Matematika 1

Grupa A

1. Odrediti koji član u razvoju binoma $(\frac{5}{7}\sqrt[5]{a}+\frac{\sqrt{7}}{5\sqrt{a}})^{14}$ ne sadrži a i naći njegovu vrijednost.

2. Metodom matematičke indukcije dokazati da jednakost $1+2^1+2^2+...+2^n=2^{n+1}-1$ vrijedi za sve prirpdne brojeve.

3. Riješiti matričnu jednačinu: $\quad (X-I)(A+2I)=B,\;$ ako je

$A=\begin{bmatrix} -4 & 0 & -1 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 0 & 3 \\ -7 & -3 & -14 \\ 3 & -2 & 0 \end{bmatrix}$, I jedinična matrica.

4. Odrediti jednačinu ravni koja sadrži pravu $a:\begin{cases} x-2y+2z-1=0 \\ y-3z-5=0 \end{cases}$ i tačku $M(1,2,6)$.

Grupa B

1. Naći sve vriednosti od z (ima ih 2) ako je $z=(\sqrt{3}-i)\sqrt{i}$

2. Diskutovati riješenja sistema u zavisnosti od parametra λ:

$$\begin{aligned} (\lambda+2)x+3y-z &= 1 \\ x+(\lambda-2)y+2z &= 1 \\ 3x+6y-z &= 3 \end{aligned}$$

3. Odrediti parametar λ tako da zapremina tetraedra $ABCD$ iznosi $\frac{17}{2}$ ako su $A(2-\lambda,2,3)$, $B(-1-\lambda,1,3)$, $C(2,2,5)$ i $D(-\lambda,-7,2)$. Za nađenu vrijednost λ izračunati površinu $\triangle ABC$.

4. Izračunati: $\quad \lim\limits_{x\to 27}\frac{\sqrt[3]{x}-3}{27-x}$.

1.[A] Odrediti koji član u razvoju binoma $\left(\frac{5}{7}\sqrt[5]{a}+\frac{\sqrt{7}}{5\sqrt{a}}\right)^{14}$ ne sadrži a i naći njegovu vrijednost.

Rj.
$$\left(\frac{5}{7}\sqrt[5]{a}+\frac{\sqrt{7}}{5\sqrt{a}}\right)^{14}=\sum_{k=0}^{14}\binom{14}{k}\left(\frac{5}{7}\sqrt[5]{a}\right)^{14-k}\cdot\left(\frac{\sqrt{7}}{5\sqrt{a}}\right)^{k}=$$

$$=\sum_{k=0}^{14}\binom{14}{k}\left(5\cdot 7^{-1}\cdot a^{\frac{1}{5}}\right)^{14-k}\cdot\left(7^{\frac{1}{2}}\cdot 5^{-1}\cdot a^{-\frac{1}{2}}\right)^{k}=$$

$$=\sum_{k=0}^{14}\binom{14}{k}\, 5^{14-k}\cdot 7^{-14+k}\cdot a^{\frac{14-k}{5}}\cdot 7^{\frac{k}{2}}\cdot 5^{-k}\cdot a^{-\frac{k}{2}}$$

$$=\sum_{k=0}^{14}\binom{14}{k}\, 5^{14-2k}\cdot 7^{-14+\frac{3k}{2}}\, a^{\frac{14-k}{5}-\frac{k}{2}}$$

Iz zadnjeg izraza vidimo da član razvoja binoma neće sadržavati a akko $\frac{14-k}{5}-\frac{k}{2}=0 \quad /\cdot 10$

$$28-2k-5k=0$$

$$28-7k=0 \Rightarrow k=4$$

Peti član u razvoju binoma ne sadrži a.
Njegova vrijednost je
$$\binom{14}{4}\cdot 5^{6}\cdot 7^{\cancel{-8}\,-7}=\frac{\cancel{14}\cdot 13\cdot\cancel{12}\cdot 11}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}\cdot 5^{6}\cdot 7^{\cancel{-8}}=13\cdot 11\cdot 5^{6}\cdot 7^{-7}=\frac{13\cdot 11\cdot 5^{6}}{7^{7}}$$

2.[A] Metodom matematičke indukcije dokazati da jednakost $1+2^1+2^2+\ldots+2^n=2^{n+1}-1$ vrijedi za sve prirodne brojeve.

Rj. $1+2^1+2^2+\ldots+2^k=2^{k+1}-1$.

BAZA INDUKCIJE

$k=1$: $\quad 1+2^1=2^{1+1}-1$ tj. $3=3$ Jednakost je tačna za $k=1$.

