 \\ 8 & -20 \end{bmatrix} \cdot \frac{1}{4} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -32 & -46 \\ -8 & -12 \end{bmatrix}$$

$$X = \begin{bmatrix} -8 & -\frac{23}{2} \\ -2 & -3 \end{bmatrix}$$ traženo rješenje

Date su tačke $A(1, -6\lambda, 2)$, $B(4, 0, 1)$, $C(5, 2, 3)$ i $D(6, 4, 4)$, $\lambda \in \mathbb{R}$. Odrediti zapreminu tetraedra $ABCD$ i visinu tetraedra koja odgovara osnovici BCD. Kolika je zapremina i visina tetraedra ako stavimo $\lambda = 8$.

R:



$B(4, 0, 1)$
$A(1, -6\lambda, 2)$
$\overrightarrow{BA} = (-3, -6\lambda, 1)$

$B(4, 0, 1)$
$D(6, 4, 4)$
$\overrightarrow{BD} = (2, 4, 3)$

$B(4, 0, 1)$
$C(5, 2, 3)$
$\overrightarrow{BC} = (1, 2, 2)$

$$V = \frac{1}{6} \left| (\overrightarrow{BC} \times \overrightarrow{BD}) \cdot \overrightarrow{BA} \right|$$ zapremina tetraedra

$$(\overrightarrow{BC} \times \overrightarrow{BD}) \cdot \overrightarrow{BA} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 4 & 3 \\ -3 & -6\lambda & 1 \end{vmatrix} \xlongequal[III_k - I_k \cdot 2]{II_k - I_k \cdot 2} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -3 & -6\lambda+6 & 7 \end{vmatrix} = (0 - (6\lambda - 6)) = 6 - 6\lambda$$

$$V = \frac{1}{6} |6 - 6\lambda| = |1 - \lambda|$$ zapremina tetraedra

$$V = \frac{B \cdot H_{BCD}}{3}$$

$$B = P_{\Delta BCD} = \frac{1}{2} \left| \overrightarrow{BC} \times \overrightarrow{BD} \right|$$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 2 & 4 & 3 \end{vmatrix} = (6-8)\vec{i} - (3-4)\vec{j} + (4-4)\vec{k} = (-2, 1, 0)$$

$$P_{\Delta BCD} = \frac{1}{2} |(-2, 1, 0)| = \frac{1}{2}\sqrt{4+1} = \frac{\sqrt{5}}{2}$$

$$|1 - \lambda| = \frac{1}{3} B \cdot H_{BCD}$$

$$|1 - \lambda| = \frac{1}{3} \cdot \frac{\sqrt{5}}{2} \cdot H_{BCD}$$

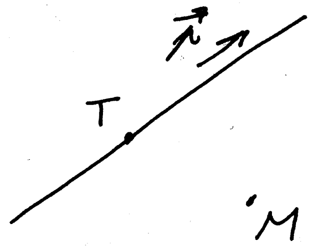
$$H_{BCD} = \frac{6|1-\lambda|}{\sqrt{5}}$$ visina tetraedra

Za $\lambda = 8$ zapremina tetraedra iznosi $V = 7$ a visina $H_{BCD} = \frac{42}{\sqrt{5}}$.

(#) Kroz tačku $M(1,2,3)$ postaviti pravu koja siječe pravu

$$\frac{x-1}{2}=\frac{y+1}{6}=\frac{z+3}{3}$$ pod uglom od $90°$.

Rj.



$\vec{p}=(2,6,3)$

Tačka M ne pripada pravoj. (Zašto?)

Pronađimo na pravoj tačku T takvu da $\vec{TM}\perp\vec{p}$.

$$\frac{x-1}{2}=\frac{y+1}{6}=\frac{z+3}{3}\ (=t)$$

$x-1=2t \qquad x=2t+1$

$y+1=6t \qquad y=6t-1$

$z+3=3t \qquad z=3t-3$

$M(1,\ 2,\ 3)$

$T(2t+1,\ 6t-1,\ 3t-3)$ (T pokretna tačka)

$\vec{MT}=(2t,\ 6t-3,\ 3t-6)$

$\vec{TM}\perp\vec{p}$ akko $\vec{TM}\cdot\vec{p}=0$

$\vec{MT}\cdot\vec{p}=(2t,6t-3,3t-6)\cdot(2,6,3)=0$

$\underline{4t}+\underline{36t}-18+\underline{9t}-18=0$

$49t=36$

$t=\frac{36}{49}$

$T\left(\frac{121}{49},\ \frac{167}{49},\ -\frac{39}{49}\right)$

$\vec{MT}\left(\frac{72}{49},\ \frac{69}{49},\ -\frac{186}{49}\right)$

$72=2^3\cdot 3^2$

$69=3\cdot 23$

$186=2\cdot 3\cdot 31$

Za $\vec{p}_a$ mogu uzeti $\vec{p}_a=(24,23,62)$

($\vec{p}_a=k\cdot\vec{MT}$, $k\in\mathbb{R}$)

$$\frac{x-x_1}{p}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$ jednačina prave kroz jednu tačku

$$\frac{x-1}{24}=\frac{y-2}{23}=\frac{z-3}{62}$$ jednačina tražene prave

(#) Ispitati i grafički predstaviti f-ju $y = \frac{2x+1}{(x+1)^2}$.

