Univerzitet u Zenici
Fakultet za metalurgiju i materijale

Zadaci sa pismenog ispita iz predmeta MATEMATIKA 1, 15.06.2009.

1. Odrediti koji član u razvoju binoma $(\sqrt[3]{x} + \dfrac{2}{\sqrt[6]{x}})^{12}$ ne sadrži x.

2. Riješiti matričnu jednačinu $AX + B = CX$
ako su $A = \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -26 & -12 \\ -2 & -4 \end{bmatrix}$ i $C = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

3. Data su tačke $A(2,1,3)$, $B(4,\lambda+1,7)$, $C(1,7,1)$ i $D(5,8,-6)$. Odrediti parametar λ tako da zapremina tetraedra $ABCD$ iznosi 45.

4. Odrediti tačku koja je simetrična tački $M(1,9,1)$ u odnosu na ravan $\alpha:\ 2x + y + 3z = 0$.

Zadaci sa pismenog ispita iz predmeta MATEMATIKA 2, 15.06.2009.

1. Ispitati i grafički predstaviti funkciju: $y = \dfrac{1-x^2}{x^2-4}$.

2. Odrediti: $\displaystyle\int \frac{x^4 - 4x^3 + 5x}{x^2 - 3x - 2}\, dx$.

3. Odrediti područje konvergencije reda $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{3^n}$.

4. Riješiti diferencijalnu jednačinu: $y' + 2xy = xe^{-x^2}$.

5. Naći ekstreme funkcije $z(x,y) = \dfrac{3}{2}x^2 + \dfrac{1}{2}xy^2 - 3xy$.

(Ova stranica je namjerno ostavljena prazna)

1. Odrediti koji član u razvoju binoma $\left(\sqrt[3]{x}+\frac{2}{\sqrt[6]{x}}\right)^{12}$ ne sadrži x.

Rj. $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$

$$\left(\sqrt[3]{x}+\frac{2}{\sqrt[6]{x}}\right)^{12}=\sum_{k=0}^{12}\binom{12}{k}\left(\sqrt[3]{x}\right)^{12-k}\left(\frac{2}{\sqrt[6]{x}}\right)^{k}$$

$$=\sum_{k=0}^{12}\binom{12}{k}x^{\frac{12-k}{3}}\left(2x^{-\frac{1}{6}}\right)^{k}=\sum_{k=0}^{12}\binom{12}{k}x^{4-\frac{k}{3}}\cdot 2^k\cdot x^{-\frac{k}{6}}$$

$$=\sum_{k=0}^{12}\binom{12}{k}2^k x^{4-\frac{k}{3}-\frac{k}{6}}$$

Za član koji ne sadrži x važi $x^{4-\frac{k}{3}-\frac{k}{6}}=1$ tj.

$$4-\frac{k}{3}-\frac{k}{6}=0 \quad /\cdot 6 \quad \Rightarrow \quad 24-2k-k=0$$

$$3k=24 \quad \Rightarrow \quad k=8$$

0, 1, 2, 3, 4, 5, 6, 7, 8, ..., 12
9 član

Deveti član u razvoju binoma ne sadrži x.

2. Riješiti matričnu jednačinu $AX+B=CX$ ako su

$$A=\begin{bmatrix}7&5\\3&1\end{bmatrix},\quad B=\begin{bmatrix}-26&-12\\-2&-4\end{bmatrix}\quad i \quad C=\begin{bmatrix}4&1\\2&3\end{bmatrix}.$$

Rj.

$$AX+B=CX$$

$$AX-CX=-B$$

$$(A-C)X=-B$$

$$A-C=\begin{bmatrix}7&5\\3&1\end{bmatrix}-\begin{bmatrix}4&1\\2&3\end{bmatrix}=\begin{bmatrix}3&4\\1&-2\end{bmatrix}$$

$$-B=-\begin{bmatrix}-26&-12\\-2&-4\end{bmatrix}=\begin{bmatrix}26&12\\2&4\end{bmatrix}$$

imamo:

$$\underbrace{\begin{bmatrix}3&4\\1&-2\end{bmatrix}}_{M}\cdot X=\underbrace{\begin{bmatrix}26&12\\2&4\end{bmatrix}}_{N}$$

