Univerzitet u Zenici
Fakultet za metalurgiju i materijale
Osjek za hemiju

Zadaci sa pismenog ispita iz predmeta Matematika 1, obe grupe, rađen 16.02.2009.

1. Koliko racionalnih članova ima u razvoju $(\sqrt[3]{3}+\sqrt[5]{2})^{27}$?

2. Naći sva rješenja od $\sqrt[3]{z}$, i predstaviti ih u kompleksnoj ravni ako je $z=(\sqrt{3}-i)^2(-\sqrt{3}+i)$.

3. Diskutovati rješenja sistema u zavisnosti od parametra λ

$$\begin{aligned} x+y+z &= 2 \\ x+(\lambda+1)y+2z &= -2 \\ x+3y+(\lambda+2)z &= -3\lambda \ . \end{aligned}$$

4. Odrediti parametar λ tako da prvršina trougla $\triangle ABC$ iznosi $\frac{15}{2}$ ako su $A(-2,2,1)$, $B(2,\lambda+2,4)$ i $C(2,7,4)$. Za nađenu vrijednost λ izraziti vektor $\vec{d}=8\vec{i}+13\vec{j}+6\vec{k}$ preko vektora $\overrightarrow{AB}$ i $\overrightarrow{AC}$.

5. Vektori $\vec{a}(3,1,1)$, $\vec{b}(3,\lambda+1,2)$ i $\vec{c}(3,4,\lambda+3)$ su ivice tetraedra.
a) Odrediti zapreminu tog tetraedra.
b) Odrediti λ tako da vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ budu komplanarni, pa za nađene vrijednosti parametra λ izraziti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

6. Napisati jednačinu ravni koja prolazi kroz tačku $M(-2,-1,2)$ i normalna je na ravnima $\alpha:\ 4x+7y+2z-3=0$ i $\beta:\ 5x+6y+2z-8=0$.

7. Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases} -x+2y+z+1=0 \\ x+3y+2z+1=0 \end{cases}$ i normalna je na ravan $x-y+3z+2=0$.

8. Izračunati: $\lim\limits_{n\to\infty}[\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+...+\frac{1}{(5n-3)\cdot(5n+2)}]$.

1. Koliko racionalnih članova ima u razvoju $(\sqrt[3]{3}+\sqrt[5]{2})^{27}$?

Rj.
$$(\sqrt[3]{3}+\sqrt[5]{2})^{27}=\sum_{k=0}^{27}\binom{27}{k}(\sqrt[3]{3})^{27-k}\cdot(\sqrt[5]{2})^{k}=\sum_{k=0}^{27}\binom{27}{k}3^{\frac{27-k}{3}}\cdot 2^{\frac{k}{5}}=$$
$$=\sum_{k=0}^{27}\binom{27}{k}3^{9-\frac{k}{3}}\cdot 2^{\frac{k}{5}}.$$

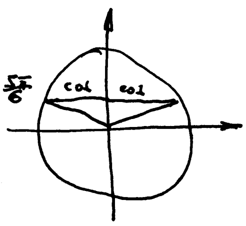
Racionalne članove ćemo imati ako su $9-\frac{k}{3}$ i $\frac{k}{5}$ cijeli brojevi tj. ako je k djeljiv sa 3 i sa 5.
Jedini brojevi $k\in\{0,1,\dots,27\}$ koji su djeljivi sa 3 i 5 su $k=0$ i $k=15$. Prema tome postoje dva člana u razvoju koji su racionalni (prvi i šesnesti član).

2. Naći sve vrijednosti od $\sqrt[3]{z}$, i predstaviti ih u kompleksnoj ravni ako je $z=(\sqrt{3}-i)^2\cdot(-\sqrt{3}+i)$.

