Univerzitet u Zenici
Mašinski fakultet
Odsjek: Opšte mašinstvo
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Pismeni ispit iz predmeta Matematika 1

1. Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} x+y+z &= 4 \\ x+\lambda y+z &= 3 \\ x+2\lambda y+z &= 4 \; . \end{aligned}$$

2. Dati su vektori $\vec{a}=(8-\lambda, 3, -1-\lambda)$, $\vec{b}=(7,1,0)$ i $\vec{c}=(7,7,0)$. Odrediti parametar λ tako da $\measuredangle(\vec{a},\vec{b})=\measuredangle(\vec{a},\vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

3. Kroz presjek ravni $4x-y+3z-1=0$ i $x+5y-z+2=0$ postaviti ravan koja je normalna na ravan $2x-y+5z-3=0$.

4. Ispitati i grafički predstaviti funkciju $y=\dfrac{x^2-5x+6}{x^2+1}$.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Riješiti sistem jednačina i diskutovati rješenja sistema u zavisnosti od parametra λ

$$\begin{aligned} x+y+z&=4\\ x+\lambda y+z&=3\\ x+2\lambda y+z&=4 \end{aligned}$$

Rj. Sistem rješavamo Cramerovom metodom

$$D=\begin{vmatrix}1&1&1\\1&\lambda&1\\1&2\lambda&1\end{vmatrix}\overset{II_v-III_v}{=\!=}\begin{vmatrix}1&1&1\\0&-\lambda&0\\1&2\lambda&1\end{vmatrix}=-\lambda\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

$$D_x=\begin{vmatrix}4&1&1\\3&\lambda&1\\4&2\lambda&1\end{vmatrix}\underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}}\begin{vmatrix}1&1-\lambda&0\\3&\lambda&1\\1&\lambda&0\end{vmatrix}=-\begin{vmatrix}1&1-\lambda\\1&\lambda\end{vmatrix}=-(\lambda-(1-\lambda))=1-\lambda-\lambda=1-2\lambda$$

$$D_y=\begin{vmatrix}1&4&1\\1&3&1\\1&4&1\end{vmatrix}\overset{III_k-I_k}{=\!=}\begin{vmatrix}1&4&0\\1&3&0\\1&4&0\end{vmatrix}=0$$

$$D_z=\begin{vmatrix}1&1&4\\1&\lambda&3\\1&2\lambda&4\end{vmatrix}\underset{III_v-II_v}{\overset{I_v-II_v}{=\!=}}\begin{vmatrix}0&1-\lambda&1\\1&\lambda&3\\0&\lambda&1\end{vmatrix}=-\begin{vmatrix}1-\lambda&1\\\lambda&1\end{vmatrix}=-(1-\lambda-\lambda)=2\lambda-1$$

Kako je $D=0$ to sistem može da ima beskonačno mnogo rješenja ili da nema rješenja.

1° $\lambda=\frac{1}{2}$

$D=0$, $D_x=0$, $D_y=0$, $D_z=0$

Sistem ćemo riješiti Gauss-ovom metodom

$$\begin{aligned} x+y+z&=4\\ x+\tfrac{1}{2}y+z&=3 \quad /\cdot 2\\ x+y+z&=4 \end{aligned}$$

$$\begin{aligned} x+y+z&=4 \quad (1)\\ 2x+y+2z&=6 \quad (2) \end{aligned}$$

$(2)-(1)$: $x+z=2$

$x=2-z$

$x+y+z=4$

$2-z+y+z=4$

$y=2$

Za $\lambda=\frac{1}{2}$ sistem ima ∞ mnogo rješenja koja su oblika $(2-t, 2, t)$ gdje je $t\in\mathbb{R}$.

2° $\lambda\neq\frac{1}{2}$

$D=0$, $D_x\neq 0$ $\Rightarrow$ sistem za $\lambda\neq\frac{1}{2}$ nema rješenja

(#) Dati su vektori $\vec{a}=(8-\lambda, 3, -1-\lambda)$, $\vec{b}=(7,1,0)$ i $\vec{c}=(7,7,0)$. Odrediti parametar λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$ (da ugao između vektora $\vec{a}$ i $\vec{b}$ bude jednak uglu između vektora $\vec{a}$ i $\vec{c}$), pa za dobijenu vrijednost λ odrediti veličinu ugla.