$M\cdot X=N \quad /\cdot M^{-1}$ sa lijeve strane

$$M^{-1}\cdot M\cdot X=M^{-1}\cdot N$$

$$X=M^{-1}\cdot N$$

$M^{-1} = \frac{1}{\det M} \cdot M_{adj}$

$\det M = \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = -6-4 = -10$

$M_{adj} = [M_{kof}]^{-1}$

$M_{kof} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

$M_{11} = (-1)^2 \cdot (-2) = -2$

$M_{12} = (-1)^3 \cdot 1 = -1$

$M_{21} = (-1)^3 \cdot 4 = -4$

$M_{22} = (-1)^4 \cdot 3 = 3$

$M^{-1} = -\frac{1}{10} \begin{bmatrix} -2 & -4 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$

$X = M^{-1} \cdot N = \frac{1}{10} \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 26 & 12 \\ 2 & 4 \end{bmatrix}$

$X = \frac{1}{10} \begin{bmatrix} 60 & 40 \\ 20 & 0 \end{bmatrix}$

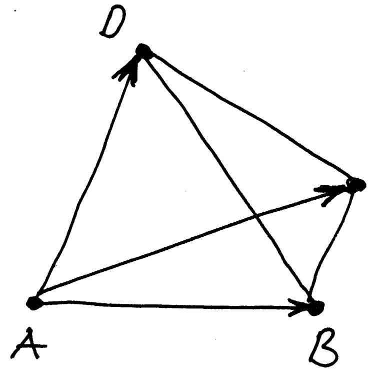
$X = \begin{bmatrix} 6 & 4 \\ 2 & 0 \end{bmatrix}$ je rešenje matrične jednačine

3. Date su tačke $A(2,1,3)$, $B(4, \lambda+1, 7)$, $C(1,7,1)$ i $D(5,8,-6)$. Odrediti parametar λ tako da zapremina tetraedra $ABCD$ iznosi 45.

Rj.

$A(2,1,3)$
$B(4, \lambda+1, 7)$
$C(1,7,1)$
$D(5,8,-6)$

$\overrightarrow{AB} = (2, \lambda, 4)$
$\overrightarrow{AC} = (-1, 6, -2)$
$\overrightarrow{AD} = (3, 7, -9)$



tetraedar (piramida)

$P_{tetraedra} = 45$

$P_{tetraedra} = \frac{1}{6} \left| (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} \right|$

$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} 2 & \lambda & 4 \\ -1 & 6 & -2 \\ 3 & 7 & -9 \end{vmatrix} \xlongequal{I_v + II_v \cdot 2} \begin{vmatrix} 0 & \lambda+12 & 0 \\ -1 & 6 & -2 \\ 3 & 7 & -9 \end{vmatrix} =$

$= -(\lambda+12) \begin{vmatrix} -1 & -2 \\ 3 & -9 \end{vmatrix} = -(\lambda+12) \cdot 15$

$\left| -15 \cdot (\lambda+12) \right| = 15 \, |\lambda+12|$

$P_{tetraedra} = \frac{1}{6} \left| (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} \right| = \frac{1}{6} \cdot 15 \, |\lambda+12| = 45$

