Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 18.06.2011.

Pismeni ispit iz predmeta **Analiza 3**

1. Razložiti funkciju $f(x,y) = arc\,tg(x^2y - 2e^{x-1})$ po formuli Tejlora u okolini tačke $M(1,3)$ do stepena drugog reda zaključno.

2. Izračunati dvostruki integral dat u polarnim koordinatama $I = \iint\limits_D \rho sin\varphi \, d\rho \, d\varphi$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho = a$, $\varphi = \frac{\pi}{2}$ i $\varphi = \pi$;
b) polukrug $\rho \leq 2acos\varphi$, $0 \leq \varphi \leq \frac{\pi}{2}$;
c) oblast između linija $\rho = 2 + cos\varphi$ i $\rho = 1$ (obavezno nacrtati izgled oblasti D u sve tri slučaja).

3. Izračunati krivoliniski integral $I = \int\limits_L (4\sqrt[3]{x} - 3\sqrt{y})dl$ između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;
b) po liniji astroide $x = cos^3t$, $y = sin^3t$.

4. Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral

$$\oiint\limits_S 4x^3dydz + 4y^3dxdz - 6z^4dxdy$$

gdje je S vanjska strana cilindra $x^2 + y^2 = a^2$ koji se nalazi između ravni $z = 0$ i $z = h$.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

Razložiti f-ju $f(x,y) = \operatorname{arctg}(x^2y - 2e^{x-1})$ po formuli Tejlova u okolini tačke $M(1,3)$ do stepena drugog reda zaključno.

Rj. Prisjetimo se kako izgleda Tejlorova formula za f-ju dvije promjenjive u okolini tačke $M_0(x_0, y_0)$

$$f(x,y) = f(x_0,y_0) + \sum_{k=1}^{n} \frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^k f(x_0,y_0) + R_n(x_0,y_0)$$

$$= f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) +$$

$$\begin{matrix} 1 \\ 1\;1 \\ 1\;2\;1 \\ 1\;3\;3\;1 \end{matrix}$$

$$+ \frac{1}{2}\left(f''_{xx}(x_0,y_0)(x-x_0)^2 + 2f''_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f''_{yy}(x_0,y_0)(y-y_0)^2\right)$$

$$+ \frac{1}{6}\left(\frac{\partial^3 f(x_0,y_0)}{\partial x^3}(x-x_0)^3 + 3\frac{\partial^3 f(x_0,y_0)}{\partial x^2 \partial y}(x-x_0)^2(y-y_0) + 3\frac{\partial^3 f(x_0,y_0)}{\partial x \partial y^2}(x-x_0)(y-y_0)^2 + \frac{\partial^3 f(x_0,y_0)}{\partial y^3}(y-y_0)^3\right)$$

$$+ \ldots$$

$$\frac{\partial f}{\partial x} = \frac{2xy - 2e^{x-1}\cdot 1}{1+(x^2y-2e^{x-1})^2} = 2\,\frac{xy - e^{x-1}}{1+(x^2y-2e^{x-1})^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^2 - 0}{1+(x^2y-2e^{x-1})^2} = \frac{x^2}{1+(x^2y-2e^{x-1})^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\left(\frac{(y-e^{x-1})\cdot(1+(x^2y-2e^{x-1})^2) - (xy-e^{x-1})\cdot 2(x^2y-2e^{x-1})\cdot(2xy-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}\right)$$

$$= 2\,\frac{y-e^{x-1}}{1+(x^2y-2e^{x-1})^2} - 8\,\frac{(xy-e^{x-1})(x^2y-2e^{x-1})(xy-e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$= 2\,\frac{y-e^{x-1}}{1+(x^2y-2e^{x-1})^2} - 8\,\frac{(xy-e^{x-1})^2(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2\,\frac{x(1+(x^2y-2e^{x-1})^2) - (xy-e^{x-1})\cdot 2(x^2y-2e^{x-1})\,x^2}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$= 2\,\frac{x}{1+(x^2y-2e^{x-1})^2} - 4\,\frac{x^2(xy-e^{x-1})(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x^2\, 2(x^2y-2e^{x-1})\cdot x^2}{(1+(x^2y-2e^{x-1})^2)^2} = -2\,\frac{x^4(x^2y-2e^{x-1})}{(1+(x^2y-2e^{x-1})^2)^2}$$

