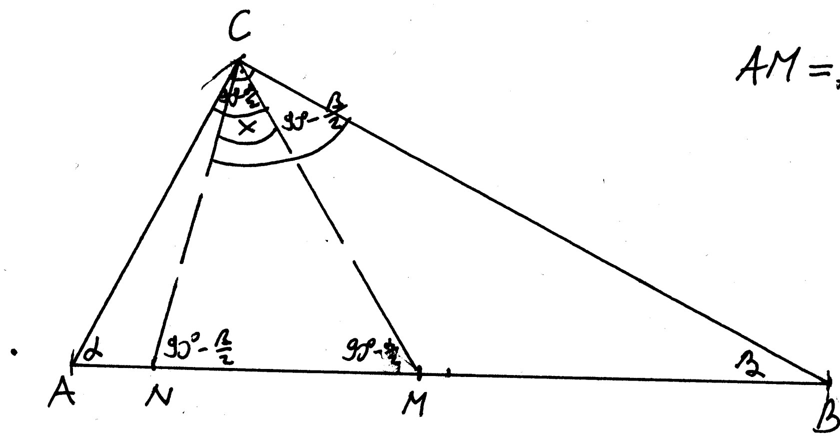
(#) Na hipotenuzi AB pravouglog trougla $\triangle ABC$ date su tačke M i N tako da je $AM = AC$, $BN = BC$ i poredak A-N-M-B. Izračunati ugao $\measuredangle MCN$.

Rj.



$AM = AC \Rightarrow \triangle AMC$ jkk

$\Rightarrow \measuredangle AMC = \measuredangle ACM = 90^\circ - \frac{\alpha}{2}$

$BN = BC \Rightarrow$

$\Rightarrow \triangle BCN$ jkk

$\Rightarrow \measuredangle BNC = \measuredangle BCN = 90^\circ - \frac{\beta}{2}$

Traženi ugao $\measuredangle MCN$ označimo sa x. Sad imamo

$$\left.\begin{array}{lll} \triangle MCN & \Rightarrow & 90^\circ - \frac{\beta}{2} + 90^\circ - \frac{\alpha}{2} + x = 180^\circ \\ \measuredangle BCA & \Rightarrow & 90^\circ - \frac{\alpha}{2} + 90^\circ - \frac{\beta}{2} - x = 90^\circ \end{array}\right\} - \Rightarrow$$

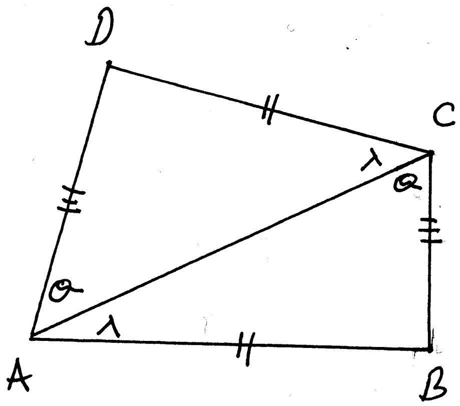
$2x = 90^\circ \Rightarrow x = 45^\circ$

$\measuredangle NCM = 45^\circ$.

(#) Definicija paralelograma: Paralelogram je četvorougao koji ima paralelne suprotne stranice.. Koristeći isključivo ovu definiciju i teoreme o podudarnosti trouglova dokazati sledeću tvrdnju: Četvorougao $\square ABCD$ je paralelogram akko ima podudarne suprotne stranice.