$\frac{1}{6}|\lambda+12|=3 \quad |\cdot 6$

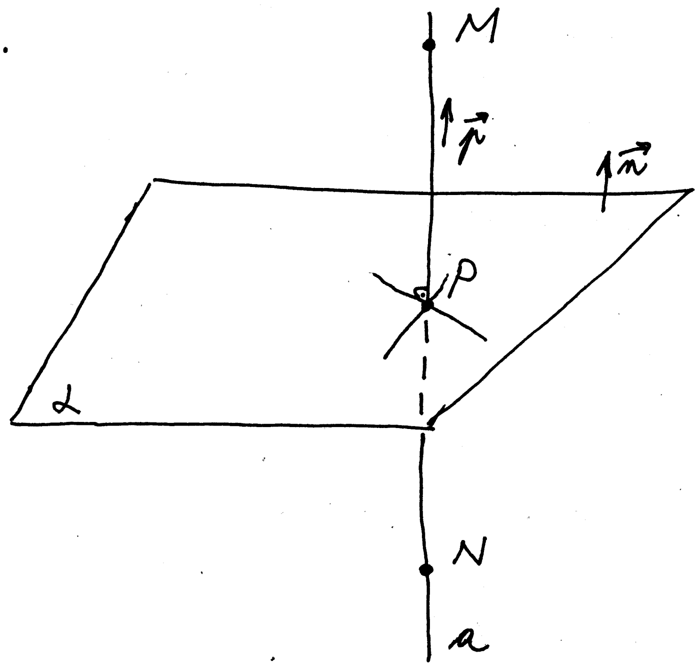
$|\lambda+12|=18$

$\lambda_1=6 \quad \lambda_2=-30$

Za vrijednosti $\lambda=6$ ili $\lambda=-30$ zapremina tetraedra ABCD iznosi 45.

4. Odrediti tačku koja je simetrična tački $M(1,9,1)$ u odnosu na ravan $\alpha: 2x+y+3z=0$.

Rj.



Kroz tačku M postavimo pravu koja je normalna na ravan α.

$$a: \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

$\vec{p}=(l,m,n)$, $\vec{n}=(2,1,3)$

$\alpha: 2x+y+3z=0$

U našem slučaju $\vec{p}=\vec{n} \Rightarrow a: \frac{x-1}{2}=\frac{y-9}{1}=\frac{z-1}{3} \; (=t)$

$\vec{p}=(2,1,3)$

$$\left.\begin{aligned} x-1&=2t\\ y-9&=t\\ z-1&=3t\end{aligned}\right. \Rightarrow \left.\begin{aligned} x&=2t+1\\ y&=t+9\\ z&=3t+1\end{aligned}\right\} \text{parametarski oblik jednačine prave}$$

Nađimo tačku P, presjeka ravni α i prave a.

$2x+y+3z=0$

$2(2t+1)+(t+9)+3(3t+1)=0$

$\underline{4t}+2+\underline{t}+9+\underline{9t}+3=0$

$14t+14=0$

$t=-1 \Rightarrow P(-1,8,-2)$

$x=2\cdot(-1)+1=-2+1$

$y=-1+9$

$z=3\cdot(-1)+1=-3+1$

Neka je N tačka simetrična tački M u odnosu na ravan α. Znamo $N \in a$ i $|\overrightarrow{MP}| = |\overrightarrow{NP}|$

$N \in a \Rightarrow N(2t+1, t+9, 3t+1)$

$M(1, 9, 1)$
$P(-1, 8, -2)$ $\Rightarrow \overrightarrow{MP} = (-2, -1, -3) \Rightarrow |\overrightarrow{MP}| = \sqrt{4+1+9} = \sqrt{14}$

$N(2t+1, t+9, 3t+1)$
$P(-1, 8, -2)$ $\Rightarrow \overrightarrow{NP} = (-2t-2, -t-1, -3t-3)$

$|\overrightarrow{NP}| = \sqrt{4t^2 + 8t + 4 + t^2 + 2t + 1 + 9t^2 + 18t + 9}$

$|\overrightarrow{NP}| = \sqrt{14t^2 + 28t + 14}$

$|\overrightarrow{MP}| = |\overrightarrow{NP}|$

$14t^2 + 28t + 14 = 14 \quad /:14$

$t^2 + 2t = 0$

$t(t+2) = 0$

$t_1 = 0 \quad t_2 = -2$

$N_1(1, 9, 1)$

$N_2(-3, 7, -5)$

$2t+1$
$t+9$
$3t+1$

Tačka koja je simetrična tački M u odnosu na ravan α je $N(-3, 7, -5)$.