$$\frac{\partial f}{\partial x}(1,3) = 2\cdot\frac{3-e^0}{2} = 2$$

$$\boxed{1+(x^2y-2e^{x-1})^2 \text{ za } x=1,\ y=3 = 1+(3-2\cdot e^0)^2 = 1+1 = 2}$$

$$\frac{\partial f}{\partial y}(1,3) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2}(1,3) = 2\cdot\frac{2}{2} - 8\cdot\frac{4\cdot(3-2)}{4} = 2-8 = -6$$

$$\frac{\partial^2 f}{\partial x\partial y}(1,3) = 2\cdot\frac{1}{2} - 4\cdot\frac{1\cdot 2\cdot 1}{4} = 1-2 = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cdot\frac{1\cdot(3-2)}{4} = -\frac{1}{2}$$

$$f(1,3) = \operatorname{arc\,tg}(3-2) = \operatorname{arc\,tg} 1 = \frac{\pi}{4}$$

$$\operatorname{arc\,tg}(x^2y-2e^{x-1}) = \frac{\pi}{4} + 2(x-1) + \frac{1}{2}(y-3) -$$

$$-3(x-1)^2 - (x-1)(y-3) - \frac{1}{4}(y-3)^2 + \dots$$

Izračunati dvostruki integral dat u polarnim koordinatama

$$I=\iint_D \rho\sin\varphi\,d\rho\,d\varphi$$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho=a$, $\varphi=\frac{\pi}{2}$ i $\varphi=\pi$

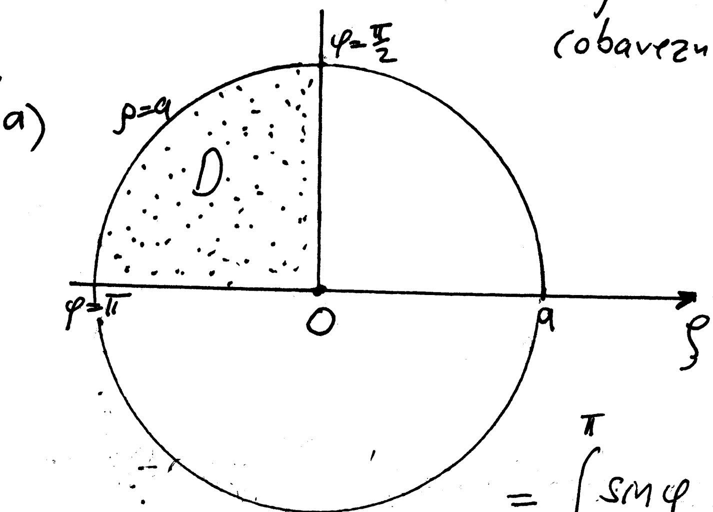
b) polukrug $\rho\le 2a\cos\varphi$, $0\le\varphi\le\frac{\pi}{2}$

c) oblast između linija $\rho=2+\cos\varphi$ i $\rho=1$.

(obavezno nacrtati izgled oblasti D)

Rj.

a)



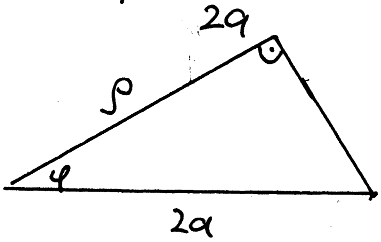
$$I=\iint_D \rho\sin\varphi\,d\rho\,d\varphi=$$

