Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 22.06.2010.

Pismeni ispit iz predmeta **Analiza 3**

1. Ako je $z = \dfrac{y}{f(x^2 - y^2)}$ gdje je f diferencijalna funkcija, izračunati $\dfrac{1}{x} \cdot \dfrac{\partial z}{\partial x} + \dfrac{1}{y} \cdot \dfrac{\partial z}{\partial y}$.

2. Izračunati $\displaystyle\iint_D dx\,dy$, ako je $D :\ y^2 - x^2 = 1,\ x^2 + y^2 = 4$.

3. Izračunati krivoliniski integral $I = \displaystyle\int_c (x^2 + y^2)\,dx + x^2 y\,dy$ gdje je c kontura trapeza koga obrazuju prave $x = 0$, $y = 0$, $x + y = 1$ i $x + y = 2$.

4. Izračunati $\displaystyle\iint_S dS$, ako je S površina djela sfere $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = a^2\}$ koja se nalazi u unutrašnjosti cilindra $S_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z \in \mathbb{R}\}$, $b < a$.

(Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Ako je $z = \frac{y}{f(x^2-y^2)}$, gde je f diferencijabilna f-ja, izračunati $\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y}$

Rj. $z = y f^{-1}(x^2-y^2) = y f^{-1}(u)$, gde je $u = x^2-y^2$

$$\frac{\partial z}{\partial x} = y(-1) f_u^{-2}(x^2-y^2)\cdot 2x = \frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \left(y f^{-1}(u)\right)'_y = 1\cdot f^{-1}(u) + y\cdot(-1) f_u^{-2}(u)\cdot(-2y) =$$

$$= \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f_u^2(x^2+y^2)}$$

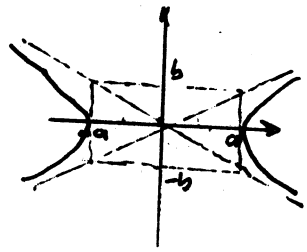
$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{-2y}{f_u^2(x^2+y^2)} + \frac{1}{y f(x^2-y^2)} + \frac{2y}{f_u^2(x^2+y^2)} =$$

$$= \frac{1}{y f(x^2-y^2)} = \frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)} = \frac{z}{y^2}$$

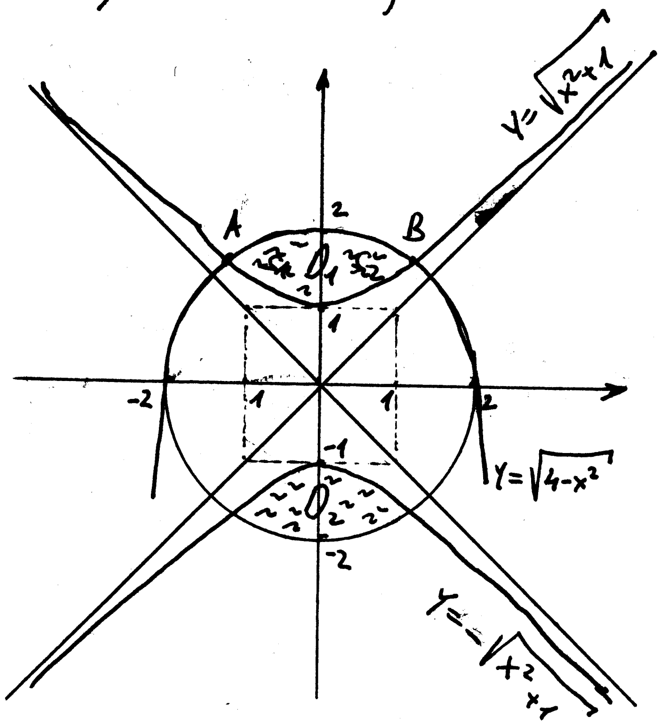
prema tome $\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$.

Izračunati $I=\iint\limits_D dx\,dy$, ako je $D:\ y^2-x^2=1,\ \ x^2+y^2=4.$

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$
je krug sa centrom u $(0,0)$
poluprečnika $r=2$

$$D=D_1\cup D_2$$

$$I=\iint\limits_D dx\,dy=\iint\limits_{D_1\cup D_2} dx\,dy=2\iint\limits_{D_1} dx\,dy$$

$y^2=4-x^2$ $\qquad$ $y^2=x^2+1$

$y=\pm\sqrt{4-x^2}$ $\qquad$ $y=\pm\sqrt{x^2+1}$

Nađimo presječnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$\sqrt{x^2+1}=\sqrt{4-x^2}\ \ /^2$

