Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
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Apsolventski pismeni ispit iz Linearne algebre

Pravila: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Neka je $\mathcal{V} = \mathbb{R}^n$ i neka je $(a_1, a_2, ..., a_n)^\top$ fiksirani vektor iz $\mathcal{V}$. Dokazati da je familija svih elemenata $(x_1, x_2, ..., x_n)^\top$ iz $\mathcal{V}$ sa osobinom $a_1x_1 + ... + a_nx_n = 0$ vektorski podprostor prostora $\mathcal{V}$. Drugim riječima da je

$$\mathcal{M} = \{(x_1, x_2, ..., x_n)^\top \in \mathcal{V} \,|\, a_1x_1 + ... + a_nx_n = 0\}$$

vektorski podprostor od $\mathcal{V}$. Odrediti bazu i dimenziju ovog podprostora.

2. Zadana je linearna transformacija $T : \mathcal{P}_2 \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$

$$T(a + bt + ct^2) = \begin{pmatrix} a - 2b & b + c \\ -2a - 4c & -2a + 4b \end{pmatrix}$$

Prikažite transformaciju T u paru standardnih baza (drugim riječima odrediti matricu koordinata $[T]_{\mathcal{S}\mathcal{S}'}$ gdje su $\mathcal{S}$ i $\mathcal{S}$' redom standardne baze za $\mathcal{P}_2$ i $\mathrm{Mat}_{2\times 2}(\mathbb{R})$) te odredite po jednu bazu za jezgru i sliku od T ($\mathcal{P}_2$ je prostor realnih polinoma stepena ≤ 2).

3. Dat je vektorski prostor $\mathcal{L}$ vektorskog prostora $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \,|\, AX - XA = \mathbf{0}, X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\}.$$

Razmatrajući standardni unutrašnji proizvod za matrice $\langle A, B\rangle = \mathrm{trag}(A^\top B)$ odrediti ortonormiranu bazu za $\mathcal{L}$.

4. Neka je $\mathcal{M} = \mathrm{span}\{a, b\}$ podprostor unitarnog prostora $\mathbb{R}^n$ (sa standardnim skalarnim proizvodom) razapet (generisan) vektorima $a = (0, 1, 2, ..., n-1)^\top$ i $b = (1, 1, 1, ..., 1)^\top$. Odrediti njegov ortogonalni komplement $\mathcal{M}^\perp$ te odredite ortogonalnu projekciju od z na $\mathcal{M}$ gdje je

$$z = (\frac{1}{2}n(3-n), \frac{1}{2}n(n-1), 0, 0, ..., 0)^\top \in \mathbb{R}^n.$$

Zadaci su skinuti sa stranice ff.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Neka je $V=\mathbb{R}^n$ i neka je $(a_1, a_2, \dots, a_n)^T$ fiksirani vektor iz V. Dokazati da je familija svih elemenata $(x_1, x_2, \dots, x_n)^T$ iz V sa osobinom $a_1x_1+\dots+a_nx_n=0$ vektorski podprostor prostora V. Drugim riječima da je

$$\mathcal{M}=\left\{\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in V \;\middle|\; a_1x_1+\dots+a_nx_n=0\right\}$$

vektorski podprostor od V. Odrediti dimenziju i bazu ovog podprostora.

Rj.

Prisjetimo se:

<u>Neprazan podskup $\mathcal{S}$ vektorskog prostora V je podprostor od V ako i samo ako</u>

<u>(A1) $\quad x, y\in\mathcal{S} \Rightarrow x+y\in\mathcal{S}$;</u>

<u>(M1) $\quad x\in\mathcal{S} \Rightarrow \alpha x\in\mathcal{S}$ za $\forall\alpha\in\mathbb{R}$</u>

Pa pokažimo da vrijede osobine (A1) i (M1).

