Univerzitet u Zenici
Mašinski fakultet
Odsjek: Opšte mašinstvo
Zenica, 01.07.2011.

Pismeni ispit iz predmeta Matematika 1

1. Riješiti matričnu jednačinu $XAB = C$, gdje su $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ i $C = \begin{bmatrix} 0 & 4 & 4 \end{bmatrix}$.

2. Dati su vektori $\vec{a} = (8 - \lambda, 3, -1 - \lambda)$, $\vec{b} = (7, 1, 0)$ i $\vec{c} = (7, 7, 0)$. Odrediti parametar λ tako da $\measuredangle(\vec{a}, \vec{b}) = \measuredangle(\vec{a}, \vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

3. Napisati jednačinu ravni koja prolazi kroz tačku $M_1(2, 0, -1)$ i normalna je na ravnima $2x - y - 3 = 0$ i $x + y - z + 1 = 0$.

4. Ispitati i grafički predstaviti funkciju $y = \dfrac{x^2 + 10}{x^2 + 4x + 4}$.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Riješiti matričnu jednačinu $XAB = C$, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 4 & 4 \end{bmatrix}$.

Rj.

$XAB = C \quad / \cdot (AB)^{-1}$ sa desne strane

$X(AB)(AB)^{-1} = C \cdot (AB)^{-1}$

$X = C \cdot (AB)^{-1}$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = (-2)(3+1) = -8$$

AB označimo sa M, nađimo M^{-1}

$M_{11} = (-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$ $\quad M_{21} = (-1)^3 \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} = -6$ $\quad M_{31} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$

$M_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$ $\quad M_{22} = (-1)^4 \begin{vmatrix} 0 & 0 \\ -1 & 3 \end{vmatrix} = 0$ $\quad M_{32} = (-1)^5 \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$

$M_{13} = (-1)^4 \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6$ $\quad M_{23} = (-1)^5 \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -2$ $\quad M_{33} = (-1)^6 \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2$

$$M_{kof} = \begin{bmatrix} 10 & -4 & 6 \\ -6 & 0 & -2 \\ 2 & 0 & -2 \end{bmatrix}, \quad M_{kof}^{T} = \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{\det M} \cdot M_{kof}^{T} = \frac{-1}{8} \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -5/4 & 3/4 & -1/4 \\ 1/2 & 0 & 0 \\ -3/4 & 1/4 & 1/4 \end{bmatrix}$$

$$X = C \cdot (AB^{-1}) = \begin{bmatrix} 0 & 4 & 4 \end{bmatrix} \cdot \left(-\frac{1}{8}\right) \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \left(-\frac{1}{8}\right) \begin{bmatrix} 8 & -8 & -8 \end{bmatrix}$$

$X = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$ rješenje matrične jednačine

$\#$ Dati su vektori $\vec{a}=(8-\lambda, 3, -1-\lambda)$, $\vec{b}=(7,1,0)$ i $\vec{c}=(7,7,0)$. Odrediti parametar λ tako da $\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b})$$

$$\vec{a}\cdot\vec{b}=(8-\lambda, 3, -1-\lambda)\cdot(7,1,0)=56-7\lambda+3=59-7\lambda$$

$$|\vec{a}|=\sqrt{(8-\lambda)^2+3^2+(-1-\lambda)^2}$$

$$|\vec{b}|=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

$$|\vec{c}|=\sqrt{49+49}=7\sqrt{2}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\cos\measuredangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$$

Kako tražimo λ tako da $\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c}) \Rightarrow$

$$\Rightarrow \cos\measuredangle(\vec{a},\vec{b})=\cos\measuredangle(\vec{a},\vec{c}) \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$$

$$\vec{a}\cdot\vec{c}=(8-\lambda, 3, -1-\lambda)(7,7,0)=56-7\lambda+21=77-7\lambda$$

$$\frac{59-7\lambda}{5\sqrt{2}}=\frac{77-7\lambda}{7\sqrt{2}} \quad /\cdot 35\sqrt{2}$$

$$413-49\lambda=385-35\lambda$$

$$14\lambda=28$$

$$\lambda=2$$

Za vrijednost $\lambda=2$ imamo $\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c})$

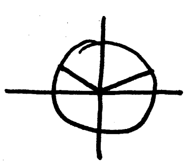
$$\lambda=2 \Rightarrow \vec{a}=(6,3,-3)$$

$$|\vec{a}|=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{(6,3,-3)\cdot(7,1,0)}{3\sqrt{6}\cdot5\sqrt{2}}=\frac{42+3}{15\sqrt{12}}=\frac{45}{15\sqrt{4\cdot3}}=$$

$$=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{2}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow$$

