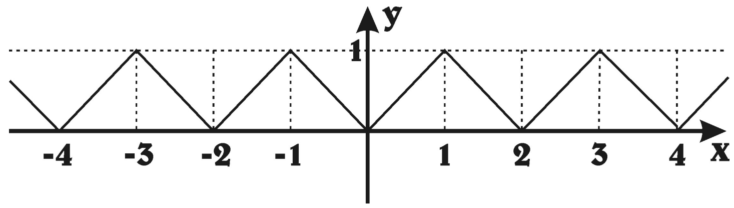
Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 06.07.2010.

Pismeni ispit iz predmeta **Analiza 3**

1. Funkciju definisanu grafikom pretvoriti u Fourier-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.



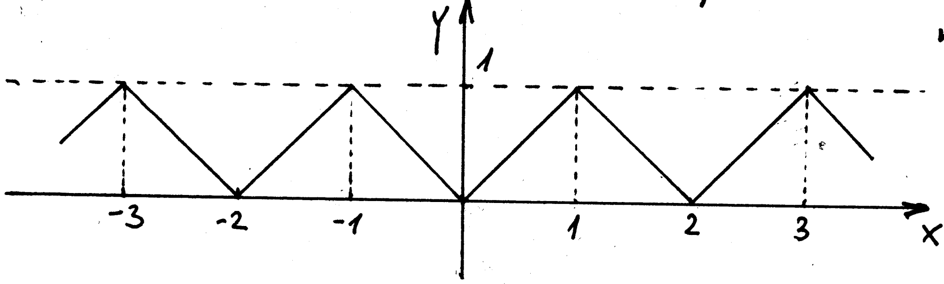
2. Ako je $u = \frac{\varphi(x-y)+\psi(x+y)}{x}$ gdje su φ i ψ diferencijalne funkcije, izračunati $\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x}) - x^2\frac{\partial^2 u}{\partial y^2}$.

3. Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

4. Pomoću Greenove formule izračunati integral $I = \int_c (xy+x+y)\,dx + (xy+x-y)\,dy$, ako je c kontura kruga $x^2+y^2=ax$ pređena u pozitivnom smislu.

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

(#) F-ju definisanu grafikom razviti u Fourierov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.



Rj: Sa grafika možemo primetiti da je f-ja parna i periodična perioda 2. F-ju je dovoljno razviti u Fourierov red u intervalu $[-1,1]$, pa kako je f-ja parna imaćemo da su $b_n=0 \; \forall n$.

Ako f-ju označim sa $f(x)$ na intervalu $[-1,1]$ imamo $f(x)=\begin{cases} x, & 0\leq x\leq 1 \\ -x, & -1\leq x\leq 0 \end{cases}$

Ako je $f(x)$ integrabilna f-ja na intervalu $[-l,l]$ Fourierove koeficijente računamo po formulama

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourierov red f-je $f(x)$ je tad oblika:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

F-ja je parna:

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=2\int_0^1 x\,dx=2\cdot\frac{1}{2}x^2\Big|_0^1=1$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=2\int_0^1 x\cos n\pi x\,dx=\begin{vmatrix} u=x & dv=\cos n\pi x\,dx \\ du=dx & v=\frac{1}{n\pi}\sin n\pi x\end{vmatrix}=$$

$$=\underbrace{\frac{2}{n\pi}x\sin n\pi x\Big|_0^1}_{=0}-\frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx=-\frac{2}{n\pi}\cdot\frac{(-1)}{n\pi}\cos n\pi x\Big|_0^1=2\cdot\frac{\cos n\pi-\cos 0}{n^2\pi^2}$$

$$a_n=2\frac{(-1)^n-1}{n^2\pi^2}, \quad b_n=0 \;\forall n \;\Rightarrow\; f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{\pi^2}\cdot\frac{-4}{(2n-1)^2}\cos(2n+1)\pi x$$

$$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n+1)^2}$$ razlaganje f-je u Fourierov red

Za $x=0$, $f(0)=0 \Rightarrow$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

