1. Nađi sve vrijednosti $\sqrt[3]{z}$ ako je $z=(\sqrt{3}-i)^2(1+i\sqrt{3})^2$.

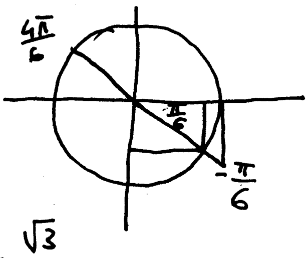
Rj. $z=z_1^2\cdot z_2^2$ $\quad tg\,30^\circ=\frac{\sqrt{3}}{3}$ $\quad \varphi_1=-\frac{\pi}{6}$ $\quad z_1=\sqrt{3}-i=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

$z_1=\sqrt{3}-i$

$|z_1|=2$

$\sin\varphi_1=\frac{-1}{2}$ $\quad tg\,\varphi_1=-\frac{\sqrt{3}}{3}$

$\cos\varphi_1=\frac{\sqrt{3}}{2}$



$z_2=1+i\sqrt{3}$

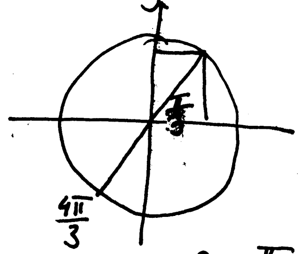
$|z_2|=2$

$\cos\varphi_2=\frac{1}{2}$

$\sin\varphi_2=\frac{\sqrt{3}}{2}$

$tg\,\varphi_2=\sqrt{3}$

$tg\,60^\circ=\sqrt{3}$

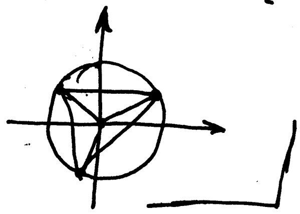


$z_2=1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ $\quad \varphi_2=\frac{\pi}{3}$

$z_1^2=2^2\left[\cos\left(-\frac{2\pi}{6}\right)+i\sin\left(-\frac{2\pi}{6}\right)\right]$ $\quad \frac{\pi}{9}=20^\circ$

$z_2^2=2^2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

rj.



$z=z_1^2\cdot z_2^2=4\cdot 4\left[\cos\left(\frac{2\pi}{3}-\frac{2\pi}{6}\right)+i\sin\left(\frac{2\pi}{3}-\frac{2\pi}{6}\right)\right]=16\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$\sqrt[3]{z}$ ima vrijednosti $z_k=\sqrt[3]{|z|}\left(\cos\frac{\varphi+2k\pi}{3}+i\sin\frac{\varphi+2k\pi}{3}\right)$, $k=0,1,2$

$z_0=\sqrt[3]{16}\left(\cos\frac{\frac{\pi}{3}}{3}+i\sin\frac{\frac{\pi}{3}}{3}\right)=\sqrt[3]{16}\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)=2\sqrt[3]{2}\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)$

$z_1=\sqrt[3]{16}\left(\cos\frac{\frac{\pi}{3}+2\pi}{3}+i\sin\frac{\frac{\pi}{3}+2\pi}{3}\right)=2\sqrt[3]{2}\left(\cos\frac{7\pi}{9}+i\sin\frac{7\pi}{9}\right)$

$z_2=\sqrt[3]{16}\left(\cos\frac{\frac{\pi}{3}+4\pi}{3}+i\sin\frac{\frac{\pi}{3}+4\pi}{3}\right)=2\sqrt[3]{2}\left(\cos\frac{13\pi}{9}+i\sin\frac{13\pi}{9}\right)$

2. Diskutovati rješenja sistema za razne vrijednosti parametra λ

$$\lambda x+2y+4z=5$$
$$3x+y+(\lambda+2)z=4$$
$$x+y+z=2$$

Rj.