Izaberimo proizvoljne elemente $\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}, \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}\in\mathcal{M}$ i $\alpha\in\mathbb{R}$. Tada

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}+\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}=\begin{pmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{pmatrix}.$$

Uvedimo oznake $a=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}$, $x=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$, $y=\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}$.

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\in\mathcal{M} \iff a_1x_1+\dots+a_nx_n=0 \iff a^Tx=0$$

$$\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}\in\mathcal{M} \iff a_1y_1+\dots+a_ny_n=0 \iff a^Ty=0$$

$$a^Tx+a^Ty=0 \iff a^T(x+y)=0 \Rightarrow x+y=\begin{pmatrix}x_1+y_1\\x_2+y_2\\\vdots\\x_n+y_n\end{pmatrix}\in\mathcal{M}$$

Prema tome vrijedi (A1)

$$\alpha \cdot x = \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$a^T x = 0 \Rightarrow a^T \alpha x = 0 \Rightarrow \alpha x \in M \quad \text{vrijedi (M1)}$$

M jest vektorski podprostor

Da bi odredili bazu i dimenziju napišimo M u drugačijem obliku

$$M = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in V \;\middle|\; a_1 x_1 + \dots + a_n x_n = 0 \right\} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; a^T x = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} =$$

$$= \ker\left(\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \right).$$

Prema tome $M = \ker(A)$ gdje je $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$.

Znamo da kolone iz općeg rješenja od $\ker(A)$ formiru bazu za $\ker(A)$. Pretpostavimo da je $a \neq 0$. Kako je $\text{rang}(A) = 1$ ako posmatramo sistem $Ax = 0$ to moramo uzeti $n-1$ promjenjivu proizvoljno. Pretpostavimo da je $a_1 \neq 0$, kako je $a \neq 0$ to $\exists a_i \neq 0$

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

$$x_1 = \frac{a_2}{a_1} x_2 + \dots + \frac{a_n}{a_1} x_n$$

$$x = \begin{pmatrix} \frac{a_2}{a_1} x_2 + \dots + \frac{a_n}{a_1} x_n \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Prema tome $\dim(M) = n-1$ i

$$B = \left\{ \begin{pmatrix} \frac{a_2}{a_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{a_3}{a_1} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \frac{a_n}{a_1} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

je baza za M.

Zadana je linearna transformacija $T: \mathcal{P}_2 \to Mat_{2\times2}(\mathbb{R})$

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Prikažite transformaciju T u paru standardnih baza (drugim riječima odrediti matricu koordinata $[T]_{\mathcal{S}\mathcal{S}'}$ gdje su $\mathcal{S}$ i $\mathcal{S}'$ redom baze za $\mathcal{P}_2$ i $Mat_{2\times2}(\mathbb{R})$) te odredite po jednu bazu za jezgru i sliku od T ($\mathcal{P}_2$ je prostor realnih polinoma stupnja ≤ 2).

Rj.

Prisjetimo se

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_n\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Tada je (za $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$)

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}.$$

Standardna baza za $\mathcal{P}_2$ je $\mathcal{S}=\{1,t,t^2\}$, a standardna baza za $Mat_{2\times2}(\mathbb{R})$ je $\mathcal{S}'=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$.

Prvo odredimo $T(1)$, $T(t)$ i $T(t^2)$:

$$T(1)=\begin{pmatrix}1&0\\-2&-2\end{pmatrix},\quad T(t)=\begin{pmatrix}-2&1\\0&4\end{pmatrix}\quad i\quad T(t^2)=\begin{pmatrix}0&1\\-4&0\end{pmatrix}.$$

Sad nije teško odrediti $[T(1)]_{\mathcal{S}'}$, $[T(t)]_{\mathcal{S}'}$ i $[T(t^2)]_{\mathcal{S}'}$ pa imamo

$$[T]_{\mathcal{S}\mathcal{S}'} = \begin{pmatrix}1&-2&0\\0&1&1\\-2&0&-4\\-2&4&0\end{pmatrix}.$$