(#) Ako je $u = \frac{\varphi(x-y) + \psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}\left(\varphi(x-y) + \psi(x+y)\right) = x^{-1}\left(\varphi(x-y) + \psi(x+y)\right)$$

$$u_x' = \frac{\partial u}{\partial x} = (-1)x^{-2}\left(\varphi(\underbrace{x-y}_{s}) + \psi(\underbrace{x+y}_{t})\right) + \frac{1}{x}\left(\varphi_s' \cdot s_x' + \psi_t' \cdot t_x'\right) =$$

$$= \frac{-1}{x^2}\left[\varphi(x-y) + \psi(x+y)\right] + \frac{1}{x}\left(\varphi_s' \cdot 1 + \psi_t' \cdot 1\right)$$

$$x^2 \frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x\left(\varphi_s' + \psi_t'\right) \quad \text{gdje su } s = x-y \text{ i } t = x+y$$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = \underline{-\varphi_s' \cdot 1} \; \underline{\underline{-\psi_t' \cdot 1}} \; \underline{+1 \cdot (\varphi_s'} \; \underline{\underline{+\psi_t'}}) + x\left(\varphi_{ss}'' \cdot 1 + \psi_{tt}'' \cdot 1\right)$$

$$= x\left(\varphi_{ss}'' + \psi_{tt}''\right) \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}\left(\varphi_s' \cdot s_y' + \psi_t' \cdot t_y'\right) = \frac{1}{x}\left(\varphi_s' \cdot (-1) + \psi_t' \cdot 1\right) = \frac{1}{x}\left(-\varphi_s' + \psi_t'\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}\left(-\varphi_{ss}'' \cdot s_y' + \psi_{tt}'' \cdot t_y'\right) = \frac{1}{x}\left(\varphi_{ss}'' + \psi_{tt}''\right)$$

$$x^2 \frac{\partial^2 u}{\partial y^2} = x\left(\varphi_{ss}'' + \psi_{tt}''\right) \qquad \ldots (2)$$

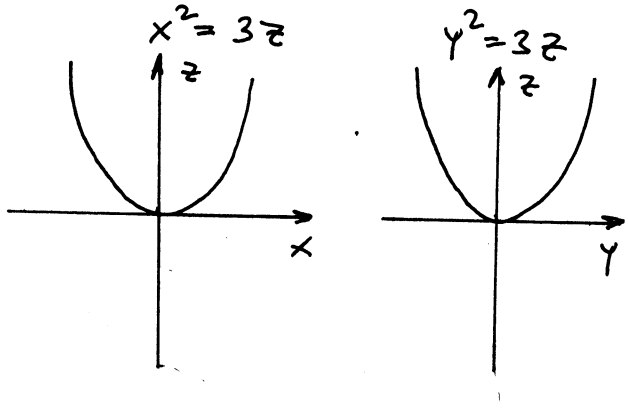
$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2} \overset{(1) \text{ i } (2)}{=\!=} 0$$ traženo rješenje

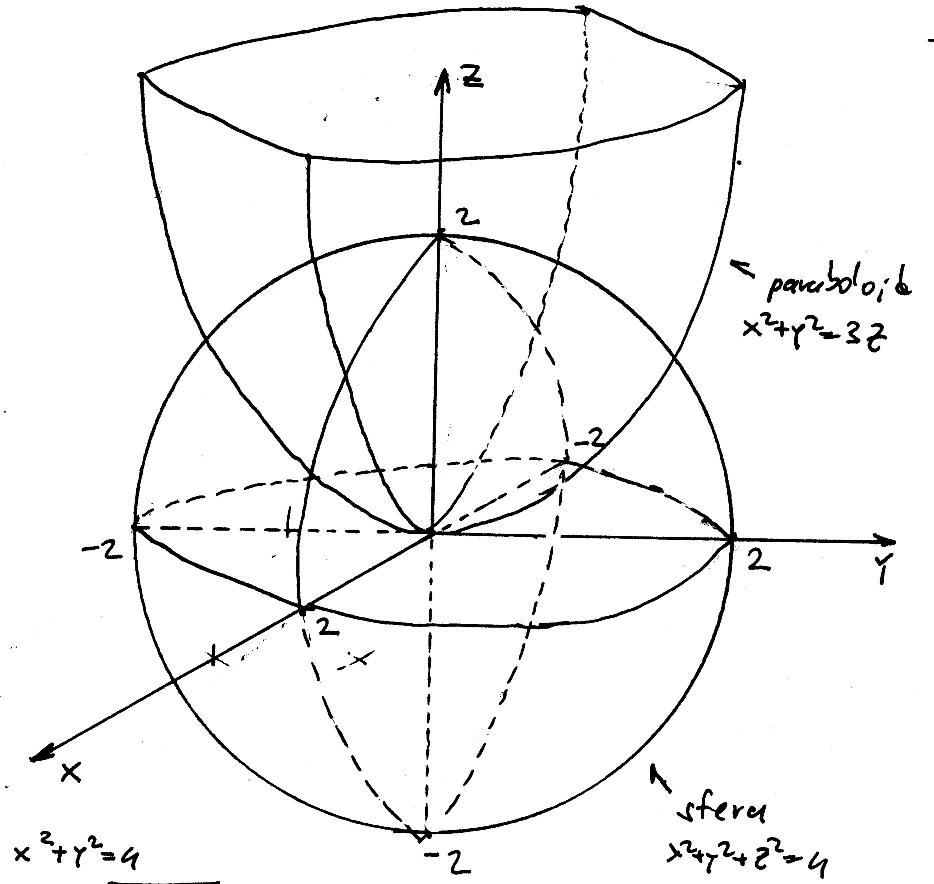
Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$x^2+y^2=4$