1. Ispitati i grafički predstaviti f-ju $y=\frac{1-x^2}{x^2-4}$.

Rj. $y=\frac{1-x^2}{x^2-4}$, def. podr. $\mathcal{D}: x\in\mathbb{R}\setminus\{-2,2\}$ – $x\in(-\infty,-2)\cup(-2,2)\cup(2,+\infty)$

$\mathcal{D}$ simetrično, $f(-x)=\frac{1-(-x)^2}{(-x)^2-4}=\frac{1-x}{x^2-4}=f(x)$ $\Rightarrow$ f-ja je parna (simetrična u odnosu na y-osu)

F-ju je dovoljno ispitati za $y>0$:

$f(0)=-\frac{1}{4}$

$(0,-\frac{1}{4})$ je tačka presjeka sa y-osom

$-2 \quad -1 \quad 0 \quad 1 \quad 2$ ← prekidi f-je + nule f-je y

$y=0 \Rightarrow 1-x^2=0$

$x=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je

	$(0,1)$	$(1,2)$	$(2,+\infty)$
$1-x^2$	+	–	–
x^2-4	–	–	+
y	–	+	–

znak f-je

$x=2$ je tačka u kojoj f-ja ima prekid

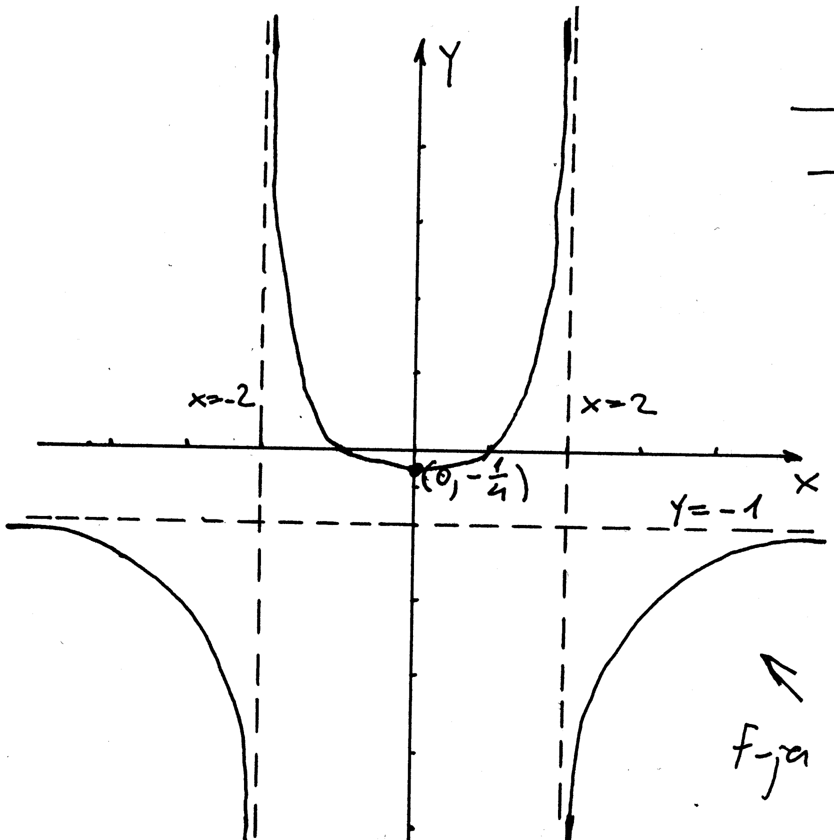
$$\lim_{x\to 2-0} f(x)=\lim_{x\to 2-0}\frac{1-x^2}{x^2-4}=\frac{1-(2-0)^2}{(2-0)^2-4}=\frac{1-(4-0)}{(4-0)-4}=\frac{-3+0}{-0}=+\infty \Rightarrow$$

$\Rightarrow x=2$ je V. A.

$\lim_{x\to 2+0} f(x)=-\infty \Rightarrow x=2$ je V. A.

$\lim_{x\to\pm\infty} f(x)=-1 \Rightarrow y=-1$ je H. A.