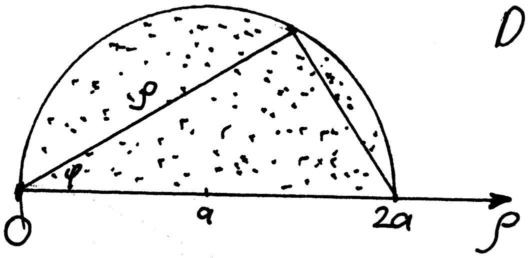
$$=\int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,d\varphi\int_0^a\rho\,d\rho=$$

$$=\int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,\frac{\rho^2}{2}\Big|_0^a\,d\varphi=\frac{a^2}{2}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right)=\frac{a^2}{2}$$

b) $\rho=2a\cos\varphi$

$\cos\varphi=\frac{\rho}{2a}$





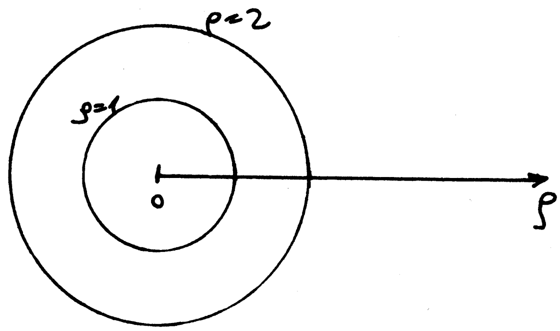
$$D:\begin{cases}0\le\rho\le 2a\cos\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$

$$I=\iint_D \rho\sin\varphi\,d\rho\,d\varphi=\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho=\int_0^{\frac{\pi}{2}}\frac{1}{2}\rho^2\Big|_0^{2a\cos\varphi}\sin\varphi\,d\varphi=$$

$$=\frac{1}{2}\cdot 4a^2\int_0^{\pi/2}\sin\varphi\cos^2\varphi\,d\varphi=\left|\begin{matrix}\cos\varphi=t\\ -\sin\varphi\,d\varphi=dt\\ \varphi\Big|_0^{\frac{\pi}{2}}\Rightarrow t\Big|_1^0\end{matrix}\right|=2a^2\left(-\int_1^0 t^2\,dt\right)=$$

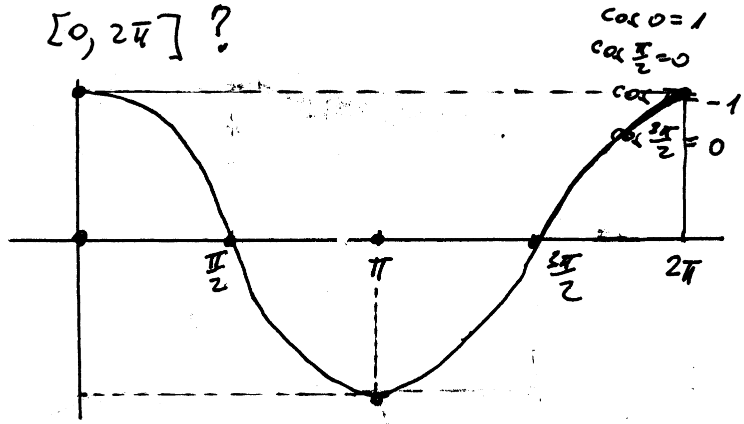
$$=2a^2\int_0^1 t^2\,dt=2a^2\cdot\frac{t^3}{3}\Big|_0^1=\frac{2}{3}a^2$$ traženo rješenje

c) linije $\rho=1$ i $\rho=2$ nije teško nacrtati

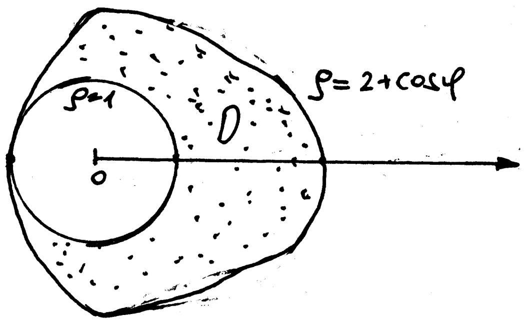


Problem predstavlja linija
$\rho = 2+\cos\varphi$

Kako izgleda $\cos\varphi$ na intervalu $[0, 2\pi]$?