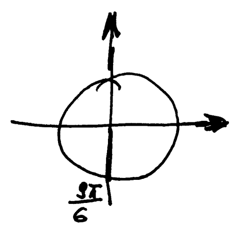
Rj.
$$(\sqrt{3}-i)^2=3-2i\sqrt{3}+i^2=2-2i\sqrt{3}$$
$$z=(\sqrt{3}-i)^2\cdot(-\sqrt{3}+i)=(2-2i\sqrt{3})(-\sqrt{3}+i)=\underline{-2\sqrt{3}}+2i+6i\underline{-2i^2\sqrt{3}}=8i$$
$$z=8i \qquad |z|=8$$
$$z=8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$\sqrt[3]{z}$ tražimo u obliku $z_k=\sqrt[3]{|z|}\left(\cos\frac{\varphi+2k\pi}{3}+i\sin\frac{\varphi+2k\pi}{3}\right)$, $k=0,1,2$

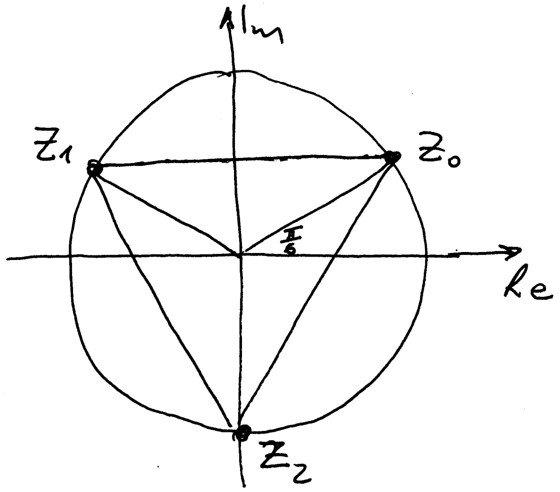
$$z_0=\sqrt[3]{8}\left(\cos\frac{\frac{\pi}{2}}{3}+i\sin\frac{\frac{\pi}{2}}{3}\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$
$$z_0=\sqrt{3}+i$$
$$z_1=\sqrt[3]{8}\left(\cos\frac{\frac{\pi}{2}+2\pi}{3}+i\sin\frac{\frac{\pi}{2}+2\pi}{3}\right)=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=2\left(-\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$
$$=2\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-\sqrt{3}+i$$



$$z_1=-\sqrt{3}+i$$



$$z_2=\sqrt[3]{8}\left(\cos\frac{\frac{\pi}{2}+4\pi}{3}+i\sin\frac{\frac{\pi}{2}+4\pi}{3}\right)=2\left(\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}\right)=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$$
$$z_2=-2i$$



vrijednosti z_0, z_1 i z_2 predstavljene u kompleksnoj ravni

3. Diskutovati rješenja sistema u zavisnosti od parametra λ

$$\begin{aligned} x + y + z &= 2 \\ x + (\lambda+1)y + 2z &= -2 \\ x + 3y + (\lambda+2)z &= -3\lambda . \end{aligned}$$

Rj.

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda+1 & 2 \\ 1 & 3 & \lambda+2 \end{vmatrix} \underset{III_v - I_v}{\overset{II_v - I_v}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 2 & \lambda+1 \end{vmatrix} = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$$

$$D_x = \begin{vmatrix} 2 & 1 & 1 \\ -2 & \lambda+1 & 2 \\ -3\lambda & 3 & \lambda+2 \end{vmatrix} \underset{III_v - II_v}{\overset{I_v - II_v \cdot 2}{=}} \begin{vmatrix} 0 & 1 & 0 \\ -2\lambda-4 & \lambda+1 & -\lambda+1 \\ -3\lambda-6 & 3 & \lambda-1 \end{vmatrix} = -\begin{vmatrix} -2\lambda-4 & -\lambda+1 \\ -3\lambda-6 & \lambda-1 \end{vmatrix} \overset{I_v + II_v}{=}$$

$$= -\begin{vmatrix} -5\lambda-10 & 0 \\ -3\lambda+6 & \lambda-1 \end{vmatrix} = -(-5\lambda^2 - 10\lambda + 5\lambda + 10) = 5(\lambda^2 + \lambda - 2) = 5(\lambda+2)(\lambda-1)$$