$x^2+1=4-x^2$

$2x^2-3=0$ $\qquad x_1=-\sqrt{\frac{3}{2}}$

$2x^2=3$ $\qquad x_2=\sqrt{\frac{3}{2}}$

$x^2=\frac{3}{2}$

$x_1=-\sqrt{\frac{3}{2}} \Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$

Presječne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primjetimo da je oblast D_1 simetrična

$$\iint\limits_{D_1} dx\,dy=\iint\limits_{S_1} dx\,dy+\iint\limits_{S_2} dx\,dy=2\iint\limits_{S_2} dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}} dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}} dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2}\quad \text{(sami)}$$

(metoda Ostrogradskog)

$$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}dx=(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}}\quad /'$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \lambda\frac{1}{\sqrt{x^2+1}} \quad \Big/ \cdot \sqrt{x^2+1}$$

$$x^2+1 = \underline{a}(x^2+1) + \underline{a}x^2 + \underline{\underline{bx}} + \lambda$$

$$2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$

$$b = 0$$

$$a+\lambda = 1 \qquad \lambda = \frac{1}{2}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}| + C$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{4-x^2}\,dx = 2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \quad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{x^2+1}\,dx = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2}\ln\left|\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

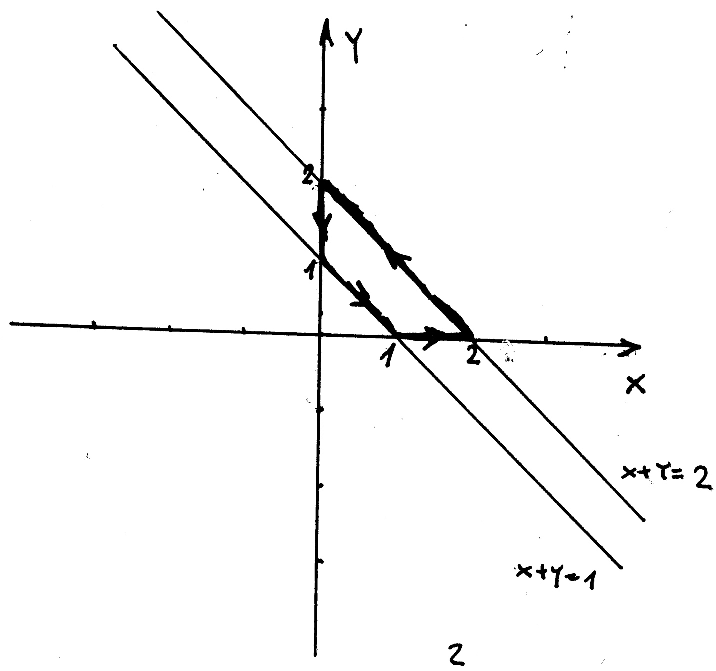
$$I = \iint_D dx\,dy = 2\iint_{D_1} dx\,dy = 2\cdot 2\iint_{S_2} dx\,dy = 4\left(2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \right.$$

$$\left.\frac{\sqrt{15}}{4} - \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|\right) = 8\arcsin\frac{\sqrt{6}}{4} - 2\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

traženo rješenje.

(#) Izračunati krivolinijski integral $I=\int\limits_c (x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koja obrazuju prave $x=0,\ y=0,\ x+y=1,\ x+y=2.$

Rj.



Ako je $c:\ y=\eta(x),\ a\le x\le b$

$$\int\limits_c P(x,y)\,dx+Q(x,y)\,dy=\int\limits_a^b [P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)]\,dx$$

U našem slučaju postoje 4 krive

$C_1:\ y=0,\ 1\le x\le 2$

$C_1:\ y=-x+2,\ 2\ge x\ge 0$

$C_3:\ x=0,\ 2\ge y\ge 1$

$C_4:\ y=-x+1,\ 0\le x\le 1$

$$I=I_1+I_2+I_3+I_4,\quad I_1=\int\limits_1^2 (x^2+x^2\cdot 0)\,dx=\int\limits_1^2 x^2\,dx=\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}(8-1)=\frac{7}{3}$$

$$I_2=\int\limits_2^0 \left(x^2+(-x+2)^2+x^2(-x+2)\cdot(-1)\right)dx=\int\limits_2^0 (\underline{\underline{x^2}}+\underline{\underline{x^2}}-4x+4\ \underline{+x^3}\ \underline{\underline{-2x^2}})\,dx=$$

$$=\int\limits_2^0 (x^3-4x+4)\,dx=\frac{1}{4}x^4\Big|_2^0-4\cdot\frac{1}{2}x^2\Big|_2^0+4x\Big|_2^0=-4+8-8=-4$$

$$I_3=\int\limits_2^1 (y^2\cdot 0+0\cdot y)\,dy=0$$

$$I_4=\int\limits_0^1 \left(x^2+(-x+1)^2+x^2(-x+1)\cdot(-1)\right)dx=\int\limits_0^1 (x^2+\underline{x^2}-2x+1\ \underline{\underline{+x^3}}\ \underline{-x^2})\,dx=$$

$$=\int\limits_0^1 (x^3+x^2-2x+1)\,dx=\frac{1}{4}x^4\Big|_0^1+\frac{1}{3}x^3\Big|_0^1-2\cdot\frac{1}{2}x^2\Big|_0^1+x\Big|_0^1=\frac{1}{4}+\frac{1}{3}-1+1=\frac{7}{12}$$

$$I=I_1+I_2+I_3+I_4=\frac{7}{3}+(-4)+\frac{7}{12}=\frac{28-48+7}{12}=-\frac{13}{12}$$ vrijednost krivolinijskog integrala

II način: Greenova formula ...