R: postavka zadatka

$$\text{„}\Leftarrow\text{"}: \left.\begin{array}{l} \square ABCD \text{ četvorougao} \\ AB \cong DC,\ AD \cong BC \end{array}\right\} \Rightarrow \square ABCD \text{ paralelogram}$$



Posmatrajmo $\triangle ABC$ i $\triangle ADC$

$$\left.\begin{array}{l} AB \cong CD \\ BC \cong AD \\ AC \cong AC \end{array}\right\} \xrightarrow{SSS} \triangle ABC \cong \triangle ADC$$

$$\Downarrow$$

$$\measuredangle CAB \cong \measuredangle ACD = \lambda$$
$$\text{i } \measuredangle ACB \cong \measuredangle CAD = \theta$$

Na pravoj $p(A,C)$ imamo

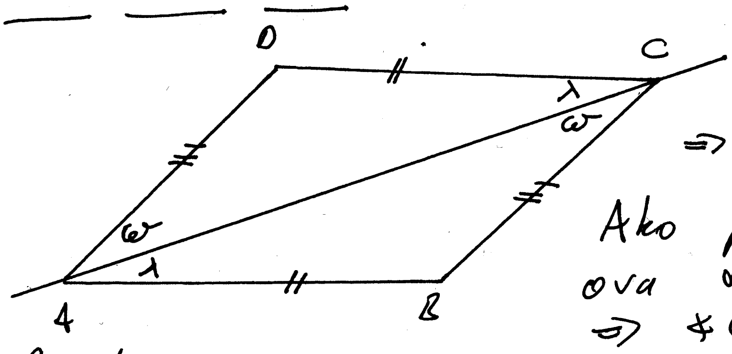
$$\left.\begin{array}{l} \measuredangle ACD \cong \measuredangle CAB = \lambda \Rightarrow p(A,B) \parallel p(C,D) \\ \text{i } \measuredangle CAD \cong \measuredangle ACB = \theta \Rightarrow p(A,D) \parallel p(B,C) \end{array}\right\} \Rightarrow$$

$$\Rightarrow AB \parallel CD \text{ i } AD \parallel BC \Rightarrow \square ABCD \text{ paralelogram}$$

q.e.d.

postavka zadatka

$$\text{„}\Rightarrow\text{"}\ \square ABCD \text{ paralelogram} \Rightarrow AB \cong DC \text{ i } AD \cong BC.$$



$\square ABCD$ paralelogram $\Rightarrow AB \parallel CD$ i $BC \parallel AD$

$\Rightarrow p(A,B) \parallel p(C,D)$ i $p(B,C) \parallel p(A,D)$

Ako posmatramo $p(A,C)$ kao transferzalu ova dva para paralelnih pravih

$\Rightarrow \measuredangle CAB = \measuredangle ACD = \lambda$ i $\measuredangle ACB \cong \measuredangle CAD = \omega$

Posmatrajmo $\triangle ABC$ i $\triangle ACD$

$$\left.\begin{array}{l} \measuredangle BAC \cong \measuredangle ACD = \lambda \\ AC \cong AC \\ \measuredangle ACB \cong \measuredangle CAD = \omega \end{array}\right\} \xrightarrow{USU} \triangle ABC \cong ADC$$

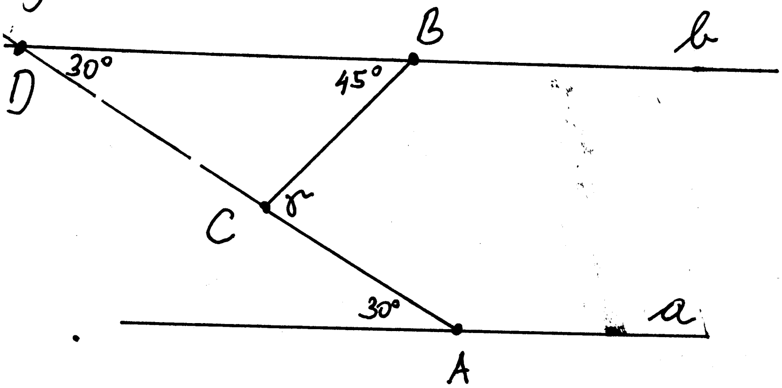
$$\Downarrow$$

$$AB \cong DC \text{ i } AD \cong BC$$

q.e.d.

(#) Date su dvije paralelne prave a i b, tačke $A \in a$, $B \in b$ i C koja se nalazi "između" pravih a i b. Ako je $\measuredangle CAa = 30°$ i $\measuredangle CBb = 45°$ izračunati ugao $\measuredangle ACB$.