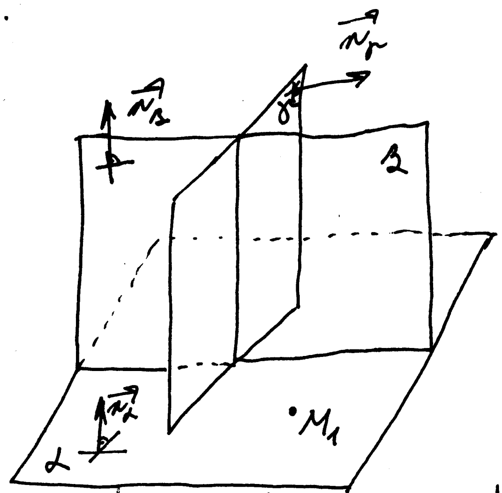


$$\measuredangle(\vec{a},\vec{b})=\frac{\pi}{6}=30^\circ \quad \text{ili} \quad \measuredangle(\vec{a},\vec{b})=\frac{11\pi}{6}=330^\circ$$

veličina ugla

(#) Napisati jednačinu ravni koja prolazi kroz tačku $M_1(2,0,-1)$ i normalna je na ravnima $2x-y-3=0$ i $x+y-z+1=0$.

Rj.



$\alpha: ?$

$\beta: 2x-y-3=0, \quad \vec{n_\beta}=(2,-1,0)$

$\gamma: x+y-z+1=0, \quad \vec{n_\gamma}=(1,1,-1)$

$$\left.\begin{array}{c}\text{Ako } M_1 \text{ uvrstim u } \beta \text{ imam} \\ 2\cdot 2-0-3\neq 0 \\ \text{Ako } M_1 \text{ uvrstim u } \gamma \text{ imam} \\ 2+0+1+1\neq 0\end{array}\right\} \Rightarrow \begin{array}{l} M_1\notin\beta \\ \text{i } M_1\notin\gamma\end{array}$$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

jednačina tražene ravni

$$\left.\begin{array}{l}\vec{n_\alpha}\perp\vec{n_\beta} \\ \vec{n_\alpha}\perp\vec{n_\gamma}\end{array}\right\} \Rightarrow \vec{n_\alpha} \parallel \vec{n_\beta}\times\vec{n_\gamma}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}: \vec{n_\alpha}=k(\vec{n_\beta}\times\vec{n_\gamma})$$

$$\vec{n_\beta}\times\vec{n_\gamma}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 1 & -1\end{vmatrix}=\vec{i}(1-0)-\vec{j}(-2-0)+\vec{k}(2+1)=\vec{i}+2\vec{j}+3\vec{k}$$
$$=(1,2,3)$$

$$\vec{n_\alpha}=k(1,2,3)=(\overset{A}{k},\overset{B}{2k},\overset{C}{3k})$$ gdje je k neki realan broj, $k\neq 0$

$$k(x-2)+2k(y-0)+3k(z+1)=0 \quad /:k$$

$$x+2y+3z-2+3=0$$

$$x+2y+3z+1=0$$

jednačina tražene ravni

(#) Ispitati f-ju i nacrtati njen grafik $y=\dfrac{x^2+10}{x^2+4x+4}$.

Rj. $y=\dfrac{x^2+10}{x^2+4x+4}=\dfrac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2\neq 0$

$x\neq -2$ $\quad \mathcal{D}: x\in(-\infty,-2)\cup(-2,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično $\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

$-2 \quad 0 \quad 2 \quad 4$

nule, presjek sa y-osom i znak f-je

$y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0 \ \forall x\in\mathcal{D}$ to f-ja nema nule

$f(0)=\dfrac{0+10}{(0+2)^2}=\dfrac{10}{4}=\dfrac{5}{2}$

$(0,\frac{5}{2})$ je presjek sa y-osom

$x^2+10>0 \ \forall x\in\mathcal{D}$ $\quad$ f-ja je

$(x+2)^2>0 \ \forall x\in\mathcal{D}$ $\quad$ uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim\limits_{x\to-2-0} f(x)=\lim\limits_{x\to-2-0}\dfrac{x^2+10}{(x+2)^2}=\dfrac{(-2-0)^2+10}{(-2-0+2)^2}=\dfrac{14+0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa lijeve strane)

$\lim\limits_{x\to-2+0} f(x)=\lim\limits_{x\to-2+0}\dfrac{x^2+10}{(x+2)^2}=\dfrac{(-2+0)^2+10}{(-2+0+2)^2}=\dfrac{14-0}{+0}=+\infty \Rightarrow x=-2$ je V. A. (sa desne strane)

$\lim\limits_{x\to\mp\infty} f(x)=\lim\limits_{x\to\mp\infty}\dfrac{x^2+10 \ :x^2}{x^2+4x+4 \ :x^2}=\lim\limits_{x\to\mp\infty}\dfrac{1+\frac{10}{x^2}\to 0}{1+\underbrace{\frac{4}{x}}_{\to 0}+\underbrace{\frac{4}{x^2}}_{\to 0}}=1 \Rightarrow y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