$$D=\begin{vmatrix}\lambda & 2 & 4\\ 3 & 1 & \lambda+2\\ 1 & 1 & 1\end{vmatrix}\ \underset{III_k-II_k}{\overset{I_k-II_k}{=\!=}}\ \begin{vmatrix}\lambda-2 & 2 & 2\\ 2 & 1 & \lambda+1\\ 0 & 1 & 0\end{vmatrix}=(-1)\begin{vmatrix}\lambda-2 & 2\\ 2 & \lambda+1\end{vmatrix}=(-1)\cdot\left[(\lambda-2)(\lambda+1)-4\right]$$

$$=(-1)(\lambda^2-\lambda-2)=(-1)(\lambda+2)(\lambda-3)$$

$$D_x = \begin{vmatrix} 5 & 2 & 4 \\ 4 & 1 & \lambda+2 \\ 2 & 1 & 1 \end{vmatrix} = (-1)(\lambda-3) \qquad D_y = \begin{vmatrix} \lambda & 5 & 4 \\ 3 & 4 & \lambda+2 \\ 1 & 2 & 1 \end{vmatrix} = -(2\lambda+1)(\lambda-3)$$

$$D_z = \begin{vmatrix} \lambda & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = (-2)(\lambda-3)$$

Diskusija:

1° $D \neq 0 \Rightarrow \lambda \neq 3$ i $\lambda \neq -2$

Sistem ima jedinstveno rješenje $x = \frac{D_x}{D} = \frac{(-1)(\lambda-3)}{(-1)(\lambda+2)(\lambda-3)} = \frac{1}{\lambda+2}$

$$y = \frac{D_y}{D} = \frac{(-1)(\lambda-3)(2\lambda+1)}{(-1)(\lambda+2)(\lambda-3)} = \frac{2\lambda+1}{\lambda+2}$$

$$z = \frac{D_z}{D} = \frac{(-2)(\lambda-3)}{(-1)(\lambda-3)(\lambda+2)} = \frac{1}{\lambda+2}$$ rj. $\left(\frac{1}{\lambda+2}, \frac{2\lambda+1}{\lambda+2}, \frac{1}{\lambda+2}\right)$, $\lambda \in \mathbb{R}$

2° $\lambda = 3 \Rightarrow D = D_x = D_y = D_z = 0$

sistem postaje
$$\begin{aligned} 3x + 2y + 4z &= 5 \\ 3x + y + 5z &= 4 \\ x + y + z &= 2 \quad / \cdot 3 \end{aligned}$$

$$\begin{aligned} 3x+2y+4z &= 5 \\ 3x+y+5z &= 4 \\ 3x+3y+3z &= 6 \\ \hline -y+z &= -1 \\ -2y+2z &= -2 \\ \hline y &= z+1 \end{aligned}$$

$x = 2 - z - y$

$x = 2 - z - z - 1$

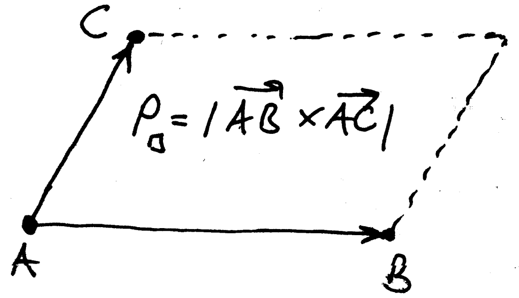
$x = -2z + 1$

sistem ima ∞ mnogo rješenja oblika $(-2t+1, t+1, t)$

3° $\lambda = -2 \Rightarrow D = 0, D_x \neq 0$ sistem nema rješenja

(3.) Odrediti parametar λ tako da $\triangle ABC$ ima površinu 14 ako su $A(1,2,-1)$, $B(4,3,-1)$, i $C(-3\lambda+4, 2, 2\lambda-3)$. Pored toga odrediti i visinu trougla koja odgovara stranici AB.