$y'=\frac{6x}{(x^2-4)^2}$, $y'=0$ akko $x=0$



$-2 \quad 0 \quad 2$ ← prekidi + nule f-je y'

x	$(-2,0)$	$(0,2)$	$(2,+\infty)$
y'	–	+	+
y	↘	↗	↗

min — rast i opadanje

$(0,-\frac{1}{4})$ je tačka minimuma

$y''=(-6)\frac{3x^2+4}{(x^2-4)^3}$

$y''\neq 0\ \forall x\in\mathcal{D} \Rightarrow$ f-ja nema P. T.

f-ja $y=\frac{1-x^2}{x^2-4}$

(2.) Odrediti $\int \frac{x^4-4x^3+5x}{x^2-3x-2}\,dx$.

Rj.: $(x^4-4x^3+5x):(x^2-3x-2)=x^2-x-1-\frac{2}{x^2-3x-2}$

$$
\begin{array}{l}
-\ \underline{x^4-3x^3-2x^2} \\
\quad -x^3+2x^2+5x \\
-\ \underline{-x^3+3x^2+2x} \\
\qquad -x^2-3x \\
\qquad \underline{-x^2-3x+2} \\
\qquad\qquad -2
\end{array}
$$

$$I=\int \frac{x^4-4x^3+5x}{x^2-3x-2}\,dx =$$

$$=\int \left(x^2-x-1-\frac{2}{x^2-3x-2}\right)dx =$$

$$=\frac{x^3}{3}-\frac{x^2}{2}-x-2\int\frac{dx}{x^2-3x-2}=\frac{1}{3}x^3-\frac{1}{2}x^2-x-2I_1$$

$$x^2-3x-2=x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}-2=\left(x-\frac{3}{2}\right)^2-\frac{17}{4}$$

$$I_1=\int\frac{dx}{x^2-3x-2}=\int\frac{dx}{\left(x-\frac{3}{2}\right)-\frac{17}{4}}=\left|\begin{array}{l} x-\frac{3}{2}=\frac{\sqrt{17}}{2}t \\ dx=\frac{\sqrt{17}}{2}dt \\ t=\frac{2x-3}{\sqrt{17}}\end{array}\right|=\int\frac{\frac{\sqrt{17}}{2}dt}{\frac{17}{4}t^2-\frac{17}{4}}=$$

$$=\frac{\sqrt{17}}{2}\cdot\frac{4}{17}\int\frac{dt}{t^2-1}=\frac{2\sqrt{17}}{17}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{\sqrt{17}}{17}\ln\left|\frac{\frac{2x-3}{\sqrt{17}}-1}{\frac{2x-3}{\sqrt{17}}+1}\right|+c$$

$$I=\frac{1}{3}x^3-\frac{1}{2}x^2-x-\frac{2\sqrt{17}}{17}\ln\left|\frac{2x-3-\sqrt{17}}{2x-3+\sqrt{17}}\right|+C$$

(3.) Odrediti područje konvergencije reda $\sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{\cdot 3^n}$.

Rj.: Za stepeni red $\sum_{n=0}^{\infty}a_n(x-a)^n$ postoji broj R takav da za $|x-a|<R$ red KV a za $|x-a|>R$ red DV. Interval $(R-a, R+a)$ zove se interval konvergencije a R poluprečnik intervala KV reda.

R se određuje po formuli $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

ili po formuli $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ ako ovaj limes postoji.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-2)^n$$

I način:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{3}\right)^n}{\left(-\frac{1}{3}\right)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{-\frac{1}{3}}\right| = \lim_{n\to\infty} 3 = 3$$

$\Rightarrow R = 3$

$$|x-2| < 3$$
$$-3 < x-2 < 3$$
$$-1 < x < 5$$

Područje KV reda je $(-1, 5)$.

II način:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\left|\left(-\frac{1}{3}\right)^n\right|} = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{3}\right|^n} = \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$

$\Rightarrow \frac{1}{R} = \frac{1}{3} \Rightarrow R = 3$

$$|x-2| < 3$$
$$-3 < x-2 < 3$$

$$-3 < x-2$$
$$x-2 < 3$$
$$-1 < x$$
$$x < 5$$

Područje KV reda je $(-1, 5)$.

ili

$(R-a,\ R+a)$, $R \to 3$, $a \to 2$, $R \to 3$, $a \to 2$

④ Riješiti jednačinu $y' + 2xy = xe^{-x^2}$.