$$D_y = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 1 & -3\lambda & \lambda+2 \end{vmatrix} \underset{III_v - I_v}{\overset{II_v - I_v}{=}} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & -3\lambda-2 & \lambda+1 \end{vmatrix} = -4\lambda - 4 + 3\lambda + 2 = -\lambda - 2 = -(\lambda+2)$$

$$D_z = \begin{vmatrix} 1 & 1 & 2 \\ 1 & \lambda+1 & -2 \\ 1 & 3 & -3\lambda \end{vmatrix} \underset{III_v - I_v}{\overset{II_v - I_v}{=}} \begin{vmatrix} 1 & 1 & 2 \\ 0 & \lambda & -4 \\ 0 & 2 & -3\lambda-2 \end{vmatrix} = -3\lambda^2 - 2\lambda + 8 = -(\lambda+2)(3\lambda-4)$$

Diskusija

1° Za $\lambda \neq -2$ i $\lambda \neq 1$ sistem ima jedinstveno rješenje

$$\left(5,\ -\frac{1}{\lambda-1},\ -\frac{3\lambda-4}{\lambda-1}\right)$$

2° Za $\lambda = 1$, $D = 0$, $D_x = 0$, $D_y = -3 \Rightarrow$ sistem nema rješenja

3° Za $\lambda = -2$, $D = D_x = D_y = D_z = 0$ sistem postoje

$$\begin{aligned} x + y + z &= 2 \quad (1) \\ x - y + 2z &= -2 \quad (2) \\ x + 3y \quad &= 6 \quad (3) \end{aligned} \Rightarrow$$

$$\begin{aligned} (1)-(3):&\ 2y - z = 4 \\ (2)-(3):&\ -4y + 2z = -8 \\ \hline &\ 2y - z = 4 \end{aligned}$$

$z = 2y - 4$

$x = 6 - 3y$

Postoji ∞ mnogo rješenja oblika

$(6-3t,\ t,\ 2t-4)$

$t \in \mathbb{R}$

4. Odrediti parametar λ tako da površina ΔABC iznosi $\frac{15}{2}$ ako su $A(-2,2,1)$, $B(2,\lambda+2,4)$ i $C(2,7,4)$. Za nađenu vrijednost λ izraziti vektor $\vec{d}=8\vec{i}+13\vec{j}+6\vec{k}$ preko vektora $\overrightarrow{AB}$ i $\overrightarrow{AC}$.

Rj.

$A(-2,2,1)$
$B(2,\lambda+2,4)$
$C(2,7,4)$

$\overrightarrow{AB}(4,\lambda,3)$
$\overrightarrow{AC}(4,5,3)$

$$P_{\Delta ABC}=\frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{2} \Rightarrow 2\cdot P_{\Delta ABC}=|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & \lambda & 3\\ 4 & 5 & 3\end{vmatrix}=(3\lambda-15)\vec{i}-0\,\vec{j}+(20-4\lambda)\vec{k}$$

$$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(3\lambda-15)^2+(20-4\lambda)^2}=\sqrt{25\lambda^2-250\lambda+625}$$

$$\sqrt{25\lambda^2-250\lambda+625}=15$$

$$5\sqrt{\lambda^2-10\lambda+25}=15$$

$$5\sqrt{(\lambda-5)^2}=15$$

$$5(\lambda-5)=15$$

$$\lambda=8$$

Za $\lambda=8$ $P_{\Delta ABC}$ iznosi $\frac{15}{2}$.

$$\vec{d}=\alpha\overrightarrow{AB}+\beta\overrightarrow{AC}$$

$$(8,13,6)=\alpha(4,8,3)+\beta(4,5,3) \Rightarrow \alpha=1 \text{ i } \beta=1$$

$\vec{d}=\overrightarrow{AB}+\overrightarrow{AC}$ razlaganje vektora $\vec{d}$ preko vektora $\overrightarrow{AB}$ i $\overrightarrow{AC}$.