Rj.:



$\gamma = ?$

Neka je $p(A,C) \cap b = \{D\}$ (vidi sliku lijevo).

$a \parallel b \overset{\measuredangle CAa = 30°}{\Longrightarrow} \measuredangle CDB = 30°$

γ je vanjski ugao $\Delta BDC \Rightarrow \gamma = 75°$.

(#) Posmatrajmo površine devet različitih kvadrata $P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}$ i P_{33}. Za ove površine znamo da vrijedi

$$P_{11}+P_{12}+P_{13}=P_{21}+P_{22}+P_{23}=P_{31}+P_{32}+P_{33}=P_{11}+P_{21}+P_{31}=P_{12}+P_{22}+P_{32}=$$
$$=P_{13}+P_{23}+P_{33}=P_{11}+P_{22}+P_{33}=P_{13}+P_{22}+P_{31}.$$

Ako su $P_{12}=21$, $P_{13}=14$, $P_{23}=19$ i $P_{31}=20$ diskutovati da li se mogu odrediti površine P_{11}, P_{22} i P_{33}.

R:

P_{11}	P_{12}	P_{13}
$Y=x-1$	21	14
P_{21}	P_{22}	P_{23}
	x	19
P_{31}	P_{32}	P_{33}
20		$Z=x+1$

Površinu P_{22} označimo sa x, P_{11} sa Y. Kako je

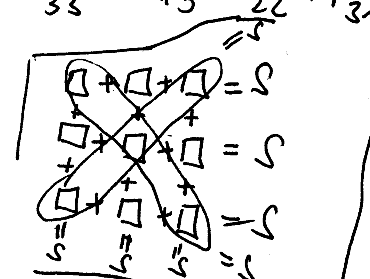
$$P_{11}+P_{12}+P_{13}=P_{13}+P_{22}+P_{31}$$

to $Y+21+\cancel{14}=\cancel{14}+x+20$

$$Y=x-1.$$

Kako je

$$P_{13}+P_{23}+P_{33}=P_{13}+P_{22}+P_{31}$$



to $$14+x+20=14+19+Z$$

$$Z=x+1$$

Sad kako je $P_{11}+P_{22}+P_{33}=P_{13}+P_{22}+P_{31}$ to

$$x-1+x+x+1=20+x+14$$

$$3x-x=34$$

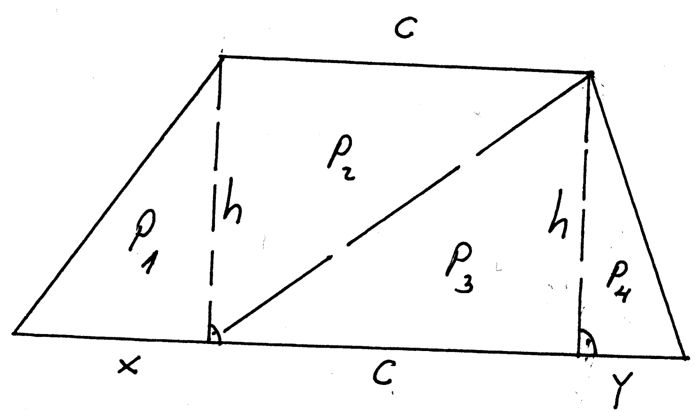
$$x=17$$

$$P_{11}=16,\quad P_{22}=17,\quad P_{33}=18.$$

Tražene površine se mogu odrediti kao i P_{21} i P_{32} (15 i 13).

(#) Definicija trapeza: Trapez je četverougao koji ima tačno jedan par paralelnih stranica. Obrazložiti odgovor na pitanje: Da li je paralelogram trapez? Koristeći isključivo formulu za površinu pravouglog trougla ($P=\frac{a\cdot b}{2}$) izvesti formulu za površinu trapeza ($P=\frac{1}{2}(a+c)h$ gdje su a i c dužine dvije paralelne stranice, a h udaljenost između njih).

