Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
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Pismeni ispit iz predmeta **Analiza 3**

1. Razviti funkciju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$.

2. Izračunati trostruki integral $I = \iiint\limits_{\Omega} z\,dx\,dy\,dz$ ako je Ω oblast ograničena površinama $y = x$, $y = 2x$, $2x = 1$, $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

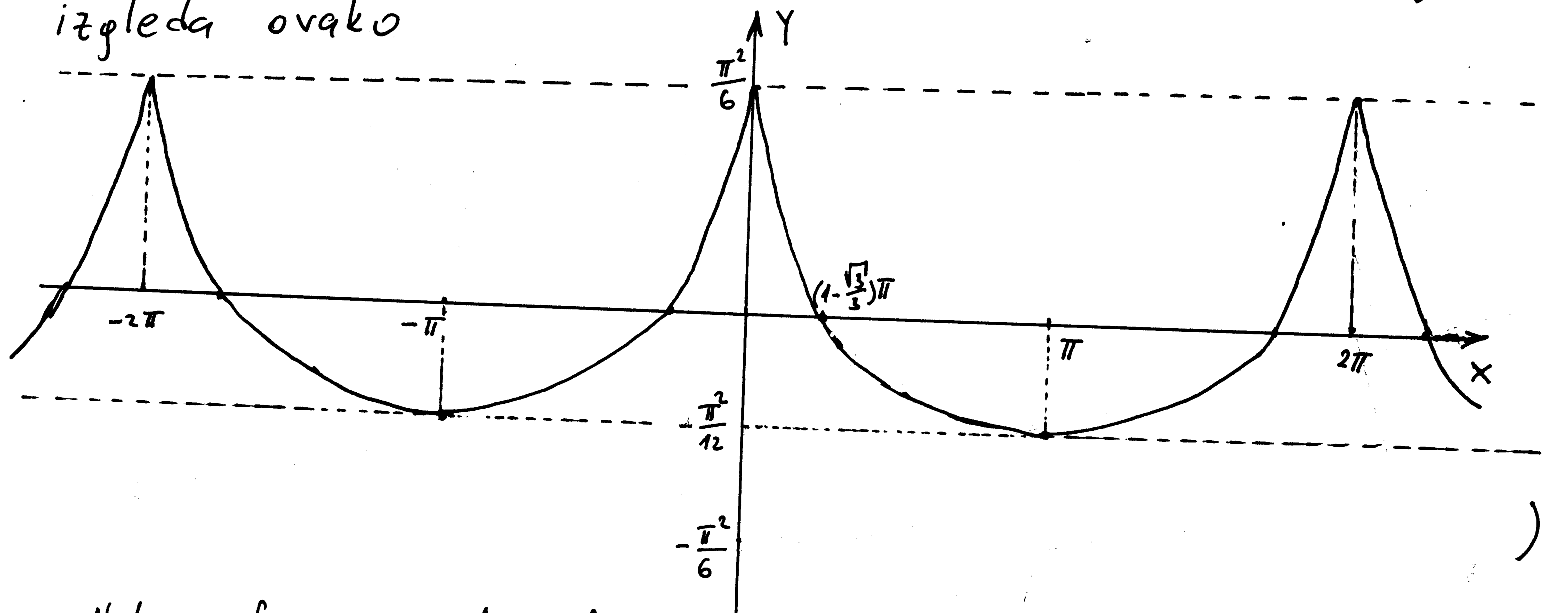
3. Izračunati krivoliniski integral $\displaystyle\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž puta koji ne siječe osu $0y$.

4. Izračunati cirkulaciju vektorskog polja $\vec{v} = (1, xy^2, yz^2)$ duž konture $x^2 + 2y^2 = 4$, $z = 2x$.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

\# Razviti f-ju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ Furijeov red f-je $f(x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$ Furijeovi koeficijenti f-je $f(x)$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow$ $b_n = 0$ $\forall n \in \mathbb{Z}$

Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow$ $a_n = 0$ $\forall n \in \mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in (0,\pi) \\ f(-x), & x \in (-\pi, 0) \end{cases} = \begin{cases} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2), & x \in (0,\pi) \\ \frac{1}{12}(3x^2 + 6\pi x + 2\pi), & x \in (-\pi, 0) \end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx \xlongequal{f^* \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{6\pi}\left(3\cdot\frac{1}{3}x^3\Big|_0^{\pi} - \right.$$

$$\left. - 6\pi\cdot\frac{1}{2}x^2\Big|_0^{\pi} + 2\pi^2\cdot x\Big|_0^{\pi}\right) = \frac{1}{6\pi}\left(\pi^3 - 3\pi^3 + 2\pi^3\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx \xlongequal{f^*(x) \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx =$$

$$= \frac{1}{6\pi}\left(3\int_0^{\pi} x^2\cos nx\,dx - 6\pi\int_0^{\pi} x\cos nx\,dx + 2\pi^2\int_0^{\pi}\cos nx\,dx\right) \overset{(*)}{=}$$

$$I_1 = \int_0^{\pi} x^2\cos nx\,dx = \left| \begin{matrix} u = x^2 & dv = \cos nx\,dx \\ du = 2x\,dx & v = \frac{1}{n}\sin nx \end{matrix} \right| = \frac{1}{n}x^2\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx =$$

$$= \left| \begin{matrix} u = x & dv = \sin nx\,dx \\ du = dx & v = -\frac{1}{n}\cos nx \end{matrix} \right| = \frac{1}{n}\underbrace{(\pi^2\sin n\pi - 0)}_{=0} - \frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi - 0) - \frac{2}{n^2}\sin nx\Big|_0^{\pi} = (-1)^n\frac{2\pi}{n^2}$$

$$I_2 = \int_0^{\pi} x\cos nx\,dx = \left| \begin{matrix} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx \end{matrix} \right| = \frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx =$$

$$= \frac{1}{n}\underbrace{(\pi\sin n\pi - 0)}_{=0} - \frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi} = \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{1}{n^2}((-1)^n - 1)$$

$$I_3 = \int_0^{\pi}\cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^{\pi} = \frac{1}{n}(\sin n\pi - \sin 0) = 0$$

$$\overset{(*)}{=} \frac{1}{\cancel{2\pi}}(-1)^n\frac{\cancel{2\pi}}{n^2} - \frac{1}{n^2}((-1)^n - 1) = \frac{1}{n^2}((-1)^n - (-1)^n + 1) = \frac{1}{n^2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx, \quad x \in (0,\pi)$$

razvoj f-je $f(x)$ u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\text{)}.$$

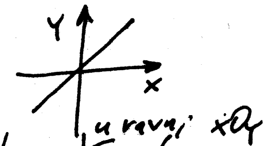
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega: y=x,\ y=2x,\ 2x=1,\ x^2+y^2+z^2=1,\ z\geq 0$

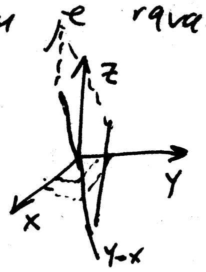
(oblast Ω je ograničena ovim površinama).

Rj: Komentarišimo površi koje čine Ω.

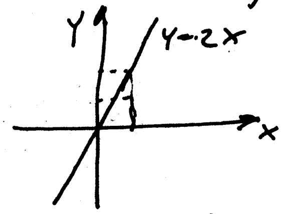
$y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=x$

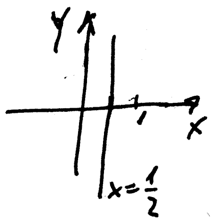


$y=2x$ u ravni je prava

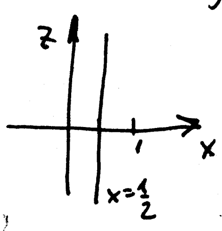


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

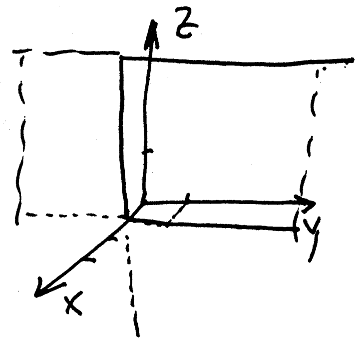
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