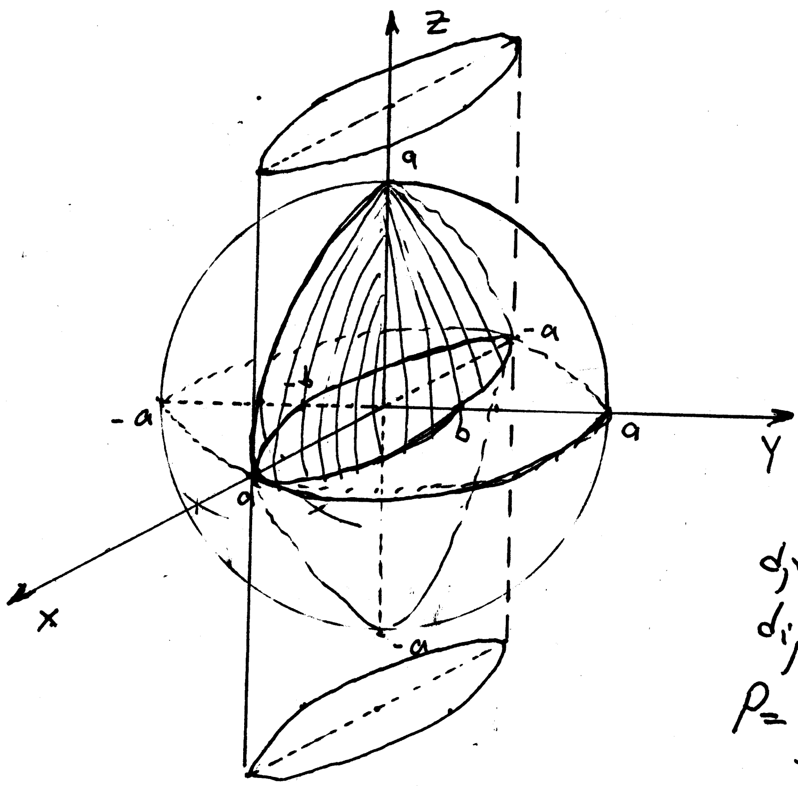
(#) Izračunati površinu djela sfere

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2\}$$

koji se nalazi u unutrašnjosti cilindra

$$S_1=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z\in\mathbb{R}\},\quad b\le a$$

R: Skicirajmo sferu S i cilindar S_1.



Cilindrična površ u presjeku sa sferom, isjeca iz nje simetričnu površ u odnosu na ravan xOy. Ta dva simetrična dijela označimo sa l_1 i l_2. Svaku od ova dva dijela, koordinatne ravni xOz i yOz ih dijele na četiri jednaka dijela.

$P=\iint\limits_S dS$ gdje je S površina djela sfere ograničena cilindrom.

$S:\ x^2+y^2+z^2=a^2$

$z=\pm\sqrt{a^2-x^2-y^2}$

Zbog navedene simetričnosti posmatraću sferu samo u prvom oktantu

$z=\sqrt{a^2-x^2-y^2}\qquad z'_x=\dfrac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}$

$\iint\limits_S dS=\iint\limits_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$ gdje je D projekcija površi S na xOy ravan

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$P=8\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ a\ge 0,\ b\ge 0}}\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=8a\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ a\ge 0,\ b\ge 0}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}=8a\iint\limits_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

gdje je $D: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \\ x \ge 0,\ y \ge 0 \end{cases}$ ili drugačije napisano $D: \begin{cases} 0 \le x \le a \\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2} \end{cases}$

$$\frac{y^2}{b^2} \le 1-\frac{x^2}{a^2}$$

$$y^2 \le \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$$P = 8a\int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{dy}{\sqrt{\underbrace{a^2-x^2}_{\text{ovo je broj za } dy}-y^2}} = 8a\int_0^a \left(\arcsin \frac{y}{\sqrt{a^2-x^2}}\Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}\right)dx$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$$

$$= 8a\int_0^a \left(\arcsin\frac{b}{a} - \underbrace{\arcsin 0}_{=0}\right)dx =$$

$$= 8a \arcsin\frac{b}{a}\int_0^a dx = 8a^2 \arcsin\frac{b}{a}$$ tražena površina