Rj. $\left.\begin{matrix} A(1,2,-1) \\ B(4,3,-1) \end{matrix}\right\} \overrightarrow{AB} = (3,1,0)$ $\qquad \left.\begin{matrix} A(1,2,-1) \\ C(-3\lambda+4, 2, 2\lambda-3) \end{matrix}\right\} \overrightarrow{AC} = (-3\lambda+3, 0, 2\lambda-2)$



$$P_{\Delta ABC} = \frac{P_{\square}}{2} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ -3\lambda+3 & 0 & 2\lambda-2 \end{vmatrix} = (2\lambda-2)\vec{i} - 6(\lambda-1)\vec{j} + -(-3\lambda+3)\vec{k} =$$

$$= (2(\lambda-1), -6(\lambda-1), 3(\lambda-1))$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{4(\lambda-1)^2 + 36(\lambda-1)^2 + 9(\lambda-1)^2}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{49(\lambda-1)^2}$$

$$|\vec{AB} \times \vec{AC}| = 7|\lambda-1|$$

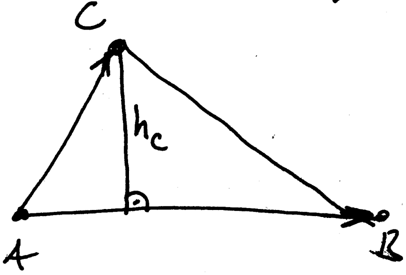
$$P_{\Delta ABC} = \frac{|\vec{AB} \times \vec{AC}|}{2}$$

$$14 = \frac{7|\lambda-1|}{2}$$

$$|\lambda-1| = 4$$

$$\lambda_1 = -3 \quad \lambda_2 = 5$$

Za vrijednost parametra $\lambda=-3$ ili $\lambda=5$ površina ΔABC je 14.



$$P_{\Delta ABC} = \frac{|\vec{AB}| \cdot h_c}{2}$$

$$|\vec{AB}| = \sqrt{9+1} = \sqrt{10}$$

$$14 = \frac{\sqrt{10} \cdot h_c}{2}$$

$$\sqrt{10}\, h_c = 28$$

$$h_c = \frac{28\sqrt{10}}{10}$$

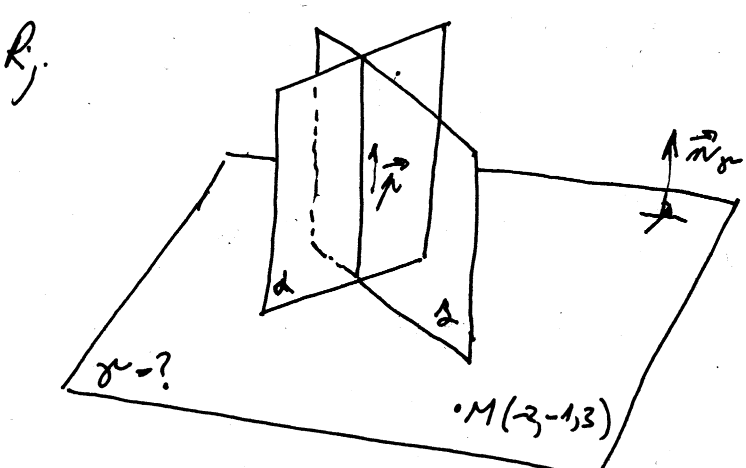
Visina trougla koja odgovara stranici AB iznosi $\frac{28\sqrt{10}}{10}$.

4. Napisati jednačinu ravni koja prolazi kroz tačku $M(-2,-1,3)$ i normalna je na ravnima

$\alpha: -3x+2y-2z+4=0$

$\beta: x-2y+z-5=0.$

Rj.



$$\vec{n}_\alpha = (-3, 2, -2)$$

$$\vec{n}_\beta = (1, -2, 1)$$

$$\left.\begin{array}{l} \vec{n}_\alpha \perp \vec{p} \\ \vec{n}_\beta \perp \vec{p} \end{array}\right\} \Rightarrow \vec{p} \parallel \vec{n}_\alpha \times \vec{n}_\beta$$

$$\Downarrow$$

$$\vec{p} = k \cdot (\vec{n}_\alpha \times \vec{n}_\beta)$$

$$\vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & -2 \\ 1 & -2 & 1 \end{vmatrix} = (2-4)\vec{i} - (-3+2)\vec{j} + (6-2)\vec{k} = (-2, 1, 4)$$

Za vektor normale tražene ravni mogu uzeti $\vec{n_\gamma}=(-2,1,4)$

$\vec{n_\gamma}=(A,B,C)$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$$-2(x-(-2))+1(y-(-1))+4(z-3)=0$$

$$-2(x+2)+(y+1)+4(z-3)=0$$

$$-2x+y+4z-4+1-12=0$$

$$-2x+y+4z-15=0$$ jednačina tražene ravni

(1.) Ispitati i grafički predstaviti f-ju $y=\frac{x^2-9}{1+x}$.

