Univerzitet u Zenici
Mašinski fakultet
Odsjek: Opšte mašinstvo
Zenica, 11.02.2011.

Pismeni ispit iz predmeta Matematika 1

1. Riješiti matričnu jednačinu $XA^{-1} = B^{-1}$ ako su $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix}$ i $B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

2. Date su tačke $A(3,2,1)$, $B(4,1,-2)$, $C(-5,-4,8)$ i $D(6,3,7)$. Izračunati zapreminu tetraedra $ABCD$ i visinu tetraedra koja odgovara osnovici BCD.

3. Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases} x - y + z + 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$ a normalna je na ravan $2x - y + 5z - 3 = 0$.

4. Ispitati i grafički predstaviti funkciju $y = \dfrac{3\,x}{1 + x^3}$.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Riješiti matričnu jednačinu $X \cdot A^{-1} = B^{-1}$ ako su

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

Rj.

$X \cdot A^{-1} = B^{-1}$ / $\cdot A$ sa desne strane

$$X \underbrace{A^{-1} \cdot A}_{I} = B^{-1} \cdot A$$

$$X = B^{-1} \cdot A$$

$$B^{-1} = \frac{1}{\det B} B_{kof}^{T}$$

$$\det B = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} \underline{\underline{I_R - II_R}}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1$$

$$\det B = 1$$

$$B_{11} = (-1)^2 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 \qquad B_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1 \qquad B_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$$

$$B_{12} = (-1)^3 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0 \qquad B_{22} = (-1)^4 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1 \qquad B_{32} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$$

$$B_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1 \qquad B_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = -1 \qquad B_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$B_{kof} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad B_{kof}^{T} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$X = B^{-1} \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} 2-3-1 & 0+3+2 & 2-3+0 \\ 3-2-1 & 0+2+2 & 3-2+0 \\ 4-1+4 & 0+1-8 & 4-1+0 \end{matrix}$$

$$X = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

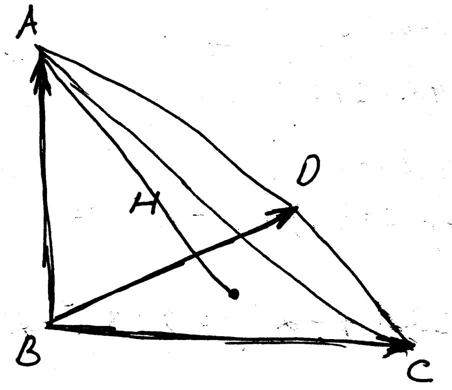
traženo rješenje

3. Date su tačke $A(3,2,1)$, $B(4,1,-2)$, $C(-5,-4,8)$ i $D(6,3,7)$. Odrediti:

a) zapreminu tetraedra $ABCD$.

b) visinu tetraedra koja odgovara osnovici BCD.

Rj.



$$\left.\begin{array}{l} B(4,1,-2) \\ A(3,2,1) \end{array}\right\} \Rightarrow \overrightarrow{BA}(-1,1,3)$$

$$D(6,3,7) \Rightarrow \overrightarrow{BD}(2,2,9)$$

$$C(-5,-4,8) \Rightarrow \overrightarrow{BC}(-9,-5,10)$$

a) $$V=\frac{1}{6}\left|(\overrightarrow{BC}\times\overrightarrow{BD})\cdot\overrightarrow{BA}\right|=\frac{1}{6}\left|\begin{vmatrix} -9 & -5 & 10 \\ 2 & 2 & 9 \\ -1 & 1 & 3 \end{vmatrix}\right| \underset{III_k - II_k\cdot 3}{\overset{I_k + II_k}{=\!=\!=}} \frac{1}{6}\left|\begin{vmatrix} -14 & -5 & 25 \\ 4 & 2 & 3 \\ 0 & 1 & 0 \end{vmatrix}\right|$$

$$=\frac{1}{6}\left|\begin{vmatrix} -14 & 25 \\ 4 & 3 \end{vmatrix}\right| = \frac{1}{6}\left|-42-100\right| = \frac{142}{6} = \frac{71}{3}$$

Zapremina tetraedra $ABCD$ iznosi $\frac{71}{3}$.

b) Zapremina piramide $V=\frac{B\cdot H_{BCD}}{3}$

$$B = P_{\triangle ACD} = \frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BD}\right| = \frac{1}{2}\sqrt{4225+10\,201+64} = \frac{1}{2}\sqrt{9\cdot 1610} = \frac{3}{2}\sqrt{1610}$$

$$\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -9 & -5 & 10 \\ 2 & 2 & 9 \end{vmatrix} = (-45-20)\vec{i}-(-81-20)\vec{j}+(-18+10)\vec{k}$$
$$=(-65, 101, -8)$$