$x=\pm\sqrt{4-y^2}$

$y=\pm\sqrt{4-x^2}$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz$$

Primetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz.

Prema tome

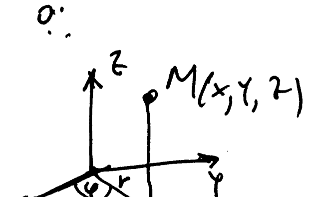
$$V=4\iiint\limits_{\Omega_1} dx\,dy\,dz \quad \text{gdje je}$$

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1:\begin{cases} 0\le x\le 2\\ 0\le y\le\sqrt{4-x^2}\\ 0\le z\le \frac{1}{3}(x^2+y^2)\end{cases}$$

$$V=4\int_0^2 dx\int_0^{\sqrt{4-x^2}}dy\int_0^{\frac{1}{3}(x^2+y^2)}dz=4\int_0^2 dx\int_0^{\sqrt{4-x^2}}\frac{1}{3}(x^2-y^2)\,dy$$

$$=\frac{4}{3}\int_0^2\left(x^2\,y\Big|_0^{\sqrt{4-x^2}}+\frac{1}{3}y^3\Big|_0^{\sqrt{4-x^2}}\right)dx=\underset{\text{komplikovano}}{\ldots}=\frac{8\pi}{3}$$



II način:

Uvedimo cilindrične koordinate $\begin{aligned}x&=r\cos\varphi\\ y&=r\sin\varphi\\ z&=z\end{aligned}$ $\quad dx\,dy\,dz=r\,dr\,d\varphi\,dz$

Oblast $\Omega_1 \xrightarrow{\text{transformise}} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 dr \int_0^{\frac{1}{3}r^2} r\, dz = 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r \cdot \frac{1}{3} r^2\, dr = \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^2 d\varphi = \frac{1}{3} \cdot 16 \cdot \frac{\pi}{2} = \frac{8\pi}{3}$$

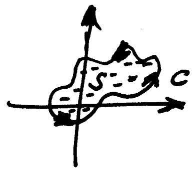
$V = \frac{8\pi}{3}$ tražena zapremina

(#) Pomoću Greenove formule izračunati integral

$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy,$$ ako je c kontura kružnice $x^2+y^2=ax$ prijeđena u pozitivnom smislu.

Rj: Greenova formula

$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$



$$P(x,y)=xy+x+y \qquad \frac{\partial P}{\partial y}=x+1,\quad \frac{\partial Q}{\partial x}=y+1$$

$$Q(x,y)=xy+x-y$$

$$x^2+y^2=ax$$

$$x^2-ax+y^2=0$$

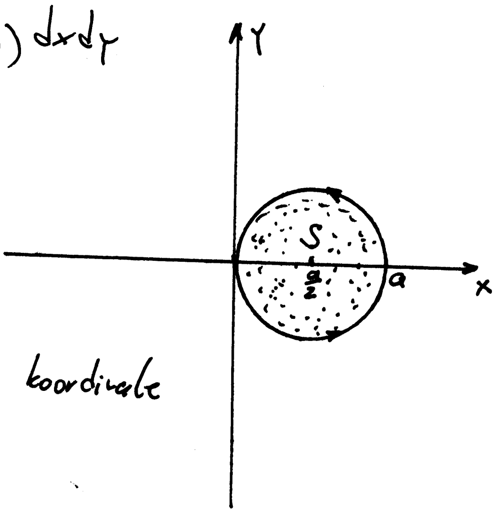
$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$$I=\iint_S(y+1-(x+1))\,dx\,dy$$

$$I=\iint_S(y-x)\,dx\,dy$$



uvedimo polarne koordinate

$$x=\frac{a}{2}+r\sin\varphi$$

$$y=r\cos\varphi$$

$$dx\,dy=r\,dr\,d\varphi$$

$$S\xrightarrow{\text{transformiš}}S':\begin{cases}0\le r\le\frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\iint_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$$

$$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$$

$$=-\frac{a^3\pi}{8}$$ traženo rješenje