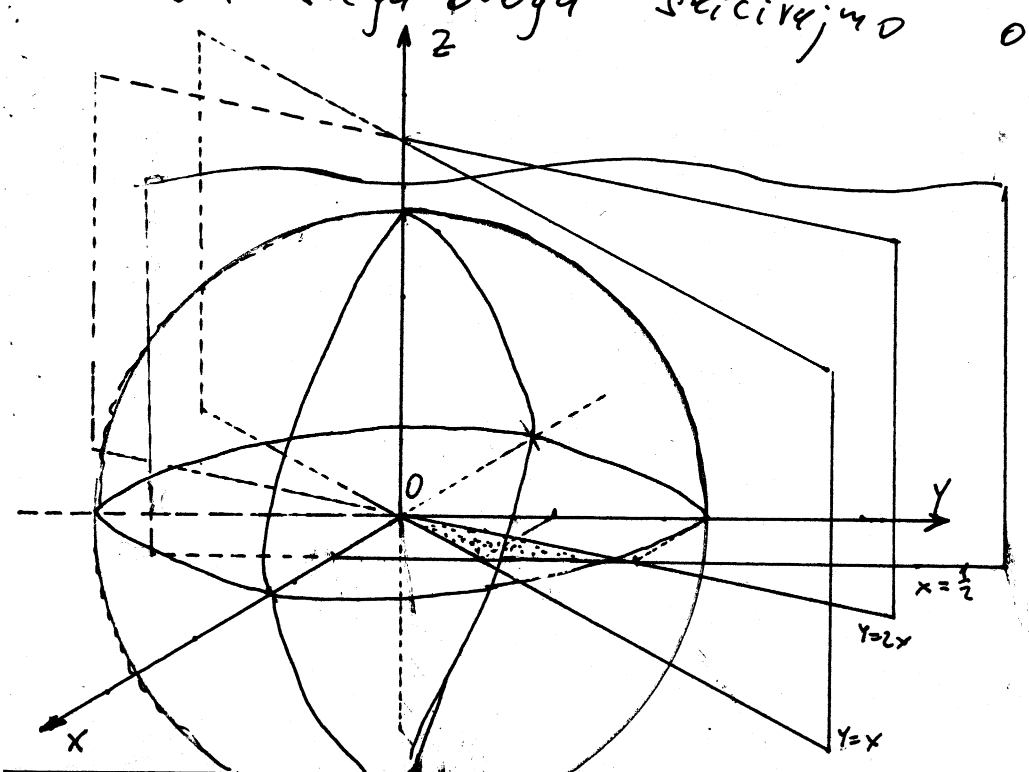
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOY ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa YOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega:\begin{cases}0\leq x\leq \frac{1}{2}\\ x\leq y\leq 2x\\ 0\leq z\leq \sqrt{1-x^2-y^2}\end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}dy\int_0^{\sqrt{1-x^2-y^2}}z\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}\frac{1}{2}z^2\Big|_0^{\sqrt{1-x^2-y^2}}dy=$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}dx\int_x^{2x}(1-x^2-y^2)\,dy=\frac{1}{2}\int_0^{1/2}\left(y\Big|_x^{2x}-x^2y\Big|_x^{2x}-\frac{1}{3}y^3\Big|_x^{2x}\right)dx=$$

$$=\frac{1}{2}\int_0^{1/2}\left(x-x^3-\frac{1}{3}7x^3\right)dx=\frac{1}{2}\int_0^{1/2}\left(x-\frac{10}{3}x^3\right)dx=\frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^{\frac{1}{2}}-\frac{\overset{5}{\cancel{10}}}{3}\cdot\frac{1}{\underset{2}{\cancel{4}}}x^4\Big|_0^{\frac{1}{2}}\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{4}-\frac{5}{6}\cdot\frac{1}{16}\right)=\frac{1}{2}\left(\frac{1}{8}-\frac{5}{\ }\right)=\frac{1}{2}\cdot\frac{12-5}{96}=\frac{7}{192}$$

(#) Izračunati krivoliniski integral $\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž putanje koja ne siječe osu Oy.

Rj:
Vrijednost integrala $I = \int_c P(x,y)\,dx + Q(x,y)\,dy$ ne zavisi od vrste konture c ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

U našem slučaju $I = \int_{(2,1)}^{(1,2)} \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$ $\quad P(x,y) = \frac{y}{x^2}, \quad Q(x,y) = -\frac{1}{x}$

$$\frac{\partial P}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial Q}{\partial x} = \frac{1}{x^2}$$

Prema tome vrijednost integrala ne zavisi od vrste krive linije c koja spaja tačke $(2,1)$ i $(1,2)$.