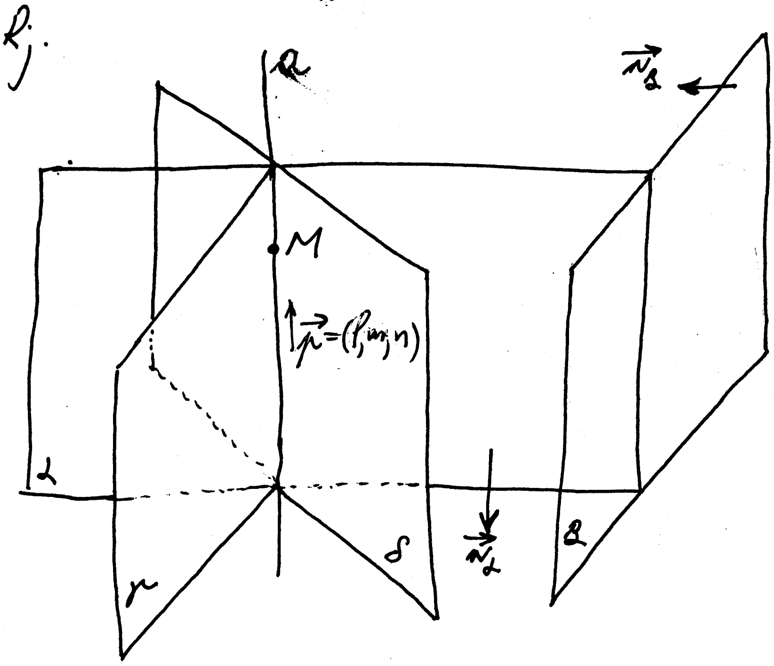
$$\frac{71}{3} = \frac{\frac{3}{2}\sqrt{1610}\cdot H_{BCD}}{3} \quad /\cdot 3\cdot 2$$

$$3\sqrt{1610}\cdot H_{BCD} = 142$$

$$H_{BCD} = \frac{142}{3\sqrt{1610}}$$ je visina tetraedra koja odgovara osnovici BCD.

Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases}$ a normalna je na ravan $2x-y+5z-3=0$.

Rj.



α: $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

β: $2x-y+5z-3=0$

pramen ravni:

$$A_1x+B_1y+C_1z+D_1+$$
$$+\lambda(A_2x+B_2y+C_2z+D_2)=0$$

gdje su

$$A_1x+B_1y+C_1z+D_1=0 \quad i$$
$$A_2x+B_2y+C_2z+D_2=0$$

dvije neparalelne ravni koje se sijeku po pravoj.

$$x-y+z+1+\lambda(x+y-z+1)=0$$
$$x+\lambda x-y+\lambda y+z-\lambda z+1+\lambda=0$$
$$x(1+\lambda)+y(-1+\lambda)+z(1-\lambda)+(1+\lambda)=0$$

pramen ravni koje prolaze kroz pravu a

$\vec{n}_\alpha=(1+\lambda, -1+\lambda, 1-\lambda)$

$\vec{n}_\beta=(2,-1,5)$

$\vec{n}_\alpha \perp \vec{n}_\beta \Rightarrow \vec{n}_\alpha \cdot \vec{n}_\beta=0$

$$(1+\lambda, -1+\lambda, 1-\lambda)(2,-1,5)=0$$
$$2+\underline{2\lambda}+1-\underline{\lambda}+5-\underline{5\lambda}=0$$
$$-4\lambda+8=0$$
$$\lambda=2$$

$\vec{n}_\alpha=(3,1,-1)$

Treba nam još tačka $M\in a$

$$a=\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases} \qquad (M\in\gamma\wedge\delta)$$

$$2x+2=0$$
$$x=-1$$

$M(-1,0,0)$

$$3(x+1)+1(y-0)-1(z-0)=0$$
$$3x+y-z+3=0 \quad \text{jednačina tražene ravni}$$

Ispitati f-ju i nacrtati joj grafik $y=\frac{3x}{1+x^3}$.

Rj. definiciono područje

$1+x^3 \neq 0$

$x^3 \neq -1$

$x \neq -1$

$D: x \in (-\infty,-1) \cup (-1,+\infty)$

parnost, neparnost, periodičnost

$f(-x)=\frac{3\cdot(-x)}{1+(-x^3)} = -\frac{3x}{1-x^3}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$

$\frac{3x}{1+x^3}=0$

$x=0$

(0,0) je nula f-je i presjek sa y-osom

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
$3x$	$-$	$-$	$+$
$1+x^3$	$-$	$+$	$+$
y	$+$	$-$	$+$

znak f-je

$-1 \quad 0 \quad 1$

ponašanje na krajevima intervala definisanosti i asimptote

za vrijednost $x=-1$ f-ja ima prekid

$\lim_{x\to -1-0} f(x) = \lim_{x\to -1-0}\frac{3x}{1+x^3} = \frac{3(-1-0)}{1+(-1-0)^3} = \frac{3(-1-0)}{1-1-0} = \frac{-3-0}{-0} = +\infty \Rightarrow x=-1$ je V.A.