INDUKCISKI KORAK

Pretpostavimo da je jednakost tačna za $k=1,2,\ldots,n$.
Dokažimo da jednakost vrijedi i za $n+1$.

$$\underbrace{1+2^1+2^2+\ldots+2^n}+2^{n+1} \xlongequal[\text{pretpostavci}]{\text{prema}} 2^{n+1}-1+2^{n+1}=$$

$$=2\cdot 2^{n+1}-1=2^{n+2}-1$$

Prema tome $1+2^1+2^2+\ldots+2^n+2^{n+1}=2^{n+2}-1$

što je i trebalo pokazati

ZAKLJUČAK

Jednakost $1+2^1+2^2+\ldots+2^n=2^{n+1}-1$ je tačna za sve prirodne brojeve.

(3.)[A] Riješiti matričnu jednačinu $(X-I)(A+2I)=B$ ako je
$$A=\begin{bmatrix}-4&0&-1\\1&-2&2\\0&1&1\end{bmatrix},\quad B=\begin{bmatrix}0&0&3\\-7&-3&-14\\3&-2&0\end{bmatrix},$$
I jedinična matrica.

Rj: $(X-I)(A+2I)=B \quad /\cdot(A+2I)^{-1}$ sa desne strane

$$X-I=B\cdot(A+2I)^{-1}$$

uvešću oznaku $C=A+2I$

imam $X-I=B\cdot C^{-1}$

$$X=B\cdot C^{-1}+I$$

$$C=\begin{bmatrix}-2&0&-1\\1&0&2\\0&1&3\end{bmatrix}$$

$$C^{-1}=\frac{1}{\det C}\cdot C_{adj}.$$

$$C_{adj}=C_{kof}^T$$

$$C_{kof}=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{bmatrix}$$

$$C_{11}=(-1)^2\begin{vmatrix}0&2\\1&3\end{vmatrix}=-2$$

$$C_{12}=(-1)^3\begin{vmatrix}1&2\\0&3\end{vmatrix}=-3$$

$$\vdots \qquad \vdots$$

$$C_{kof}=\begin{bmatrix}-2&-3&1\\-1&-6&2\\0&3&0\end{bmatrix}\Rightarrow C_{adj}=\begin{bmatrix}-2&-1&0\\-3&-6&3\\1&2&0\end{bmatrix}$$

$$\det C=\begin{vmatrix}-2&0&-1\\1&0&2\\0&1&3\end{vmatrix}=3 \Rightarrow C^{-1}=\begin{bmatrix}-\frac{2}{3}&-\frac{1}{3}&0\\-1&-2&1\\\frac{1}{3}&\frac{2}{3}&0\end{bmatrix}$$

$$B\cdot C^{-1}=\begin{bmatrix}1&2&0\\3&-1&-3\\0&3&-2\end{bmatrix}$$

$$X=\begin{bmatrix}2&2&0\\3&0&-3\\0&3&-1\end{bmatrix}$$

4 A Odrediti jednačinu ravni koja sadrži pravu

$$a: \begin{cases} x - 2y + 2z - 1 = 0 \\ y - 3z - 5 = 0 \end{cases}$$

i tačku $M(1, 2, 6)$.

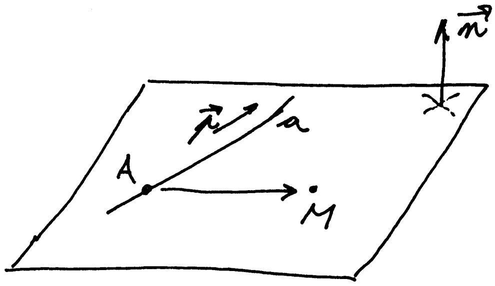
Rj.

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$\vec{n}(A, B, C)$

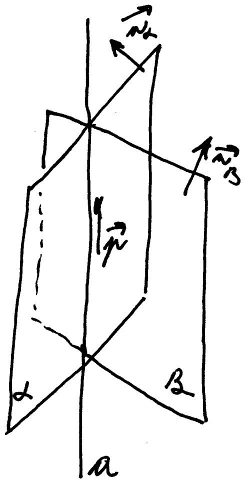
$M(x_1, y_1, z_1)$

jednačina ravni kroz tačku M



$M \notin a$

$\vec{p} = ?$



$\alpha: \; x - 2y + 2z - 1 = 0$

$\beta: \; y - 3z - 5 = 0$

$$\left.\begin{matrix} \vec{p} \perp \vec{n_\alpha} \\ \vec{p} \perp \vec{n_\beta} \end{matrix}\right. \Rightarrow \vec{p} \parallel \vec{n_\alpha} \times \vec{n_\beta} \Rightarrow \vec{p} = k \cdot (\vec{n_\alpha} \times \vec{n_\beta})$$