Rj. Odgovor na pitanje: Da li je paralelogram trapez? ostavljamo za vježbu. (uputa: nije).



Uvedimo oznake kao na slici $(a=x+c+Y)$.

$$P_{trapeza}=P_1+P_2+P_3+P_4$$

$$=\frac{x\cdot h}{2}+\frac{h\cdot c}{2}+\frac{c\cdot h}{2}+\frac{h\cdot Y}{2}$$

$$=\frac{(x+c+Y)h+c\cdot h}{2}=\frac{ah+ch}{2}$$

Prema tome $P=\frac{1}{2}(a+c)h$.

(#) Dokazati da je svaka ravan incidentna sa najmanje tri tačke.

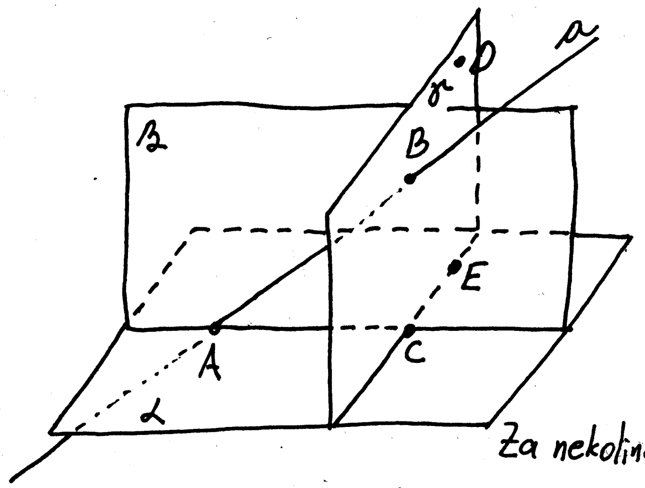
Rj. postavka zadatka

ravan α $\Longrightarrow$ $\exists$ tačke A, B i C: $A \in \alpha$, $B \in \alpha$ i $C \in \alpha$.

Za datu ravan α prema aksiomi I_4 $\exists A: A \in \alpha$.

Za ravan α, prema I_8 $\exists B: B \notin \alpha$.

Za A i B prema I_1, I_2 $\exists!\, a: A \in a$ i $B \in a$.



Za pravu a prema I_3 $\exists C: C \notin a$

Moguća su dva slučaja:

1° $C \in \alpha$

2° $C \notin \alpha$

1° $C \in \alpha$

Za nekolinea. A, B, C prema I_4, I_5 $\exists!\, \beta: A \in \beta, B \in \beta$ i $C \in \beta$

Za β prema I_8 $\exists D: D \notin \beta$. Moguća su dva slučaja

a) $D \in \alpha$

b) $D \notin \alpha$.

Ako bi bilo da $D \in \alpha$, problem je riješen: Tri tražene tačke koje pripadaju ravni α su A, C i D.

Ako $D \notin \alpha$, tad za nekolinear. C, B i D prema I_4, I_5 $\exists!\, \gamma: B \in \gamma, C \in \gamma$ i $D \in \gamma$.

Iz $D \in \alpha, D \in \gamma$ prema aksiomu I_7 $\exists E: E \in \alpha$ i $E \in \gamma$