Rj. $\mathcal{D}$: $x\in(-\infty,-1)\cup(-1,+\infty)$

nije ni parna ni neparna

nije periodična

$(0,-9)$ presjek sa y-osom

$(-3,0)$ i $(3,0)$ nule f-je

-1 je tačka prekida f-je

$\lim\limits_{x\to-1-0} f(x)=+\infty$ $\Rightarrow$ $x=-1$ je V.A.

$\lim\limits_{x\to-1+0} f(x)=-\infty$

$-3 \quad -1 \quad 3$

x	$(-\infty,-3)$	$(-3,-1)$	$(-1,3)$	$(3,+\infty)$
$(x-3)$	$-$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$	$+$
$1+x$	$-$	$-$	$+$	$+$
y	$-$	$+$	$-$	$+$

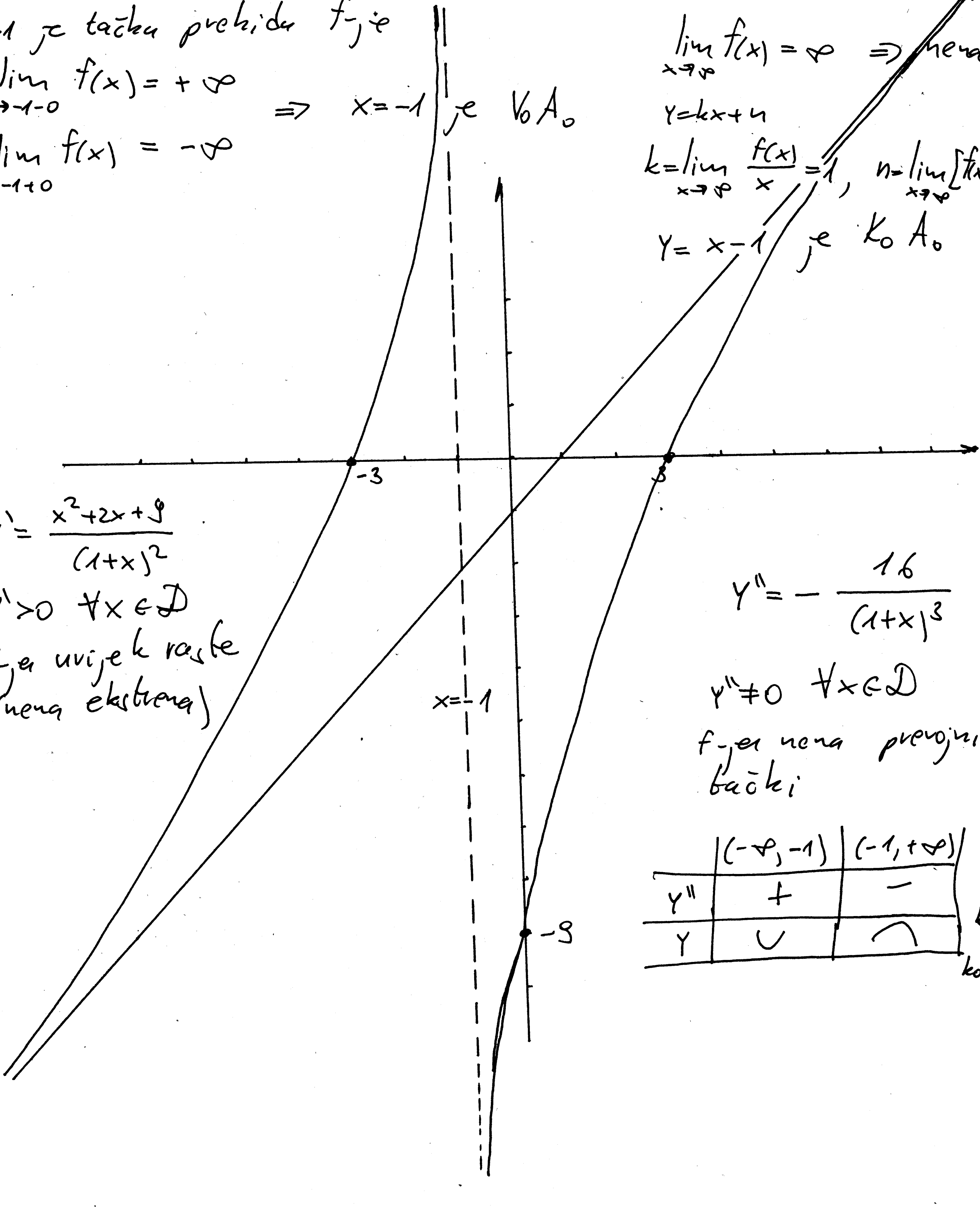
znak f-je

$\lim\limits_{x\to\infty} f(x)=\infty$ $\Rightarrow$ nema H.A.

$y=kx+n$

$k=\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$, $n=\lim\limits_{x\to\infty}[f(x)-kx]=-1$

$y=x-1$ je K.A.



$y'=\frac{x^2+2x+9}{(1+x)^2}$

$y'>0$ $\forall x\in\mathcal{D}$

f-ja uvijek raste (nema ekstrema)

$y''=-\frac{16}{(1+x)^3}$

$y''\neq0$ $\forall x\in\mathcal{D}$

f-ja nema prevojnih tački

	$(-\infty,-1)$	$(-1,+\infty)$
y''	$+$	$-$
y	$\cup$	$\cap$

konveksnost i konkavnost

2. odrediti: $\displaystyle\int \frac{2x+3}{\sqrt{x^2-x+1}}\,dx$.

Rj. I način: Metoda Ostrogradskog

II način: $$\int \frac{2x+3}{\sqrt{x^2-x+1}}\,dx = \int \frac{2x-1+4}{\sqrt{x^2-x+1}}\,dx = \int \frac{2x-1}{\sqrt{x^2-x+1}}\,dx + 4\int \frac{dx}{\sqrt{x^2-x+1}}$$

$$\int \frac{2x-1}{\sqrt{x^2-x+1}}\,dx = \left| \begin{array}{l} x^2-x+1=t^2 \\ (2x-1)\,dx = 2t\,dt \\ t=\sqrt{x^2-x+1} \end{array} \right| = \int \frac{2t\,dt}{\sqrt{t^2}} = 2t + C_1 = 2\sqrt{x^2-x+1} + C_1$$