5. Vektori $\vec{a}(3,1,1)$, $\vec{b}(3,\lambda+1,2)$ i $\vec{c}(3,4,\lambda+3)$ su ivice tetraedra.
a) odrediti zapreminu tog tetraedra
b) odrediti λ tako da vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ budu komplanarni, pa za nađene vrijednosti parametra λ izraziti vektor $\vec{a}$ preko vektora $\vec{b}$ i $\vec{c}$.

Rj. a) zapremina tetraedra $V=\frac{1}{6}|(\vec{a}\times\vec{b})\cdot\vec{c}|$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}3 & 1 & 1\\ 3 & \lambda+1 & 2\\ 3 & 4 & \lambda+3\end{vmatrix} \overset{II_v-I_v}{\underset{III_v-I_v}{=}} \begin{vmatrix}3 & 1 & 1\\ 0 & \lambda & 1\\ 0 & 3 & \lambda+2\end{vmatrix}=3(\lambda^2+2\lambda-3)=$$
$$=3(\lambda+3)(\lambda-1)$$

$$V=\frac{1}{2}(\lambda+3)(\lambda-1)$$

b) Za $\lambda=1$ i $\lambda=-3$ vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarni.

za $\lambda=1 \Rightarrow \vec{a}=\frac{3}{2}\vec{b}-\frac{1}{2}\vec{c}$; Za $\lambda=-3 \Rightarrow \vec{a}=\frac{1}{2}\vec{b}+\frac{1}{2}\vec{c}$.

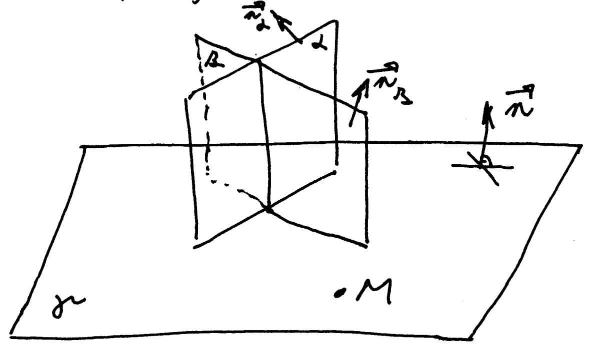
6. Napisati jednačinu ravni koja prolazi kroz tačku $M(-2,-1,2)$ i normalna je na ravnima

$\alpha: 4x+7y+2z-3=0$ i $\beta: 5x+6y+2z-8=0$.

Rj.

γ je tražena ravan $A(x-x_1)+B(y-y_1)+C(z-z_1)$

gdje je $x_1=-2$, $y_1=-1$ i $z_1=2$.



$$\left.\begin{array}{ll}\vec{n}_\alpha(4,7,2) & \vec{n}\perp\vec{n}_\alpha\\ \vec{n}_\beta(5,6,2) & \vec{n}\perp\vec{n}_\beta\end{array}\right\}\Rightarrow \vec{n}\parallel \vec{n}_\alpha\times\vec{n}_\beta$$

$$\Downarrow$$

$$\vec{n}=k\cdot(\vec{n}_\alpha\times\vec{n}_\beta)$$

$k\in\mathbb{R}$

$$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 7 & 2\\ 5 & 6 & 2\end{vmatrix}=(2,2,-11)$$

$$\vec{n}(2,2,-11)\Rightarrow \gamma:\quad 2(x+2)+2(y+1)-11(z-2)=0$$

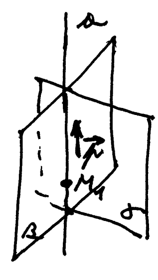
$$2x+2y-11z+16=0$$ jednačina tražene ravni

7. Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases}-x+2y+z+1=0\\ x+3y+2z+1=0\end{cases}$ i normalna je na ravan $x-y+3z+2=0$.