$\lim_{x\to -1+0} f(x) = \lim_{x\to -1+0}\frac{3x}{1+x^3} = \frac{3(-1+0)}{1+(-1+0)^3} = \frac{-3+0}{1-1+0} = \frac{-3+0}{+0} = -\infty \Rightarrow x=-1$ je V.A.

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\frac{3x \; :x}{1+x^3 \; :x} = \lim_{x\to -\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2} = 0 \Rightarrow y=0$ je H.A.

$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty}\frac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2} = 0 \Rightarrow y=0$ je H.A. f-ja nema K.A.

rast i opadanje

$y' = \left(\frac{3x}{1+x^3}\right)' = 3\cdot\frac{1\cdot(1+x^3)-x\cdot 3x^2}{(1+x^3)^2} = 3\frac{1+x^3-3x^3}{(1+x^3)^2}$

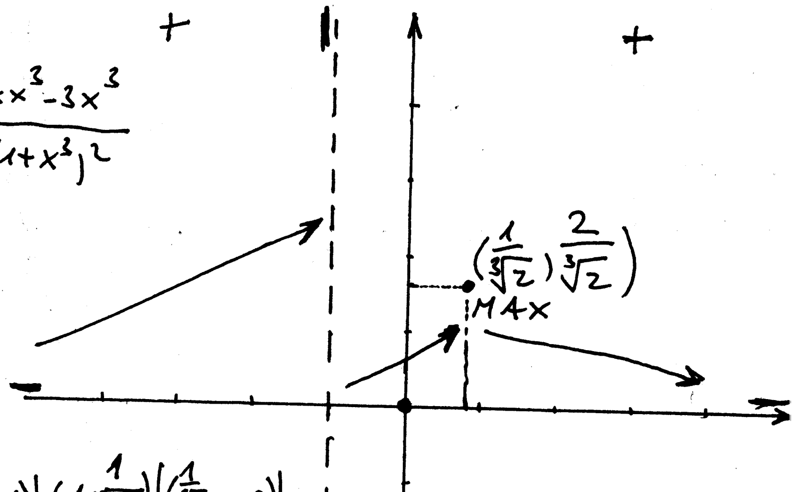
$y' = 3\cdot\frac{1-2x^3}{(1+x^3)^2}$

$y'=0$ akko $1-2x^3=0$

$2x^3=1$

$x^3=\frac{1}{2}$

$x=\frac{1}{\sqrt[3]{2}} \approx 0{,}8$



x	$(-\infty,-1)$	$(-1,\frac{1}{\sqrt[3]{2}})$	$(\frac{1}{\sqrt[3]{2}},+\infty)$
y'	$+$	$+$	$-$
y	↗	↗	↘

MAX

$-1 \quad \frac{1}{\sqrt[3]{2}}$

prekidi y + nule y'

ekstremi f-je

Na osnovu tabele

$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{\frac{3}{\sqrt[3]{2}}}{1+\frac{1}{2}} = \frac{\frac{3}{\sqrt[3]{2}}}{\frac{3}{2}} = \frac{2}{\sqrt[3]{2}} \approx 1{,}6$

$\left(\frac{1}{\sqrt[3]{2}}, \frac{2}{\sqrt[3]{2}}\right)$ je tačka maksimuma prevojne tačke i intervali konveksnosti i konkavnosti

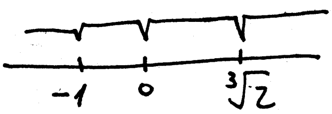
$$y' = 3\cdot\frac{1-2x^3}{(1+x^3)^2}\,,\quad y'' = 3\cdot\frac{-6x^2\cdot(1+x^3)^{\cancel{2}} - (1-2x^3)\cdot 2\cancel{(1+x^3)}\cdot 3x^2}{(1+x^3)^3\cdot\cancel{(1+x^3)}} =$$

$$= 3\cdot\frac{-6x^2-6x^5-6x^2+12x^5}{(1+x^3)^3} = 3\cdot\frac{6x^5-12x^2}{(1+x^3)^3}$$

$$y'' = 18\cdot\frac{x^5-2x^2}{(1+x^3)^3} = \frac{18x^2(x^3-2)}{(1+x^3)^3}$$

$y''=0$ akko $x=0$ ili $x^3-2=0$

$x_1=0$ $\quad$ $x_2=\sqrt[3]{2}\approx 1{,}3$



x	$(-\infty,-1)$	$(-1,0)$	$(0,\sqrt[3]{2})$	$(\sqrt[3]{2},+\infty)$
y''	+	−	−	+
y	$\cup$	$\cap$	$\cap$	$\cup$

PoTo

$$f(\sqrt[3]{2}) = \frac{3\sqrt[3]{2}}{1+2} = \sqrt[3]{2}$$

$(\sqrt[3]{2},\sqrt[3]{2})$ je prevojna tačka

grafik