Rj: ovo je linearna diferencijalna jednačina, $y' + f(x)y = g(x)$.

uvodimo smjenu $y = u\cdot v$, $y' = u'\cdot v + u\cdot v'$

$$u'\cdot v + u\cdot v' + 2x\cdot uv = xe^{-x^2}$$
$$u'\cdot v + u\underbrace{(v' + 2xv)}_{=0} = xe^{-x^2}$$

$$v' + 2xv = 0$$
$$v' = -2xv$$

$$\frac{dv}{dx} = -2xv \quad / \cdot \frac{dx}{v}$$
$$\frac{dv}{v} = -2x\,dx \quad / \int$$
$$\ln v = -2\cdot\frac{x^2}{2} \Rightarrow \ln v = -x^2 \Rightarrow v = e^{-x^2}$$

$u' \cdot v = x e^{-x^2}$

$u' \cdot e^{-x^2} = x \cdot e^{-x^2} \quad / : e^{-x^2}$

$u' = x$

$\frac{du}{dx} = x, \quad du = x\,dx \quad / \int$

$u = \frac{x^2}{2} + c$

$y = u \cdot v = e^{-x^2}\left(\frac{1}{2}x^2 + c\right)$

$y = e^{-x^2}\left(\frac{1}{2}x^2 + c\right)$ opšte rješenje dif. jedn.

provjera:

$y' = \underline{e^{-x^2}}(-2x)\underline{\left(\frac{1}{2}x^2+c\right)} + e^{-x^2} \cdot x$

$y' = -2x \cdot y + e^{-x^2} \cdot x$

$y' + 2xy = x e^{-x^2}$

5. Naći ekstreme f-je $z = \frac{3}{2}x^2 + \frac{1}{2}xy^2 - 3xy$.

Rj.

$\frac{\partial z}{\partial x} = 3x + \frac{1}{2}y^2 - 3y$

$\frac{\partial z}{\partial y} = xy - 3x$

$3x + \frac{1}{2}y^2 - 3y = 0 \quad / \cdot 2$

$xy - 3x = 0$

$6x + y^2 - 6y = 0$

$x(y-3) = 0 \Rightarrow x = 0$ ili $y = 3$

$x = 0$: $y^2 - 6y = 0$

$y(y-6) = 0 \Rightarrow y_1 = 0,\ y_2 = 6$

$y = 3$: $6x + 9 - 18 = 0$

$6x = 9 \Rightarrow x = \frac{9}{6} = \frac{3}{2}$

Stacionarne tačke su: $M_1(0,0)$, $M_2(0,6)$, $M_3\left(\frac{3}{2}, 3\right)$.

$\frac{\partial^2 z}{\partial x^2} = 3$

$\frac{\partial^2 z}{\partial x \partial y} = y - 3$

$\frac{\partial^2 z}{\partial y^2} = x$

Za tačku $M_1(0,0)$

$A = 3, \quad B = -3, \quad C = 0$

$D = AC - B^2 = -9 < 0 \Rightarrow$ F-ja z u tački M_1 nema ekstrem.

Za tačku $M_2(0,6)$

$A = 3, \quad B = 3, \quad C = 0$

$D = AC - B^2 = -9 \Rightarrow$ f-ja z u tački M_2 nema ekstrem

Za tačku $M_3\left(\frac{3}{2}, 3\right)$, $A = 3, \quad B = 0, \quad C = \frac{3}{2}$

$D = AC - B^2 = \frac{9}{2} > 0 \Rightarrow$ f-ja u tački M_3 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}\left(\frac{3}{2}, 3\right) = \frac{3}{2} \cdot \left(\frac{3}{2}\right)^2 + \frac{1}{2} \cdot \frac{3}{2} \cdot 3^2 - 3 \cdot \frac{3}{2} \cdot 3 = \frac{27}{8} + \frac{27}{4} - \frac{27}{2} = -3\frac{3}{8}$