Rj. $\beta: -x+2y+z+1=0$

$\gamma: x+3y+2z+1=0$

α je tražena ravan



$$\left.\begin{array}{l}\vec{p}\perp\vec{n}_\beta\\ \vec{p}\perp\vec{n}_\gamma\end{array}\right.\Rightarrow \vec{p}\parallel\vec{n}_\beta\times\vec{n}_\gamma\Rightarrow \vec{p}=k(\vec{n}_\beta\times\vec{n}_\gamma)$$

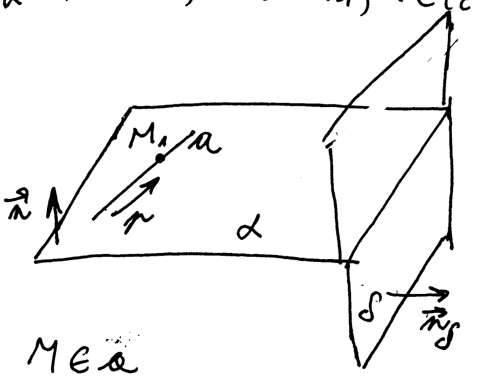
$k\in\mathbb{R}$

$$\vec{n}_\beta\times\vec{n}_\gamma=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & 1\\ 1 & 3 & 2\end{vmatrix}=(1,3,-5)$$

$$\Downarrow$$

$$\vec{p}(1,3,-5)$$

$\alpha: A(x-x_1)+B(y-y_1)+C(z-z_1)=0$



$\delta: x-y+3z+2=0$

$$\left.\begin{array}{l}\vec{n}\perp\vec{n}_\delta\\ \vec{n}\perp\vec{p}\end{array}\right.\Rightarrow \vec{n}\parallel\vec{n}_\delta\times\vec{p}\Rightarrow \vec{n}=k(\vec{n}_\delta\times\vec{p})$$

$k\in\mathbb{R}$

$$\vec{n}_\delta\times\vec{p}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 3\\ 1 & 3 & -5\end{vmatrix}=(-4,8,4)\Rightarrow \vec{n}(\overset{A}{-1},\overset{B}{2},\overset{C}{1})$$

$M\in a$

$\beta+\gamma: 5y+3z+2=0$

$z=1$, $y=-1$ $\Rightarrow$ $x=0$ $\quad M(0,-1,1)$ $\quad -1(x-0)+2(y+1)+1(z-1)=0$

$$-x+2y+z+1=0$$ jednačina tražene ravni

8. Izračunati: $\lim\limits_{n\to\infty}\left[\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\dots+\frac{1}{(5n-3)\cdot(5n+2)}\right]$.

Rj.

$$\frac{1}{(5n-3)(5n+2)}=\frac{A}{5n-3}+\frac{B}{5n+2} \quad /\cdot(5n-3)(5n+2)$$

$$1=A(5n+2)+B(5n-3)$$

$$5A+5B=0$$
$$2A-3B=1$$

$$A+B=0$$
$$2A-3B=1$$

$$2A+2B=0 \quad (1)$$
$$-\quad 2A-3B=1 \quad (2)$$

(1)-(2): $5B=-1$

$$B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$$

$$\frac{1}{(5n-3)(5n+2)}=\frac{\frac{1}{5}}{5n-3}+\frac{-\frac{1}{5}}{5n+2}=\frac{1}{5}\left(\frac{1}{5n-3}-\frac{1}{5n+2}\right)$$

$$\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\dots+\frac{1}{(5n-3)(5n+2)}=$$

$$=\frac{1}{5}\left(\frac{1}{2}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}-\cancel{\frac{1}{12}}+\dots+\cancel{\frac{1}{5n-3}}-\frac{1}{5n+2}\right)=$$

$$=\frac{1}{5}\left(\frac{1}{2}-\frac{1}{5n+2}\right)$$

$$\lim_{n\to\infty}\left[\frac{1}{2\cdot 7}+\frac{1}{7\cdot 12}+\dots+\frac{1}{(5n-3)(5n+2)}\right]=$$

$$=\lim_{n\to\infty}\frac{1}{5}\left(\frac{1}{2}\underbrace{-\frac{1}{5n+2}}_{\to 0}\right)=\frac{1}{10}$$

tražena vrijednost