Ako liniji $\rho=2$ dodamo $\cos\varphi$ imamo otprilike sljedeću sliku:



$$D: \begin{cases} 1 \leq \rho \leq 2+\cos\varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$I=\iint_D \rho \sin\varphi \, d\rho \, d\varphi = \int_0^{2\pi} \sin\varphi \, d\varphi \int_1^{2+\cos\varphi} \rho \, d\rho = \int_0^{2\pi} \frac{\rho^2}{2}\Big|_1^{2+\cos\varphi} \sin\varphi \, d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi} \left((2+\cos\varphi)^2 - 1^2 \right) \sin\varphi \, d\varphi = \frac{1}{2}\int_0^{2\pi} (4+4\cos\varphi+\cos^2\varphi-1) \sin\varphi \, d\varphi$$

$$= -\frac{1}{2}\int_0^{2\pi} (\cos^2\varphi + 4\cos\varphi + 3) \, d\cos\varphi = \left(-\frac{1}{2}\right)\left(\frac{\cos^3\varphi}{3}\Big|_0^{2\pi} + 4\,\frac{\cos^2\varphi}{2}\Big|_0^{2\pi} + 3\cos\varphi\Big|_0^{2\pi} \right)$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{3}(1-1) + 2(1-1) + 3(1-1) \right) = 0$$ traženo rješenje

Izračunati krivoliniski integral $I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$

između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;

b) po liniji astroide $x=\cos^3 t$, $y=\sin^3 t$.

Rj.

$$I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl$$

Ovo je krivoliniski integral prve vrste. Prisjetimo se

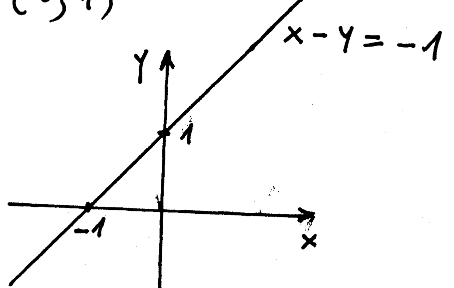
Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ tada

$$\int_L f(x,y)\,dl=\int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases} x=\mu(t) \\ y=\theta(t) \end{cases}$ gdje $t_1\le t\le t_2$

$$\int_L f(x,y)\,dl=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{(\mu'(t))^2+(\theta'(t))^2}\,dt$$

a) $E(-1;0)$
$F(0;1)$



$-y=-x-1,\quad x\in[-1,0]$ tj. $y=x+1$

$y'=1 \Rightarrow dl=\sqrt{1+1^2}\,dx=\sqrt{2}\,dx$

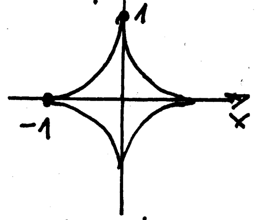
$$I=\int_L (4\sqrt[3]{x}-3\sqrt{y})\,dl=\int_{-1}^{0}(4x^{\frac{1}{3}}-3(x+1)^{\frac{1}{2}}\sqrt{2}\,dx$$

$$=4\sqrt{2}\int_{-1}^{0}x^{\frac{1}{3}}dx-3\sqrt{2}\int_{-1}^{0}(x+1)\,dx=$$

$$=4\sqrt{2}\cdot\frac{3}{4}x^{\frac{4}{3}}\Big|_{-1}^{0}-3\sqrt{2}\int_{-1}^{0}(x+1)^{\frac{1}{2}}d(x+1)=3\sqrt{2}\,(0-1)-3\sqrt{2}\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_{-1}^{0}=-5\sqrt{2}$$

↑ traženo rješenje

b)



astroida

$x=\cos^3 t,\quad x'=-3\cos^2 t\sin t$

$y=\sin^3 t,\quad y'=3\sin^2 t\cos t$

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\sqrt{\cos^2 t \sin^2(t)\underbrace{(\cos^2 t + \sin^2 t)}_{=1}} =$$

$$= 3\,|\sin t \cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π, pa je

$dl = -3\sin t \cos t\, dt$

$$I = \int_L \left(4\sqrt[3]{x} - 3\sqrt{y}\right) dl = \int_{\frac{\pi}{2}}^{\pi} \left(4\sqrt[3]{\cos^3 t} - 3\sqrt{\sin^3 t}\right)(-3\sin t \cos t)\, dt$$

$$= -12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t \sin t\, dt + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t \cos t\, dt =$$