$$\int \frac{dx}{\sqrt{x^2-x+1}} = \left| x^2-x+1 = x^2 - 2\cdot\tfrac{1}{2}x + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 + 1 = \left(x-\tfrac{1}{2}\right)^2 + \tfrac{3}{4} \right| =$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}} = \left| \begin{array}{l} x-\frac{1}{2} = \frac{\sqrt{3}}{2}t \\ dx = \frac{\sqrt{3}}{2}\,dt \\ t = \frac{2x-1}{\sqrt{3}} \end{array} \right| = \int \frac{\frac{\sqrt{3}}{2}\,dt}{\sqrt{\frac{3}{4}t^2 - \frac{3}{4}}} =$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{t^2-1}} = \ln\left| t + \sqrt{t^2-1} \right| + C_2$$

$$= \ln\left| \frac{2x-1}{\sqrt{3}} + \sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2 - 1} \right| + C_2$$

$$\int \frac{2x+3}{\sqrt{x^2-x+1}}\,dx = 2\sqrt{x^2-x+1} + \ln\left| \frac{\sqrt{3}}{3}(2x-1) + \sqrt{\frac{1}{3}(2x-1)^2 - 1} \right| + C$$

3. Odrediti područje konvergencije reda $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

Rj. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(x-1)^n$ ovo je stepeni red $\left(\sum_{n=1}^{\infty} a_n (x-a)^n\right)$,

ovaj red za $|x-a|<R$ KV

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\right| = \lim_{n\to\infty}\left|\sqrt{1+\frac{1}{n}}\right| = 1$$

$R=1$

$|x-1|<1$

$-1<x-1<1$

$x\in(0,2)$

Ispitajmo još konvergenciju reda u graničnim tačkama.