Rj.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\sphericalangle(\vec{a},\vec{b})$$

$$\vec{a}\cdot\vec{b}=(8-\lambda, 3, -1-\lambda)\cdot(7,1,0)=56-7\lambda+3=59-7\lambda$$

$$|\vec{a}|=\sqrt{(8-\lambda)^2+3^2+(-1-\lambda)^2}$$

$$|\vec{b}|=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

$$|\vec{c}|=\sqrt{49+49}=7\sqrt{2}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\cos\sphericalangle(\vec{a},\vec{c})=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$$

Kako tražimo λ tako da $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c}) \Rightarrow$

$$\Rightarrow \cos\sphericalangle(\vec{a},\vec{b})=\cos\sphericalangle(\vec{a},\vec{c}) \Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}$$

$$\vec{a}\cdot\vec{c}=(8-\lambda, 3, -1-\lambda)(7,7,0)=56-7\lambda+21=77-7\lambda$$

$$\frac{59-7\lambda}{5\sqrt{2}}=\frac{77-7\lambda}{7\sqrt{2}} \quad /\cdot 35\sqrt{2}$$

$$413-49\lambda=385-35\lambda$$

$$14\lambda=28$$

$$\lambda=2$$

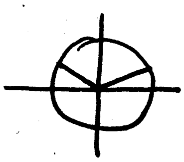
Za vrijednost $\lambda=2$ imamo $\sphericalangle(\vec{a},\vec{b})=\sphericalangle(\vec{a},\vec{c})$

$$\lambda=2 \Rightarrow \vec{a}=(6,3,-3)$$

$$|\vec{a}|=\sqrt{36+9+9}=\sqrt{54}=3\sqrt{6}$$

$$\cos\sphericalangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{(6,3,-3)\cdot(7,1,0)}{3\sqrt{6}\cdot5\sqrt{2}}=\frac{42+3}{15\sqrt{12}}=\frac{45}{15\sqrt{4\cdot3}}=$$

$$=\frac{3}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{2} \qquad \cos\sphericalangle(\vec{a},\vec{b})=\frac{\sqrt{3}}{2} \Rightarrow$$

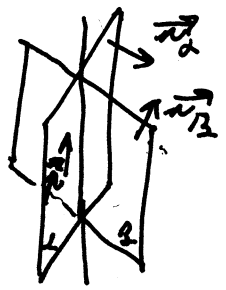


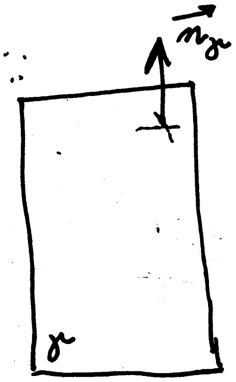
$$\sphericalangle(\vec{a},\vec{b})=\frac{\pi}{6}=30^\circ \quad \text{ili} \quad \sphericalangle(\vec{a},\vec{b})=\frac{11\pi}{6}=330^\circ$$

veličina ugla

(#) Kroz presjek ravni $4x-y+3z-1=0$ i $x+5y-z+2=0$ postaviti ravan koja je normalna na ravan $2x-y+5z-3=0$.

Rj. I način:





$\alpha: 4x-y+3z-1=0$

$\beta: x+5y-z+2=0$

$\gamma: 2x-y+5z-3=0$

$\vec{n}_\alpha=(4,-1,3)$

$\vec{n}_\beta=(1,5,-1)$

$\vec{n}_\gamma=(2,-1,5)$

$$\left.\begin{array}{l}\vec{p}\perp\vec{n}_\alpha\\ \vec{p}\perp\vec{n}_\beta\end{array}\right\} \Rightarrow \vec{p}\parallel\vec{n}_\alpha\times\vec{n}_\beta \Downarrow \vec{p}=k\cdot(\vec{n}_\alpha\times\vec{n}_\beta),\ k\in\mathbb{R}$$

$$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -1 & 3\\ 1 & 5 & -1\end{vmatrix}=(1-15)\vec{i}-(-4-3)\vec{j}+(20+1)\vec{k}=(-14,7,21)$$

pa za $\vec{p}$ mogu uzeti $\vec{p}=(-2,1,3)$

$$\left.\begin{array}{l}\vec{n}\perp\vec{p}\\ \vec{n}\perp\vec{n}_\gamma\end{array}\right\} \Rightarrow \vec{n}\parallel\vec{p}\times\vec{n}_\gamma, \quad \vec{p}\times\vec{n}_\gamma=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 1 & 3\\ 2 & -1 & 5\end{vmatrix}=(8,16,0) \Rightarrow$$