Rj. definiciono područje $x+1 \neq 0$

$x \neq -1$

$D: x \in \mathbb{R} \setminus \{-1\}$

$x \in (-\infty, -1) \cup (-1, +\infty)$

• parnost (neparnost), periodičnost

D nije simetrično

f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom znak f-je

$y=0$ akko $2x+1=0 \Rightarrow x=-\frac{1}{2}$

$(-\frac{1}{2}, 0)$ je nula f-je

$f(0) = \frac{1}{(0+1)^2} = 1$ $(0,1)$ je presjek sa y-osom

$(x+1)^2 > 0 \quad \forall x \in D$

nule + prekidi f-je: -1, $-\frac{1}{2}$

x	$(-\infty, -1)$	$(-1, -\frac{1}{2})$	$(-\frac{1}{2}, +\infty)$
$2x+1$	$-$	$-$	$+$
y	$-$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-1$ f-ja ima prekid

$\lim_{x\to -1-0} f(x) = \lim_{x\to -1-0} \frac{2x+1}{(x+1)^2} = \frac{2(-1-0)+1}{(-1-0+1)^2} = \frac{-1-0}{+0} = -\infty$

$\lim_{x\to -1+0} f(x) = \lim_{x\to -1+0} \frac{2x+1}{(x+1)^2} = \frac{2(-1+0)+1}{(-1+0+1)^2} = \frac{-1+0}{+0} = -\infty$

$\Rightarrow x=-1$ je V. A.

$\lim_{x\to \mp\infty} f(x) = \lim_{x\to \mp\infty} \frac{2x+1}{(x+1)^2} = \lim_{x\to \mp\infty} \frac{2x+1 \;/:x}{x^2+2x+1 \;/:x} = 0 \Rightarrow y=0$ je H. A.

F-ja nema kose asimptote.

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

$y' = \left(\frac{2x+1}{(x+1)^2}\right)' = \frac{2\cdot(x+1)^{\cancel{2}} - (2x+1)\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}}$

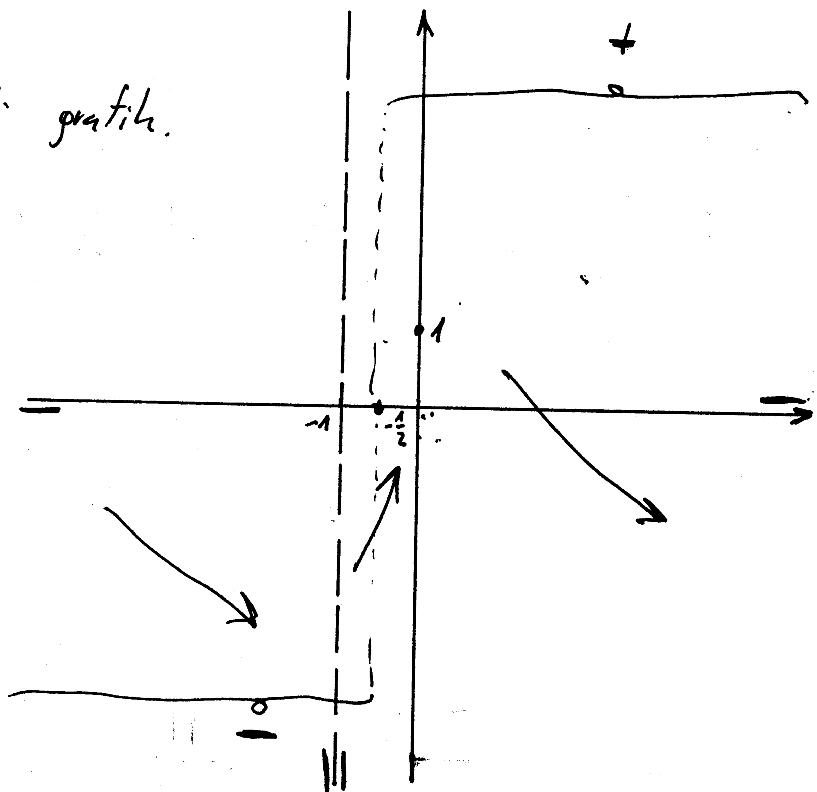
$= \frac{2x+\cancel{2}-4x-\cancel{2}}{(x+1)^3} = -\frac{2x}{(x+1)^3}$

$y'=0$ akko $2x=0 \Rightarrow x=0$

-1, 0

prekidi od y + nule y'

x	$(-\infty, -1)$	$(-1, 0)$	$(0, +\infty)$
y'	$-$	$+$	$-$
y	↘	↗	↘



ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da f-ja ima ekstrem za $x=0$.