Za $x=0$ imamo red $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Ovo je naizmenični red.

Teorema: Neka je dat naizmenični red $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. Ako je $\{a_n\}$ opadajući niz koji konvergira nuli (0) tad naizmenični red KV.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$. Kako je $\left\{\frac{1}{\sqrt{n}}\right\}$ opadajući niz koji KV 0

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \quad \text{KV}.$$

Za $x=2$ red izgleda $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Ovo je red sa pozitivnim članovima. Možemo primjeniti kriterij poređenja

$n>\sqrt{n} \Rightarrow \frac{1}{n}<\frac{1}{\sqrt{n}}$, kako $\sum_{n=1}^{\infty}\frac{1}{n}$ DV to i $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DV.

Područje KV reda je $[0,2)$.

4. Riješiti diferencijalnu jednačinu $y' + \frac{2}{x} y = \frac{2\sqrt{y}}{\cos^2 x}$.

Rj: $y' + \frac{2}{x} y = \frac{2}{\cos^2 x} \sqrt{y}$ ovo je Bernulijeva difer. jedn. uvodimo smjenu $y = u \cdot v$, $y' = u'v + u \cdot v'$

$$u' \cdot v + u \cdot v' + \frac{2}{x} uv = \frac{2}{\cos^2 x} \sqrt{y}$$

$$u' \cdot v + \underbrace{u\left(v' + \frac{2}{x} v\right)}_{=0} = \frac{2}{\cos^2 x} \sqrt{y}$$

$$v' + \frac{2}{x} v = 0$$

$$v' = -\frac{2}{x} v$$

$$\frac{dv}{dx} = -\frac{2}{x} v$$

$$\frac{dv}{v} = -2 \frac{dx}{x} \quad / \int$$

$$\int \frac{dv}{v} = -2 \int \frac{dx}{x}$$

$$\ln|v| = -2 \ln|x|$$

$$v = x^{-2}$$

$$v = \frac{1}{x^2}$$

$$u' \cdot v = \frac{2}{\cos^2 x} \sqrt{u \cdot v}$$

$$u' \cdot \frac{1}{x^2} = \frac{2\sqrt{u \cdot \frac{1}{x^2}}}{\cos^2 x}$$

$$u' \cdot \frac{1}{x^2} = \frac{2}{\cos^2 x} \sqrt{\frac{u}{x^2}}$$

$$u' \cdot \frac{1}{x^2} = \frac{2\sqrt{u}}{x \cos^2 x} \quad / \cdot x^2$$

$$u' = \frac{2x\sqrt{u}}{\cos^2 x}$$

$$\frac{du}{dx} = \frac{2x\sqrt{u}}{\cos^2 x}$$

$$\frac{du}{\sqrt{u}} = \frac{2x}{\cos^2 x} dx \quad / \int$$

$$\int \frac{du}{\sqrt{u}} = 2 \int \frac{x}{\cos^2 x} dx$$

$$\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{u}$$

$$\int \frac{x}{\cos^2 x} dx = \left| \begin{matrix} u = x & dv = \frac{dx}{\cos^2 x} \\ du = dx & v = \operatorname{tg} x \end{matrix} \right| = x \operatorname{tg} x - \int \operatorname{tg} x \, dx$$

$$\int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} dx = \left| \begin{matrix} \cos x = t \\ -\sin x \, dx = dt \end{matrix} \right| = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C$$

$$\int \frac{x}{\cos^2 x} dx = x \operatorname{tg} x + \ln|\cos x| + C$$

$$2\sqrt{u} = 2(x \operatorname{tg} x + \ln|\cos x| + C)$$

$$u = (x \operatorname{tg} x + \ln|\cos x| + C)^2$$

$$y = uv = \frac{(x \operatorname{tg} x + \ln|\cos x| + C)^2}{x^2}$$

opšte rješenje diferencijalne jednačine