$\Rightarrow \vec{n}=(1,2,0)$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina ravni kroz tačku (x_1,y_1,z_1) i vektor normale $\vec{n}=(A,B,C)$.

nađimo tačku koja pripada presjeku ravni $\alpha\cap\beta$

$$\begin{array}{l}4x-y+3z-1=0\\ x+5y-z+2=0 \quad /\cdot 3\\ \hline\end{array} \qquad \begin{array}{l}4x-y+3z-1=0\\ +\ 3x+15y-3z+6=0\\ \hline\end{array} \qquad 7x+14y+5=0$$

$x=\frac{2}{7} \Rightarrow 14y=-2-5$

$y=-\frac{1}{2}$

$4\cdot\frac{2}{7}+\frac{1}{2}+3z-1=0 \Rightarrow 3z=-\frac{8}{7}-\frac{1}{2}+1=\frac{1}{2}-\frac{8}{7}=\frac{7-16}{14}=\frac{-9}{14} \quad /\cdot\frac{1}{3}$

$\Rightarrow z=-\frac{3}{14}$

$M\left(\frac{2}{7},-\frac{1}{2},-\frac{3}{14}\right)$

$1\cdot\left(x-\frac{2}{7}\right)+2\left(y+\frac{1}{2}\right)+0\left(z+\frac{3}{14}\right)=0$

$x-\frac{2}{7}+2y+1=0 \Rightarrow 7x+14y+5=0$ jednačina tražene ravni

II način: koristimo formulu pramena ravni

$4x-y+3z-1+\lambda(x+5y-z+2)=0$

$(4+\lambda)x+(-1+5\lambda)y+(3-\lambda)z-1+2\lambda=0$

$\vec{n}=(4+\lambda,-1+5\lambda,3-\lambda)$

$\vec{n}\perp\vec{n}_\gamma = \vec{n}\cdot\vec{n}_\gamma=0 \Rightarrow \lambda=3$

$\Rightarrow 7x+14y+5=0$ jednačina tražene ravni.

Ispitati i nacrtati grafik f-je $y = \dfrac{x^2-5x+6}{x^2+1}$.

1. definiciono područje

$x^2+1 \neq 0$

$D: x \in \mathbb{R}$

$x \in (-\infty, +\infty)$

parnost/neparnost, periodičnost

D je simetrično

$$f(-x) = \frac{(-x)^2-5(-x)+6}{(-x)^2+1} = \frac{x^2+5x+6}{x^2+1} \neq f(x)$$
$$\neq -f(x)$$

f-ja nije ni parna ni neparna

f-ja nije periodična

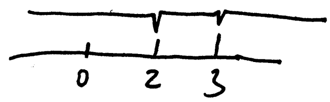
nule, presjek sa y-osom, znak

$$f(0) = \frac{6}{1} = 6$$

$(0,6)$ je presjek sa y-osom

$y=0 \Rightarrow x^2-5x+6=0$

$(x-2)(x-3)=0$



prekidi y + nule od y

$x^2+1>0 \; \forall x$

x	$(-\infty, 2)$	$(2, 2)$	$(3, +\infty)$
$x-2$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
y	$+$	$-$	$+$

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja nema prekid $\Rightarrow$ f-ja nema V. A.

$$\lim_{x\to \mp\infty} f(x) = \lim_{x\to \mp\infty} \frac{x^2-5x+6}{x^2+1} \begin{matrix} /:x^2 \\ /:x^2 \end{matrix} = 1 \Rightarrow y=1 \text{ je H. A.}$$

f-ja nema kosu asimptotu.

Nakon ovog koraka počinjemo sa skiciranjem grafa f-je

rast i opadanje

$$y' = \left(\frac{x^2-5x+6}{x^2+1}\right)' = \frac{(2x-5)(x^2+1)-(x^2-5x+6)(2x)}{(x^2+1)^2} =$$