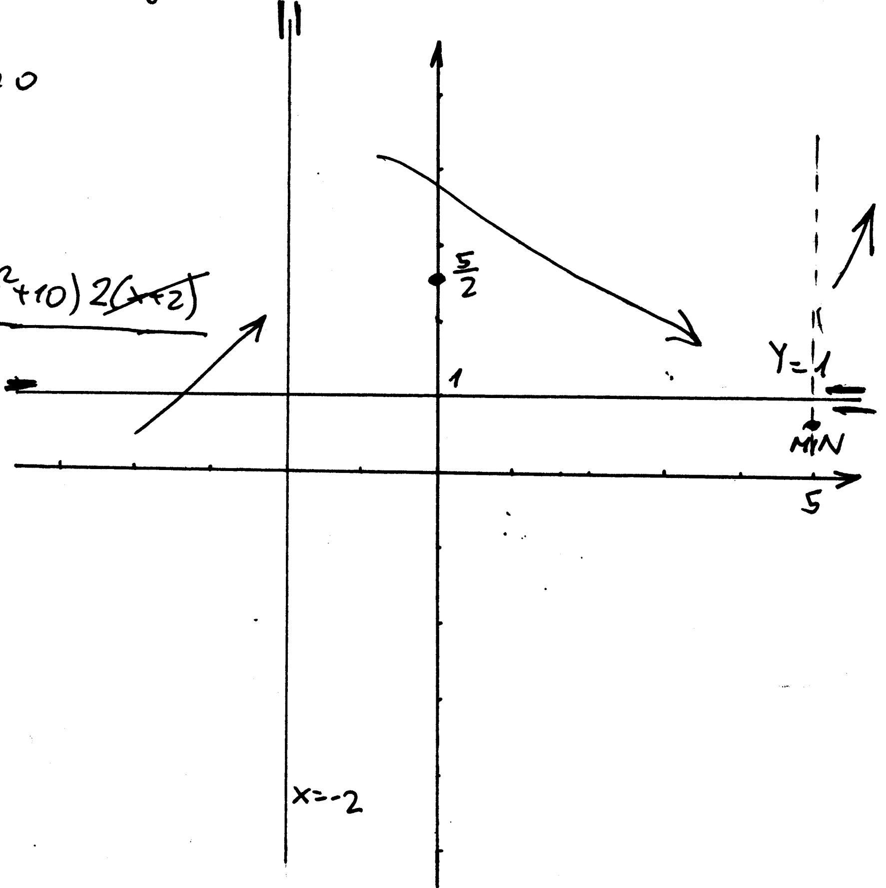
$y'=\left(\dfrac{x^2+10}{(x+2)^2}\right)'=\dfrac{2x\cdot(x+2)^{\cancel{2}}-(x^2+10)\,2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}}$

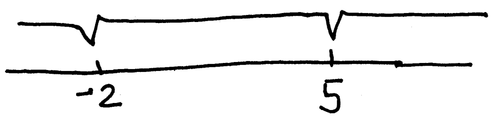
$y'=\dfrac{\cancel{2x^2}+4x-\cancel{2x^2}-20}{(x+2)^3}$

$y'=\dfrac{4x-20}{(x+2)^3}=4\dfrac{x-5}{(x+2)^3}$

$y'=0$ akko $x-5=0$

$x=5$





← prekidi Y + nule Y'

x	$(-\infty,-2)$	$(-2,5)$	$(5,+\infty)$
Y'	+	−	+
Y	↗	↘	↗

min

rast i opadanje

ekstremi f-je

Stacionarna tačka je $x=5$.

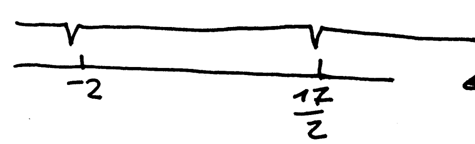
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$f(5)=\dfrac{25+10}{7^2}=\dfrac{35}{49}\approx 0{,}71$ $\quad\left(5,\dfrac{35}{49}\right)$ je tačka minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(4\,\frac{x-5}{(x+2)^3}\right)'=4\,\frac{1\cdot(x+2)^{\cancel{3}}-(x-5)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}}=4\,\frac{x+2-3x+15}{(x+2)^4}$$

$$Y''=4\,\frac{-2x+17}{(x+2)^4}=-4\,\frac{2x-17}{(x+2)^4}$$



← nule Y'' + prekidi Y

$Y''=0$ akko $2x-17=0$

$x=\dfrac{17}{2}$

x	$(-\infty,-2)$	$(-2,\frac{17}{2})$	$(\frac{17}{2},+\infty)$
Y''	+	+	−
Y	∪	∪	∩

P.T.

intervali konveksn. i konkavn.