Ostalo je još da odredimo po jednu bazu za jezgru i sliku od T.

$$\operatorname{im}(T) = \{ Tp \mid p \in \mathcal{P}_2 \} = \left\{ \begin{bmatrix} a-2b & b+c \\ -2a-4c & -2a+4b \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\} =$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} a + \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} b + \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} c \,\middle|\, a,b,c \in \mathbb{R} \right\}$$

Baza za $\operatorname{im}(T)$ je

$$\left\{ \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \right\}.$$

$$\ker(T) = \{ p \in \mathcal{P}_2 \mid T(p) = 0 \} = \left\{ a + bt + ct^2 \,\middle|\, \begin{pmatrix} a-2b & b+c \\ -2a-4c & -2a+4b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

a odavde vidimo da ćemo a, b i c odrediti iz sustava

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 0 & -4 & 0 \\ -2 & 4 & 0 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$ jednu promjenjivu uzimamo proizvoljno

$$\begin{aligned} a + 2c &= 0 \\ b + c &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= -2c \\ b &= -c \end{aligned}$$

$$\ker(T) = \{ -2\alpha - \alpha t + \alpha t^2 \mid \alpha \in \mathbb{R} \} = \operatorname{span}\{ -2 - t + t^2 \}$$

baza za $\ker(T)$.

(#) Dat je vektorski podprostor $\mathcal{L}$ vektorskog prostora $Mat_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \left\{ A \in Mat_{2\times 2}(\mathbb{R}) \;\middle|\; AX - XA = 0, \; X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

Posmatrajući standardni unutrašnji proizvod za matrice

$$\langle A, B \rangle = trag(A^T B)$$

odrediti ortonormiranu bazu za $\mathcal{L}$.

Rj.

Da bi odredili ortonormiranu bazu za $\mathcal{L}$ prvo je potrebno pronaći bilo kakvu bazu za $\mathcal{L}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left.\begin{aligned} AX &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix} \\ XA &= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix} \end{aligned}\right\} \Rightarrow AX - XA = \begin{bmatrix} 2b-c & a-d \\ 2d-2a & c-2b \end{bmatrix}$$

$$AX - XA = 0 \iff \begin{aligned} 2b - c &= 0 \\ a - d &= 0 \\ 2d - 2a &= 0 \\ c - 2b &= 0 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}}_{=B} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \underset{III_v + II_v}{\overset{IV_v + I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} a - d &= 0 \\ 2b - c &= 0 \end{aligned}$$

$$\overline{\quad}$$

$a = d$

$b = \frac{c}{2}$ $\quad c = 2b$

Odavdje sad nije teško vidjeti da se prostor $\mathcal{L}$ može napisati u obliku:

$$\mathcal{L} = \left\{ \begin{bmatrix} d & b \\ 2b & d \end{bmatrix} \middle| \ b, d \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \beta \ \middle| \ \alpha, \beta \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

Baza za $\mathcal{L}$ je $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$.

Sad iskoristimo Gram-Schmidtovu proceduru pa odredimo ortonormiranu bazu za $\mathcal{L}$.

Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$u_k \leftarrow \dfrac{u_k}{\|u_k\|}$

$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\|X_1\| = \sqrt{\text{trag}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)} = \sqrt{2}$

$U_1 \ \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$U_2 \leftarrow X_2 - \langle U_1, X_2 \rangle U_1$

$U_2 \leftarrow \dfrac{U_2}{\|U_2\|}$

$\langle U_1, X_2 \rangle = \text{trag}(U_1^T X) = \text{trag}([$

$= \text{trag}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) = 0$

$X_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$U_2 \leftarrow X_2 - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\| U_2 \| = \sqrt{\operatorname{trag}\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right)} = \sqrt{4+1} = \sqrt{5}$$