$$= \frac{\cancel{2x^3}+2x-5x^2-5-\cancel{2x^3}+10x^2-12x}{(x^2+1)^2}$$

$$y' = \frac{5x^2-10x-5}{(x^2+1)^2}$$

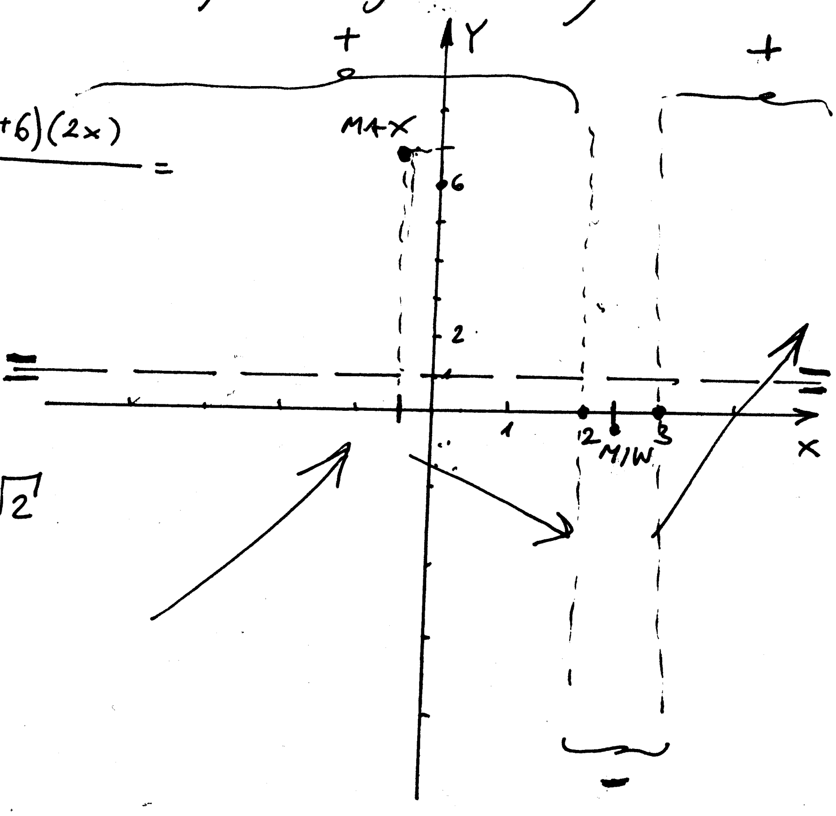
$D = 100+100$

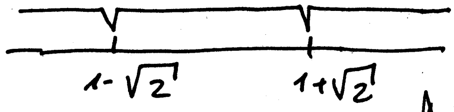
$\sqrt{200} = \sqrt{4\cdot 25\cdot 2} = 10\sqrt{2}$

$$x_{1,2} = \frac{10 \pm 10\sqrt{2}}{10} = 1 \pm \sqrt{2}$$

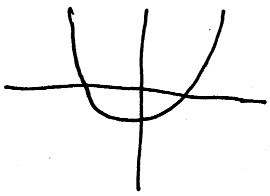
$x_1 \approx -0{,}4142$

$x_2 \approx 2{,}4142$





$5x^2-10x-5=0$



prekid y
+ nule y'

x	$(-\infty, 1-\sqrt{2})$	$(1-\sqrt{2}, 1+\sqrt{2})$	$(1+\sqrt{2}, +\infty)$
Y'	+	−	+
Y	↗	↘	↗
	MAX	MIN	

rast i opadanje

ekstremi f-je

Na osnovu tabele rasta i opadanja vidimo da f-ja ima maksimum za $x=1-\sqrt{2}$, i minimum za $x=1+\sqrt{2}$.