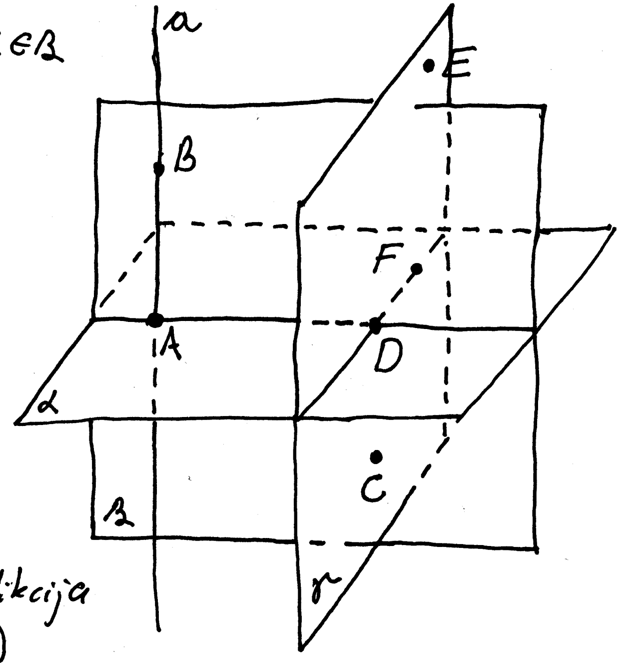
$E \notin p(A, C)$

(Ako bi $E \in p(A,C)$, $C \in \gamma$ i $E \in \gamma$ prema I_6 $p(C,E) \subseteq \gamma$, tad $A \in \gamma$.
Dalje imali bi $A, C, B \in \gamma$ pa prema I_4, I_5 (kako ravan β određuju tačke A, C i B) je $\gamma \equiv \beta \Rightarrow D \in \beta$

\# kontradikcija (sa $D \notin \beta$))

Prema tome tri tražene tačke koje su incidentne sa α su A, C i E

Ostaje nam još slučaj $2^\circ\ C \notin \alpha$



A, B, C nekolinearne, prema I_4 $\exists \beta: A \in \beta, B \in \beta$ i $C \in \beta$

$A \in \alpha$ i $A \in \beta$ prema I_7 $\exists D: D \in \alpha$ i $D \in \beta$

Za ravan β prema I_8 $\exists E: E \notin \beta$

Ako bi bilo $E \in \alpha$, zadatak je gotov.
(tačke A, D i E incidentne su sa α)

Za $E \notin \alpha$ imamo:

E, D i C nekolinearne, prema I_4, I_5 $\exists! \gamma: E, D, C \in \gamma$

$D \in \alpha$ i $D \in \gamma$ prema I_7 $\exists F: F \in \alpha$ i $F \in \gamma$

$F \notin p(A, D)$ (Ako bi $F \in p(A, D)$ imali bi

$\left.\begin{array}{l} A \in p(F, D) \subseteq \gamma,\ A, D, C \in \gamma \\ A, D, C \in \beta \end{array}\right\} \xRightarrow{I_4, I_5} \beta \equiv \gamma \Rightarrow E \in \beta$
#kontradikcija
$(E \notin \beta)$)

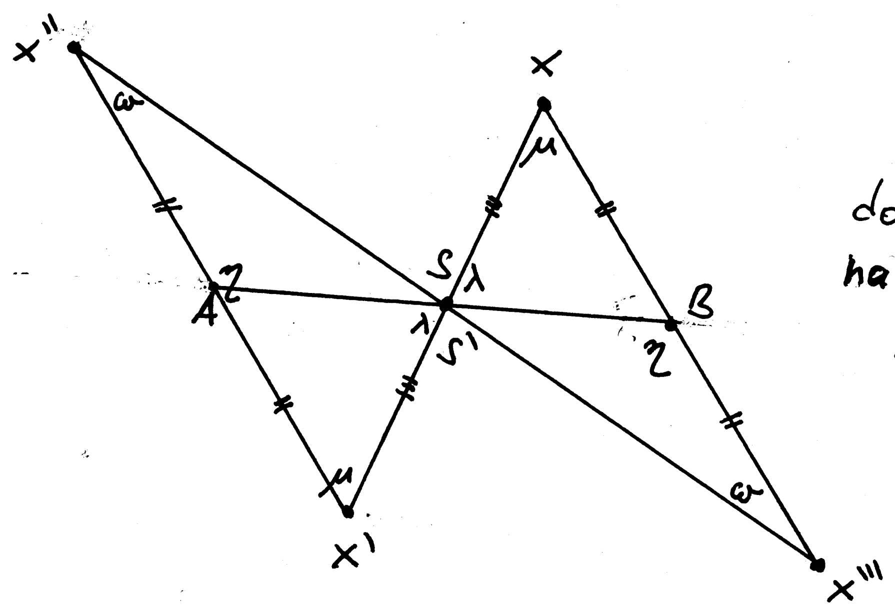
Tačke A, D i F su incidentne sa α.