Ortonormirana baza za $\mathcal{L}$ je

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}.$$

(#) Neka je $M = \mathrm{span}\{a, b\}$ podprostor unitarnog prostora $\mathbb{R}^n$ (sa standardnim skalarnim proizvodom) razapet (generiran) vektorima $a=(0,1,2,\ldots,n-1)^T$ i $b=(1,1,\ldots,1)^T$. Odrediti njegov ortogonalni komplement $M^\perp$ te odredite ortogonalnu projekciju od z na M gdje je

$$z=\left(\tfrac{1}{2}n(3-n),\quad \tfrac{1}{2}n(n-1), 0, 0, \ldots, 0\right)^T \in \mathbb{R}^n.$$

Rj.

$$M = \mathrm{span}\left\{\begin{pmatrix}0\\1\\2\\\vdots\\n-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}\right\} = \left\{\alpha\begin{pmatrix}0\\1\\2\\\vdots\\n-1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R}\right\}$$

Prisjetimo se da je prema definiciji

$$\underline{M^\perp = \{x \in V \mid \langle m, x\rangle = 0 \text{ za } \forall m \in M \subseteq V\}}$$

Trebamo odrediti vektor(e) $x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ takav(e) da vrijedi

$$\langle a, x\rangle = 0 \quad \text{i} \quad \langle b, x\rangle = 0 \quad \text{tj.}$$

$$\begin{pmatrix}0\\1\\2\\\vdots\\n-1\end{pmatrix}(x_1\, x_2 \,\ldots\, x_n) = 0 \quad \text{i} \quad \begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}(x_1 x_2 \ldots x_n) = 0$$

Drugim riječima trebamo riješiti sljedeći sistem linearnih jednačina:

$$\begin{pmatrix}0 & 1 & 2 & \ldots & n-1\\ 1 & 1 & 1 & \ldots & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & \dots & n-1 & & 0 \\ 1 & 1 & 1 & \dots & 1 & & 0 \end{array}\right] \overset{I_v \leftrightarrow II_v}{\sim} \left[\begin{array}{cccccc|c} 1 & 1 & 1 & \dots & 1 & & 0 \\ 0 & 1 & 2 & \dots & n-1 & & 0 \end{array}\right] \overset{I_v - II_v}{\sim}$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 0 & -1 & -2 & \dots & 2-n & 0 \\ 0 & 1 & 2 & 3 & \dots & n-1 & 0 \end{array}\right] \Rightarrow$$ $n-2$ promjenjive uzimamo proizvoljno npr. $x_3, x_4, \dots, x_n$

$$x_1 = x_3 + 2x_4 + \dots + (n-2)x_n$$

$$x_2 = -2x_3 - 3x_4 + \dots + (1-n)x_n$$

$$M^{\perp} = \left\{ \begin{pmatrix} x_3 + 2x_4 + 3x_5 + \dots + (n-2)x_n \\ -2x_3 - 3x_4 - 4x_5 - \dots - (n-1)x_n \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_3, x_4, \dots, x_n \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} n-2 \\ -n+1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\frac{1}{2}(n^2-3n+2)$

$\frac{1}{2}(n(n-3)+2)$

Znamo da je $1+2+3+\dots+n = \frac{1}{2}n(n+1)$ pa je $1+2+\dots+(n-2) = \frac{1}{2}(n-2)(n-1)$

$$-2-3-4-\dots-(n-1) = (-1)\left(\underbrace{2+3+4+\dots+(n-1)}_{+1-1}\right) = (-1)\left(\frac{1}{2}(n-1)n - 1\right) = 1 - \frac{1}{2}(n-1)n$$

Prema tome

$$\begin{pmatrix} \frac{1}{2}n(3-n) \\ \frac{1}{2}n(n-1) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{=b\in M} \underbrace{- \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \dots - \begin{pmatrix} n-2 \\ -n+1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\in M^{\perp}}$$

Ortogonalna projekcija od z na M je $(1,1,1,\dots,1)^T$