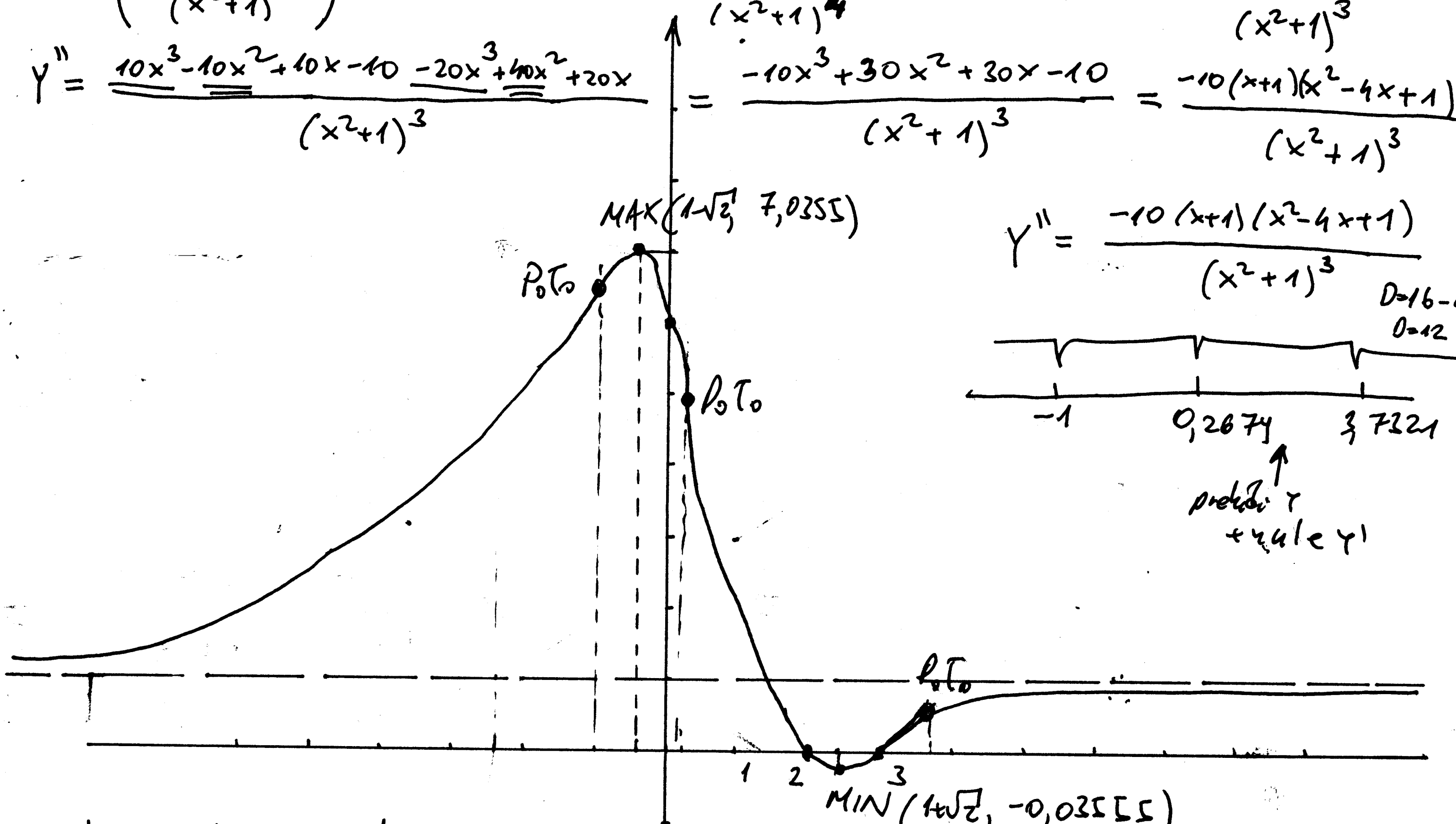
$$f(1-\sqrt{2})=\frac{(1-\sqrt{2})^2-5\cdot(1-\sqrt{2})+6}{(1-\sqrt{2})^2+1}\approx 7{,}0355 \qquad (1-\sqrt{2},\ 7{,}0355)\ \max$$

$$f(1+\sqrt{2})=\frac{(1+\sqrt{2})^2-5(1+\sqrt{2})+6}{(1+\sqrt{2})^2+1}\approx -0{,}0355 \qquad (1+\sqrt{2},\ -0{,}0355)\ \min$$

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(\frac{5x^2-10x-5}{(x^2+1)^2}\right)'=\frac{(10x-10)(x^2+1)^2-(5x^2-10x-5)2(x^2+1)\cdot 2x}{(x^2+1)^4}=\frac{10(x-1)(x^2+1)-(5x^2-10x-5)4x}{(x^2+1)^3}$$

$$Y''=\frac{10x^3-10x^2+10x-10-20x^3+40x^2+20x}{(x^2+1)^3}=\frac{-10x^3+30x^2+30x-10}{(x^2+1)^3}=\frac{-10(x+1)(x^2-4x+1)}{(x^2+1)^3}$$



x	$(-\infty,-1)$	$(-1, 0{,}2679)$	$(0{,}2679, 3{,}7321)$	$(3{,}7321, +\infty)$
Y''	+	−	+	−
Y	∪	∩	∪	∩
	PoTo	PoTo	PoTo	

konveksnost i konkavnost