$f(0)=1 \qquad (0,1)$ je ekstrem f-je (maksimum)

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(-\frac{2x}{(x+1)^3}\right)' = -\frac{2(x+1)^{\cancel{3}} - 2x\cdot 3\cancel{(x+1)^2}}{(x+1)^{\cancel{6}\,4}} = -\frac{2x+2-6x}{(x+1)^4} = \frac{4x-2}{(x+1)^4}$$

$y'' = \frac{4x-2}{(x+1)^4}$, $\quad y''=0$ akko $4x-2=0$

$x=2$

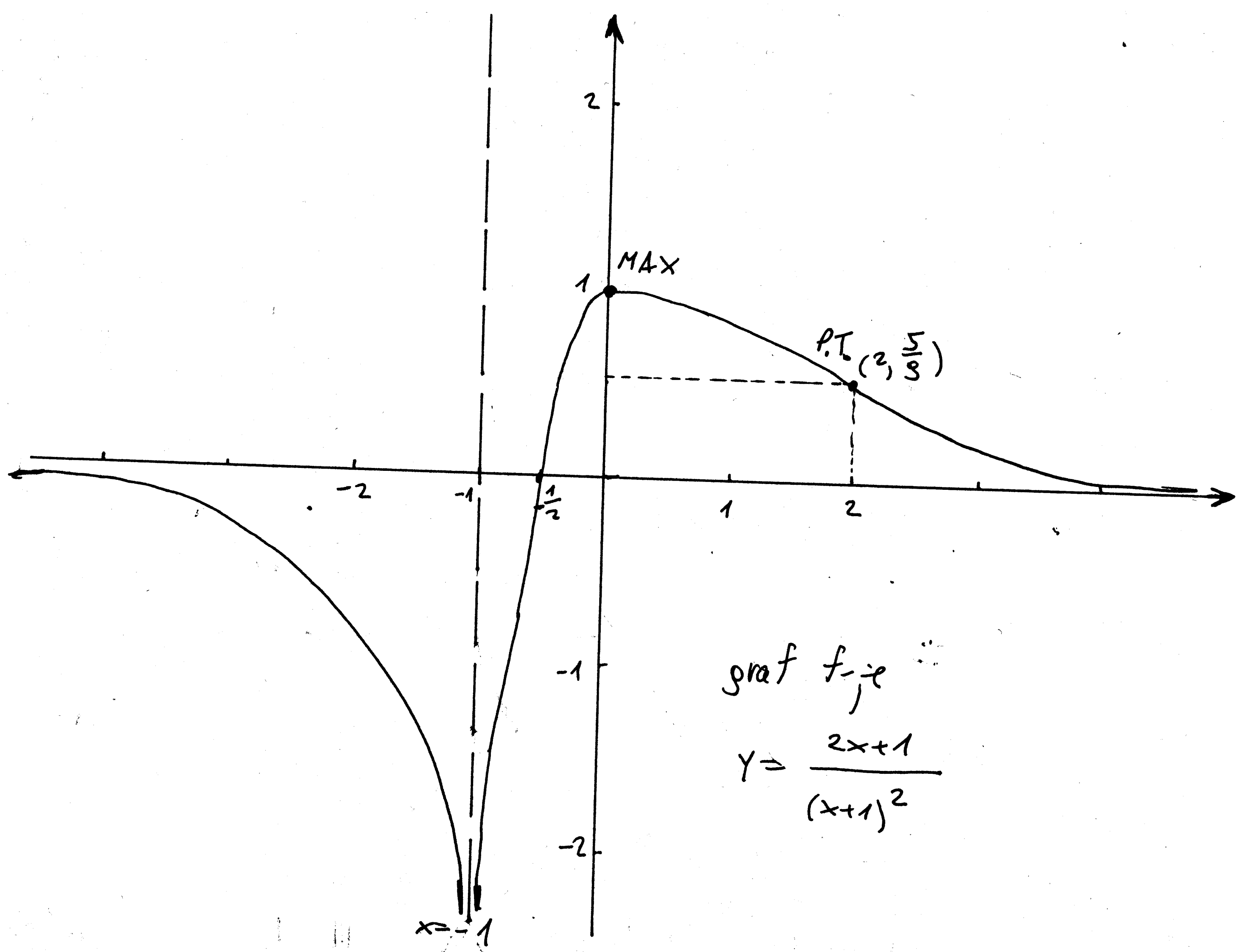
prekidi y + nule y''

$-1 \qquad 2$

x	$(-\infty,-1)$	$(-1,2)$	$(2,+\infty)$
y''	$-$	$-$	$+$
y	$\frown$	$\frown$	$\smile$

P.T.

$f(2) = \frac{5}{9}$, $\quad (2, \frac{5}{9})$ je prevojna tačka



graf f-je

$$y = \frac{2x+1}{(x+1)^2}$$