Našli smo tri tačke koje pripadaju datoj ravni.

q.e.d.

(#) Dokazati da je S sredina duži AB ako i samo ako vrijedi da je $\sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B$.

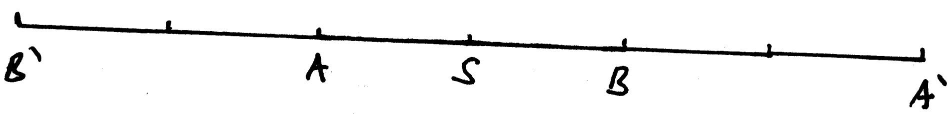
Rj. „$\Rightarrow$": S sredina duži AB $\Rightarrow$ $\sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B$.



Da bi dokazali jednakost $\sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B$ dovoljno je je dokazati na tri ne kolinearne tačke. Posmatraćemo tačke A, B i $X \notin p(A,B)$. S je sredina duži AB.

a) $(\sigma_A \circ \sigma_S)(A) = \sigma_A(\sigma_S(A)) = \sigma_A(B) = B'$ (A je sredina BB')

$(\sigma_S \circ \sigma_B)(A) = \sigma_S(\sigma_B(A)) = \sigma_S(A')$ (B je sredina AA').



$$\left.\begin{array}{l} BA \cong B'A \\ AB \cong A'B \\ AS \cong BS \end{array}\right\} B'S \cong A'S$$

$\Rightarrow \sigma_S(A') = B'$ Prema tome $(\sigma_A \circ \sigma_S)(A) = (\sigma_S \circ \sigma_B)(A)$

b)
$$\left.\begin{array}{l} (\sigma_A \circ \sigma_S)(B) = \sigma_A(\sigma_S(B)) = \sigma_A(A) = A \\ (\sigma_S \circ \sigma_B)(B) = \sigma_S(\sigma_B(B)) = \sigma_S(B) = A \end{array}\right\} \Rightarrow (\sigma_A \circ \sigma_S)(B) = (\sigma_S \circ \sigma_B)(B)$$

c) $(\sigma_A \circ \sigma_S)(X) = \sigma_A(\sigma_S(X)) = \sigma_A(X')$ ($SX \cong SX'$) (S sredina XX')

$\sigma_A(X') = X''$ ($AX' \cong AX''$) (A sredina $X'X''$)

$(\sigma_S \circ \sigma_B)(X) = \sigma_S(\sigma_B(X)) = \sigma_S(X''')$ (B sredina XX''') ($XB \cong X'''B$)

Trebamo pokazati da je S sredina $X''X'''$.

$$\left.\begin{array}{l} XS \cong X'S \\ \measuredangle XSB \cong \measuredangle X'SA = \lambda \\ AS \cong BS \end{array}\right\} \xrightarrow{SUS} \Delta XSB \cong \Delta ASX'$$

$$\Downarrow$$

$AX' \cong BX$; $\measuredangle SXB \cong \measuredangle SX'A = \mu$

Primjetimo sad da je $x'x'' \cong xx'''$, i primjetimo da je

$\mu(x',x'') \parallel \mu(x,x''') \Rightarrow \angle Ax''S \cong \angle Bx'''S$ i $\angle x''AS \cong \angle x'''BS = \eta$

Ako sa S' označim presjek $\{S'\} = AB \cap x''x'''$ iz podudarnosti SUS (ugao ω, $Ax'' \cong Bx'''$, ugao η) slijedi da je $\Delta x''AS' \cong \Delta x'''BS'$

$$\Downarrow$$

$$AS' \cong BS'$$

tj. $S \equiv S'$

Konačno iz

$$\left.\begin{array}{l} x'x'' \cong xx''' \\ \angle x''x'S = \angle Sxx''' = \mu \\ xS \cong x'S \end{array}\right\} \xRightarrow{SUS} \Delta x''x'S \cong \Delta Sxx'''$$

$$\Downarrow$$

$$x''S \cong x'''S$$

(S je sredina duži $x''x'''$)