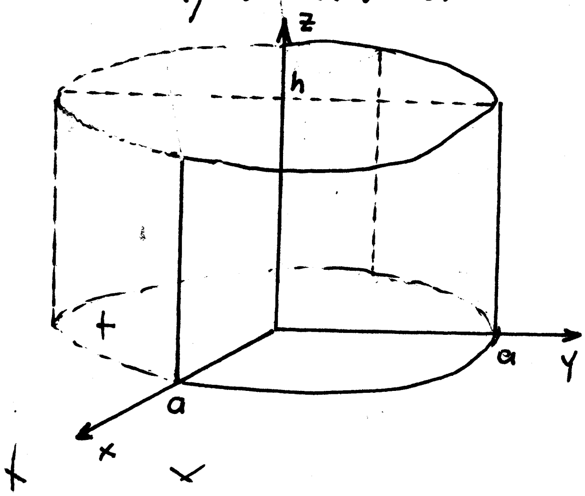
$$= +12\int_{\frac{\pi}{2}}^{\pi} \cos^2 t\, d\cos t + 9\int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t\, d\sin t = 12\cdot\frac{\cos^3 t}{3}\Big|_{\frac{\pi}{2}}^{\pi} + 9\cdot\frac{\sin^{\frac{7}{2}} t}{\frac{7}{2}}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\left((-1)^3 - 0\right) + \frac{18}{7}\left(0 - 1^{\frac{7}{2}}\right) = -4 - \frac{18}{7} = -\frac{46}{7}$$ traženo rješenje

(#) Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral $I = \oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy$

gdje je S vanjska strana cilindra $x^2+y^2=a^2$ koji se nalazi između ravni $z=0$ i $z=h$.

Rj. Skicirajmo dati cilindar



Prisjetimo se formule Gauss-Ostrogradski

$$\oiint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

Ω - unutrašnjost oblasti S

$P(x,y,z) = 4x^3 \qquad \frac{\partial P}{\partial x} = 12x^2$

$Q(x,y,z) = 4y^3 \qquad \frac{\partial Q}{\partial y} = 12y^2$

$R(x,y,z) = 6z^4 \qquad \frac{\partial R}{\partial z} = 24z^3$

$$\oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy = 12\iiint\limits_\Omega (x^2+y^2-2z^3)\,dx\,dy\,dz =$$

$$= \left| \begin{array}{ll} \text{uvedimo cilindrične koordinate} & \\ x = r\cos\varphi & \Omega \xrightarrow{\text{transformiše}} \Omega' \\ y = r\sin\varphi & 0 \le r \le a \\ z = z & 0 \le \varphi \le 2\pi \\ dx\,dy\,dz = r\,dr\,d\varphi\,dz & 0 \le z \le h \\ x^2+y^2 = \ldots = r^2 & \end{array} \right| = 12\iiint\limits_{\Omega'} (r^2 - 2z^3)\,r\,dr\,d\varphi\,dz =$$

$$= 12\int_0^{2\pi} d\varphi \int_0^a dr \int_0^h (r^3 - 2rz^3)\,dz = 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 z\Big|_0^h - 2r\cdot\frac{1}{4}z^4\Big|_0^h\right) dr$$

$$= 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 h - \frac{1}{2} r h^4\right) dr = 12\,\varphi\Big|_0^{2\pi} \left(h\frac{1}{4}r^4\Big|_0^a - \frac{1}{2}h^4\cdot\frac{1}{2}r^2\Big|_0^a\right) =$$

$$= 24\pi\cdot\frac{1}{4}h\,(a^4 - h^3a^2) = 6\pi h a^2(a^2 - h^3) \quad \text{traženo rješenje}$$