I način: Odredićemo primitivnu f-ju

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

ovo je egzaktna dif. jednačina

$$u = u(x,y)$$

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$du = \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$$

$$\frac{\partial u}{\partial x} = \frac{y}{x^2}, \quad \frac{\partial u}{\partial y} = -\frac{1}{x} \quad \ldots (1)$$

$$\partial u = \frac{y}{x^2}\,\partial x$$

$$u = \int \frac{y}{x^2}\,dx + \varphi(y) = y\,\frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$$

$$\frac{\partial u}{\partial y} = -\frac{1}{x} + \varphi'(y) \quad \ldots (2)$$

$$(1) \text{ i } (2) \Rightarrow \varphi'(y) = 0$$

$$\varphi(y) = c$$

$$u = -\frac{y}{x} + c$$

$$\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2} = \int_{(2,1)}^{(1,2)} du = -\frac{y}{x}\Big|_{(2,1)}^{(1,2)} = -\frac{2}{1} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 2 = -\frac{3}{2}$$

II način: Spojimo tačke $(2,1)$ i $(1,2)$ nekom krivom (ili pravom) ili izlomljenom pravom linijom i izračunamo integral na klasičan način.

Izračunati cirkulaciju vektorskog polja $\vec{v}=(1, xy^2, yz^2)$ duž konture $x^2+2y^2=4$, $z=2x$.

R.:

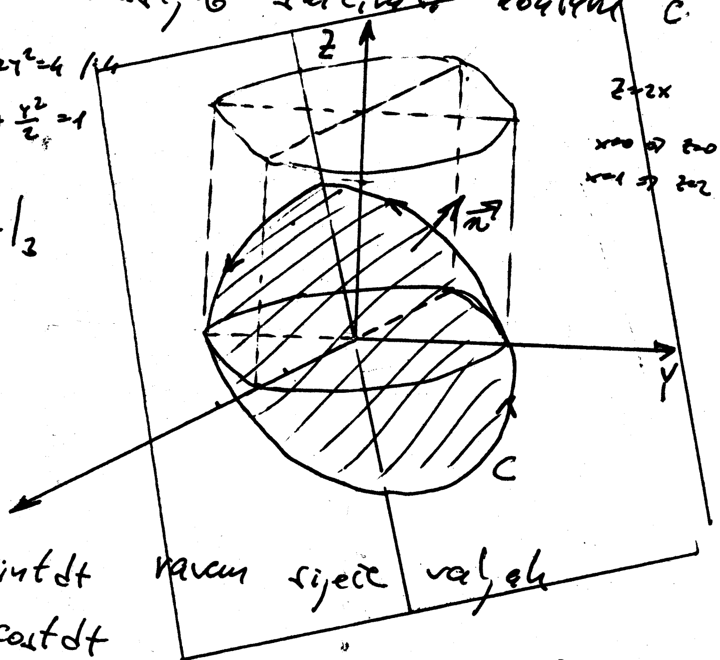
Cirkulacija vektorskog polja $\vec{v}$ $(\vec{v}=(v_x, v_y, v_z))$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r}=\int_c v_x\, dx+v_y\, dy+v_z\, dz$$

Pokušajmo skicirati konturu c.

$x^2+2y^2=4 \;/:4$

$\frac{x^2}{4}+\frac{y^2}{2}=1$



$z=2x$

$x=0 \Rightarrow z=0$

$x=1 \Rightarrow z=2$

ravan siječe valjak

U našem slučaju

$$C=\int_c dx+xy^2\, dy+yz^2\, dz=I_1+I_2+I_3$$

parametriziramo konturu c

kako je $\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{2}}\right)^2=1$ uvedimo smjene

$$\left.\begin{array}{l}\frac{x}{2}=\cos t\\ \frac{y}{\sqrt{2}}=\sin t\\ z=z\end{array}\right\} \Rightarrow \begin{array}{ll} x=2\cos t & dx=-2\sin t\, dt\\ y=\sqrt{2}\sin t & dy=\sqrt{2}\cos t\, dt\\ z=4\cos t & dz=-4\sin t\, dt\\ 0\le t\le 2\pi & \end{array}$$

$\sin^2 x+\cos^2 x=1$

$\cos^2 x-\sin^2 x=\cos 2x$

$2\sin^2 x=1-\cos 2x$

$2\cos^2 x=1+\cos 2x$

$$C=\int_0^{2\pi}\left(-2\sin t+2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t+\sqrt{2}\sin t\cdot 16\cdot\cos^2 t(-4\sin t)\right)dt$$

pojednostavimo računanje ovog integrala

$$I_1=\int_c dx=\int_0^{2\pi}-2\sin t\, dt=2\cos t\Big|_0^{2\pi}=2(1-1)=0$$

$4\cos^2 t\sin^2 t=(2\cos t\sin t)^2$

$$I_2=\int_c xy^2\, dy=\int_0^{2\pi}2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t\, dt=4\sqrt{2}\int_0^{2\pi}\cos^2 t\sin^2 t\, dt=\sqrt{2}\int_0^{2\pi}\sin^2 2t\, dt$$