Znači $\sigma_S(x''') = x''$

Dobili smo $(\sigma_A \circ \sigma_S)(x) = (\sigma_S \circ \sigma_B)(x)$

Prema tome, iz a), b) i c) $\Rightarrow \sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B$

q.e.d.

"$\Leftarrow$": $\sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B \Rightarrow S$ sredina duži AB

$\sigma_A \circ \sigma_S = \sigma_S \circ \sigma_B \quad / \circ \sigma_S$ sa desne

$\sigma_A = \sigma_S \circ \sigma_B \circ \sigma_S \qquad$ Označimo sa $\gamma = \sigma_S \circ \sigma_B \circ \sigma_S$.

Nije teško pokazati da je γ involutivna transformacija čija je jedina fiksna tačka $\sigma_S(B)$ (OVE DVIJE TVRDNJE DOKAZATI ZA VJEŽBU).

Prema tome imamo ili a) γ je identitet ili

b) γ je osna simetrija

c) γ je centralna simetrija

a) i b) nije (ZAŠTO) $\Rightarrow \gamma$ je centralna simetrija sa centrom u tački $\sigma_S(B) \Rightarrow \sigma_A = \sigma_{\sigma_S(B)} \Rightarrow A = \sigma_S(B)$

$\Rightarrow S$ je sredina duži AB

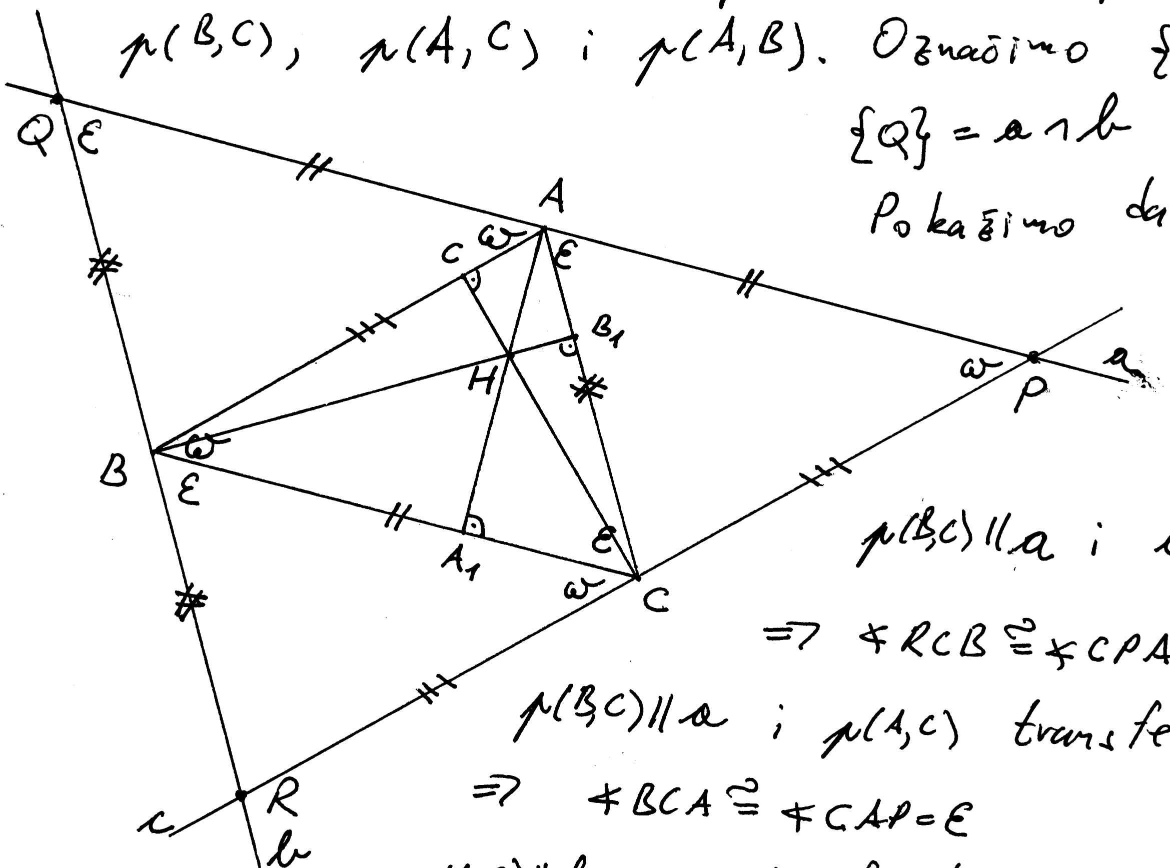
q.e.d.