$k \in R$

$\vec{n_\alpha}(1, -2, 2)$

$\vec{n_\beta}(0, 1, -3)$

$$\vec{n_\alpha} \times \vec{n_\beta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 0 & 1 & -3 \end{vmatrix} = (4, 3, 1) \Rightarrow \vec{p}(4, 3, 1)$$

tražim

$A \in a$

$A(11, 5, 0)$

tačka koja pripada pravoj A

$\vec{AM}(-10, -3, 6)$

$$\left.\begin{matrix} \vec{AM} \perp \vec{n} \\ \vec{p} \perp \vec{n} \end{matrix}\right\} \Rightarrow \vec{n} \parallel \vec{AM} \times \vec{p}$$

$$\Downarrow$$

$$\vec{n} = k(\vec{AM} \times \vec{p})$$

$k \in R$

$$\vec{AM} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -10 & -3 & 6 \\ 4 & 3 & 1 \end{vmatrix} = (-21, 34, -18)$$

$$\Rightarrow \vec{n}(-21, 34, -18)$$

$$-21(x-1) + 34(y-2) - 18(z-6) = 0$$

$$-21x + 34y - 18z + 21 - 68 + 108 = 0$$

$$-21x + 34y - 18z + 61 = 0$$

je jednačina tražene ravni

1.[B] Naći sve vrijednosti od z (ima ih 2) ako je
$z = (\sqrt{3} - i) \cdot \sqrt{i}$.

Rj: Ako uvedem oznaku $w = i$ tad $\sqrt{w}$ ima dvije vrijednosti:

$$w_k = \sqrt{|w|}\left(\cos\frac{\varphi + 2k\pi}{2} + i\sin\frac{\varphi+2k\pi}{2}\right), \quad k = 0, 1$$

$w = i \quad \sin\varphi = 1$

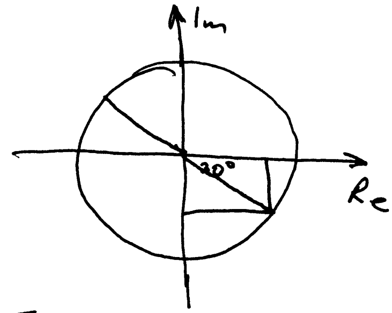
$|w| = 1 \quad \cos\varphi = 0$

$\varphi = \frac{\pi}{2}$

$$w = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$w_1 = \cos\frac{\frac{\pi}{2}+2\pi}{2} + i\sin\frac{\frac{\pi}{2}+2\pi}{2} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$



$$s = \sqrt{3} - i, \quad |s| = \sqrt{3+1} = 2$$

$$\cos\varphi = \frac{\sqrt{3}}{2}, \quad \sin\varphi = -\frac{1}{2}, \quad tg\,\varphi = -\frac{\sqrt{3}}{3} \qquad tg\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\Longrightarrow \quad \varphi = -\frac{\pi}{6} \quad \text{ili} \quad \varphi = \frac{11\pi}{6}$$

$$\sqrt{3} - i = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

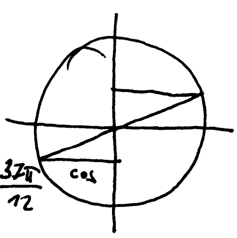
$$z_1 = s \cdot w_0 = 2\left[\cos\left(\frac{\pi}{4} + \frac{11\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{11\pi}{6}\right)\right] = 2\left(\cos\frac{35\pi}{12} + i\sin\frac{35\pi}{12}\right)$$

$$z_1 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = 2\left(\frac{\sqrt{6}-\sqrt{2}}{4} + i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$z_2 = s \cdot w_1 = 2\left[\cos\left(\frac{5\pi}{4} + \frac{11\pi}{6}\right) + i\sin\left(\frac{5\pi}{4} + \frac{11\pi}{6}\right)\right] = 2\left(\cos\frac{37\pi}{12} + i\sin\frac{37\pi}{12}\right)$$



$$z_2 = 2\left(-\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) = 2\left(-\frac{\sqrt{6}-\sqrt{2}}{4} - i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

Vrijednosti od z su:

$$z_1 = \frac{\sqrt{6}-\sqrt{2}}{2} + i\,\frac{\sqrt{6}+\sqrt{2}}{2} \quad \text{i} \quad z_2 = -\frac{\sqrt{6}-\sqrt{2}}{2} - i\,\frac{\sqrt{6}+\sqrt{2}}{2}.$$