$$=\frac{\sqrt{2}}{2}\int_0^{2\pi}(1-\cos 2\cancel{t})\,dt=\frac{\sqrt{2}}{2}\left(t\Big|_0^{2\pi}\ \frac{1}{2}\sin t\Big|_0^{2\pi}\right)=\frac{\sqrt{2}}{2}(2\pi-0)=\pi\sqrt{2}$$

$$I_3=\int_c yz^2\, dz=\int_0^{2\pi}\sqrt{2}\sin t\cdot 16\cos^2 t\cdot(-4)\sin t\, dt=\overbrace{-64\sqrt{2}}^{(16)\cdot 4}\int_0^{2\pi}\sin^2 t\cos^2 t=-16\, I_2=-16\pi\sqrt{2}$$

$$C=\pi\sqrt{2}-16\pi\sqrt{2}=-15\pi\sqrt{2}$$

II način

pomoću Stoksove formule

površinski integral

$$C=\int_C \vec{v}\,d\vec{r}=\iint_S \vec{n}\,\mathrm{rot}\,\vec{v}\,dS=\iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} dS$$

gdje je S površina koju zatvara kontura C, $\vec{n}=(\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na S

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & xy^2 & yz^2 \end{vmatrix}=(z^2-0)\vec{i}-(0-0)\vec{j}+(y^2-0)\vec{k}=(z^2,0,y^2)$$

$$C=\iint_S (z^2\cos\alpha+y^2\cos\gamma)\,dS$$

posmatramo ravan $z=2x$ i vektor normale na ovu ravan, zato što je naša elipsa unutar ove ravni

$2x-z=0 \qquad \vec{n}=(2,0,-1)$

$|\vec{n}|=\sqrt{4+1}=\sqrt{5} \qquad \vec{n}_0=\left(\frac{2}{\sqrt{5}},0,-\frac{1}{\sqrt{5}}\right)$

projekcija površi S na xOy ravan je elipsa $x^2+2y^2=4$

prema tome

$$C=\iint_S (z^2\cos\alpha+y^2\cos\gamma)\,dS=\iint_{D'} z^2\,dy\,dz-\iint_{D''} y^2\,dx\,dy$$

$x^2+2y^2=4$

$z=2x$

$\left(\frac{z}{2}\right)^2+2y^2=4 \quad /\cdot 4$

$z^2+8y^2=16 \quad /:16$

$\frac{z^2}{16}+\frac{y^2}{2}=1$

Projekcija površi S na yOz ravan je elipsa $D': \left\{\frac{z^2}{16}+\frac{y^2}{2}=1\right.$

$$\iint_{D'} z^2\,dy\,dz=\left|\begin{matrix} \frac{z}{4}=r\cos\varphi & z=4r\cos\varphi & dy\,dz=4\sqrt{2}\,r\,dr\,d\varphi \\ \frac{y}{\sqrt{2}}=r\sin\varphi & y=\sqrt{2}\,r\sin\varphi & \\ & 0\le\varphi\le 2\pi & \\ & 0\le r\le 1 & \end{matrix}\right|=\iint_{D_1'} 16r^2\cos^2\varphi\, 4\sqrt{2}\,r\,dr\,d\varphi=$$

$$=64\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\cos^2\varphi\,d\varphi=64\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\frac{1}{2}(1+\cos 2\varphi)\,d\varphi=32\sqrt{2}\cdot\frac{r^4}{4}\Big|_0^1\left(2\pi-\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}\right)=$$

$=16\sqrt{2}\,\pi$

Projekcija površi S na xOy ravan je elipsa $D'': \left\{\frac{x^2}{4}+\frac{y^2}{2}=1\right.$

$$\iint_{D''} y^2\,dx\,dy=\left|\begin{matrix} \frac{x}{2}=r\cos\varphi & x=2r\cos\varphi & dx\,dy=2\sqrt{2}\,r\,dr\,d\varphi \\ \frac{y}{\sqrt{2}}=r\sin\varphi & y=\sqrt{2}\,r\sin\varphi & 0\le\varphi\le 2\pi \\ & & 0\le r\le 1 \end{matrix}\right|=\int_0^1\int_0^{2\pi} 2r^2\sin^2\varphi\cdot 2\sqrt{2}\,r\,dr\,d\varphi=$$

$$=4\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\sin^2\varphi\,d\varphi=\ldots=\pi\sqrt{2}$$

$$C=16\pi\sqrt{2}-\pi\sqrt{2}=15\pi\sqrt{2}$$