(7) Dokazati da se visine trougla sijeku u jednoj tački H (H zovemo ortocentar trougla).

Rj: postavka zadatka:

$$\left.\begin{array}{l}\Delta ABC\\ AA_1,\ BB_1 \text{ i } CC_1 \text{ visine trougla}\end{array}\right\} \Rightarrow AA_1 \cap BB_1 \cap CC_1 = \{H\}$$

Neka prave a, b i c redom prolaze kroz tačke A, B i C i neka su redom paralelne sa pravama $p(B,C)$, $p(A,C)$ i $p(A,B)$. Označimo $\{P\} = a \cap c$, $\{Q\} = a \cap b$ i $\{R\} = c \cap b$. Pokažimo da su trouglovi ΔRCB, ΔAPC i ΔQAB podudarni.



$p(B,C) \parallel a$ i c transferzala $\Rightarrow \measuredangle RCB \cong \measuredangle CPA = \omega$

$p(B,C) \parallel a$ i $p(A,C)$ transferzala $\Rightarrow$

$\Rightarrow \measuredangle BCA \cong \measuredangle CAP = \varepsilon$

$p(A,C) \parallel b$ i a transferzala $\Rightarrow \measuredangle BQA \cong \measuredangle CAP = \varepsilon$

$p(A,B) \parallel c$ i $p(B,C)$ transferzala $\Rightarrow \measuredangle RCB \cong \measuredangle CBA = \omega$

$p(B,C) \parallel a$ i $p(A,B)$ transferzala $\Rightarrow \measuredangle CBA \cong \measuredangle BAQ = \omega$

$p(B,C) \parallel a$ i transferzala $\Rightarrow \measuredangle AQB \cong \measuredangle CBR = \varepsilon$

$$\left.\begin{array}{l}\measuredangle ABC \cong \measuredangle APC = \omega\\ \measuredangle BCA \cong \measuredangle CAP = \varepsilon\\ AC \cong AC\end{array}\right\} \overset{UUS}{\Longrightarrow} \begin{array}{c}\Delta ABC \cong \Delta CPA\\ \Downarrow\\ BC \cong AP\end{array}$$

$$\left.\begin{array}{l}\measuredangle BCA \cong \measuredangle BQA = \varepsilon\\ \measuredangle ABC \cong \measuredangle BAQ = \omega\\ AB \cong AB\end{array}\right\} \overset{UUS}{\Longrightarrow} \begin{array}{c}\Delta ABC \cong \Delta ABQ\\ \Downarrow\\ BC \cong AQ\end{array}$$

Možemo primjetiti da su ΔBRC, ΔACP i ΔQBA podudarni (zbog pravila USU)

U trouglu ΔPQR prave $p(A,A_1)$, $p(B,B_1)$ i $p(C,C_1)$ su simetrale stranica pa prema ranije urađenom zadatku one se sijeku u jednoj tački H. Prema tome $AA_1 \cap BB_1 \cap CC_1 = \{H\}$ q.e.d.