2. [B] Diskutovati rješenja sistema u zavisnosti od parametra λ:

$$(\lambda+2)x + 3y - z = 1$$
$$x + (\lambda-2)y + 2z = 1$$
$$3x + 6y - z = 3.$$

Rj:

$$D = \begin{vmatrix} \lambda+2 & 3 & -1 \\ 1 & \lambda-2 & 2 \\ 3 & 6 & -1 \end{vmatrix} \underset{II_v + I_v \cdot 2}{\overset{III_v - I_v}{=}} \begin{vmatrix} \lambda+2 & 3 & -1 \\ 2\lambda+5 & \lambda+4 & 0 \\ -\lambda+1 & 3 & 0 \end{vmatrix} = (-1)(\lambda^2 + 9\lambda + 11)$$

$$D_x = \begin{vmatrix} 1 & 3 & -1 \\ 1 & \lambda-2 & 2 \\ 3 & 6 & -1 \end{vmatrix} \underset{II_v - I_v}{\overset{III_v - I_v \cdot 3}{=}} \begin{vmatrix} 1 & 3 & -1 \\ 0 & \lambda-5 & 3 \\ 0 & -3 & 2 \end{vmatrix} = 2\lambda - 1$$

$$D_y = (-7)(\lambda+1), \quad D_z = 3(\lambda+1)(\lambda-4)$$

Diskusija:

1° $\lambda \neq \frac{-9-\sqrt{37}}{2}$; $\lambda \neq \frac{-9+\sqrt{37}}{2}$ Zadatak ima jedinstveno rješenje

$$\left(-\frac{2\lambda-1}{\lambda^2+9\lambda+11}, \frac{7(\lambda+1)}{\lambda^2+9\lambda+11}, \frac{-3(\lambda+1)(\lambda-4)}{\lambda^2+9\lambda+11}\right)$$

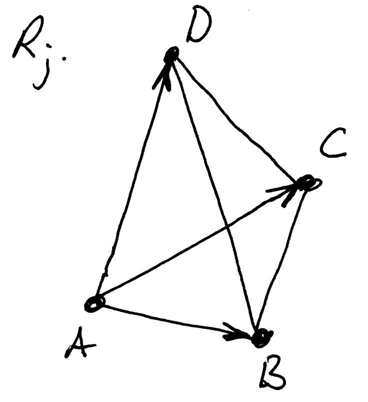
2° $\lambda = -\frac{9-\sqrt{37}}{2}$

$D_x \neq 0$ sistem nema rješenja

3° $\lambda = \frac{-9+\sqrt{37}}{2}$, $\Rightarrow D_y \neq 0$ sistem nema rješenja

3. [B] Odrediti parametar λ tako da zapremina tetraedra $ABCD$ iznosi $\frac{17}{2}$ ako su $A(2-\lambda, 2, 3)$, $B(-1-\lambda, 1, 3)$, $C(2, 2, 5)$ i $D(-\lambda, -7, 2)$. Za nađenu vrijednost λ izračunati površinu ΔABC.

Rj.



$$\overrightarrow{AB}(-3, -1, 0)$$
$$\overrightarrow{AC}(\lambda, 0, 2)$$
$$\overrightarrow{AD}(-2, -9, -1)$$

$$V = \frac{1}{6}\left|(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}\right|$$

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} -3 & -1 & 0 \\ \lambda & 0 & 2 \\ -2 & -9 & -1 \end{vmatrix}$$

$$\underline{\underline{I_k - II_k \cdot 3}} \begin{vmatrix} 0 & -1 & 0 \\ \lambda & 0 & 2 \\ 25 & -9 & -1 \end{vmatrix} = \begin{vmatrix} \lambda & 2 \\ 25 & -1 \end{vmatrix} = -\lambda - 50$$

$$V = \frac{1}{6}|-\lambda - 50|$$

$$V = \frac{1}{6} \cdot (\lambda + 50)$$

$$\frac{17}{2} = \frac{\lambda+50}{6} \quad / \cdot 6$$

$$51 = \lambda + 50$$

$$\Rightarrow \lambda = 1$$

$\lambda = 1 \implies \overrightarrow{AB}(-3,-1,0)$

$\overrightarrow{AC}(1,0,2)$

$\overrightarrow{AD}(-2,-9,-1)$

$$P_{\Delta ABC} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = (-2, 6, 1)$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{4+36+1} = \sqrt{41} \implies P_{\Delta ABC} = \frac{\sqrt{41}}{2}.$$

4. [B] Izračunati: $\lim\limits_{x \to 27} \frac{\sqrt[3]{x} - 3}{27 - x}$

Rj. $$\lim_{x\to 27} \frac{\sqrt[3]{x} - 3}{27 - x} \cdot \frac{(\sqrt[3]{x^2} + 3\sqrt[3]{x} + 9)}{(\sqrt[3]{x^2} + 3\sqrt[3]{x} + 9)} = \lim_{x \to 27} \frac{\cancel{x-27}}{-\cancel{(x-27)}(\sqrt[3]{x^2} + 3\sqrt[3]{x} + 9)} =$$

$$\boxed{(a-b)(a^2+ab+b^2) = a^3 - b^3}$$

$$= -\frac{1}{9+9+9} = -\frac{1}{27}$$