Pismeni ispit iz predmeta **Analiza 3**, 27.01.2012.

1. Razviti funkciju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$. Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$.

2. Izračunati trostruki integral $K = \iiint\limits_T ydxdydz$ gdje je oblast T ograničena površinama $y = \sqrt{x^2 + z^2}$ i $y = h$, $h > 0$.

3. Izračunati krivoliniske integrale a) $I = \int\limits_{-l} 2xdx - (x + 2y)dy$; b) $I = \int\limits_{+l} ycosxdx + sinxdy$; gdje je l kontura trougla čiji su vrhovi $A(-1; 0)$, $B(0; 2)$ i $C(2; 0)$.

4. Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_{\sigma} \sqrt[4]{x^2 + y^2}dxdy$ gdje je σ donja strana kruga $x^2 + y^2 \leq a^2$.

Pismeni ispit iz predmeta **Analiza 3**, 27.01.2012.

1. Razviti funkciju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$. Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$.

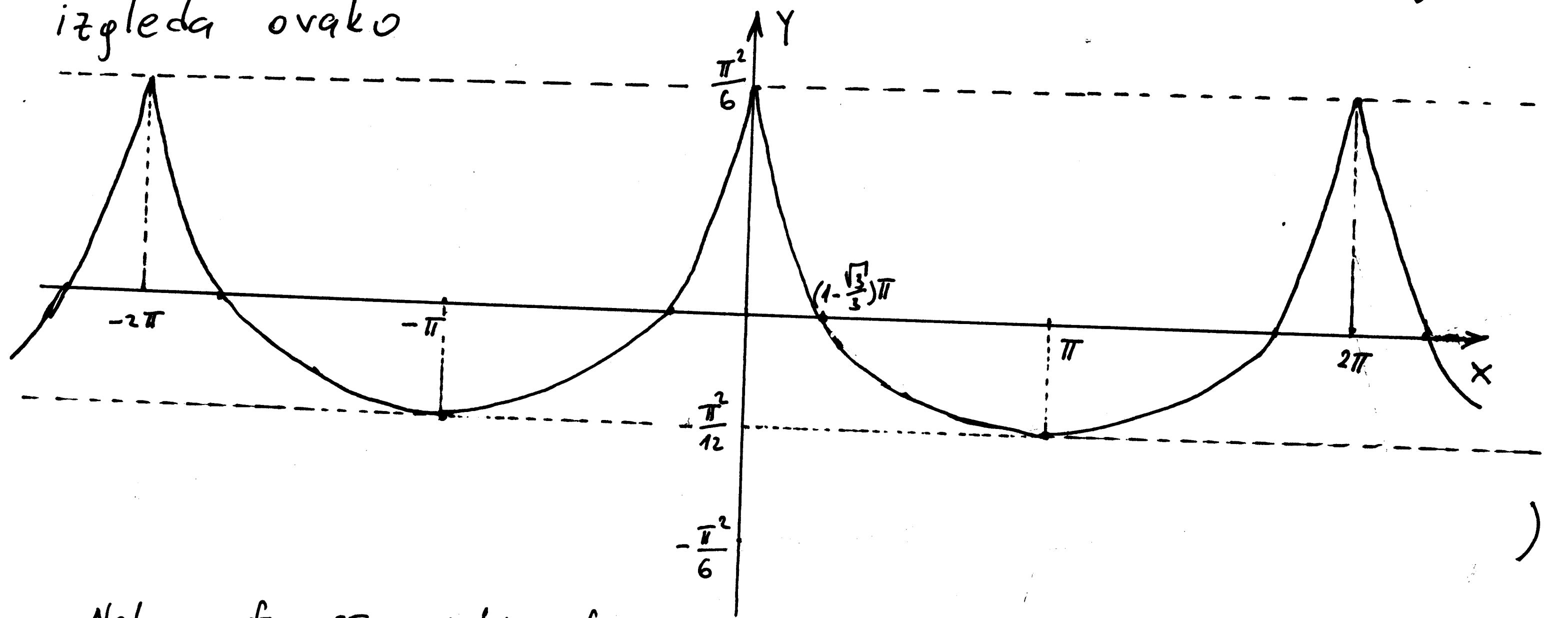
2. Izračunati trostruki integral $K = \iiint\limits_T ydxdydz$ gdje je oblast T ograničena površinama $y = \sqrt{x^2 + z^2}$ i $y = h$, $h > 0$.

3. Izračunati krivoliniske integrale a) $I = \int\limits_{-l} 2xdx - (x + 2y)dy$; b) $I = \int\limits_{+l} ycosxdx + sinxdy$; gdje je l kontura trougla čiji su vrhovi $A(-1; 0)$, $B(0; 2)$ i $C(2; 0)$.

4. Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_{\sigma} \sqrt[4]{x^2 + y^2}dxdy$ gdje je σ donja strana kruga $x^2 + y^2 \leq a^2$.

Razviti f-ju $f(x) = \dfrac{3x^2 - 6\pi x + 2\pi^2}{12}$ u red po kosinusima u intervalu $(0, \pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{Furijeov red f-je } f(x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad \text{Furijeovi koeficijenti f-je } f(x)$$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow b_n = 0 \quad \forall n \in \mathbb{Z}$
Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow a_n = 0 \quad \forall n \in \mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in (0,\pi) \\ f(-x), & x \in (-\pi, 0) \end{cases} = \begin{cases} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2), & x \in (0,\pi) \\ \frac{1}{12}(3x^2 + 6\pi x + 2\pi), & x \in (-\pi, 0) \end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx \xrightarrow{f^* \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{6\pi}\left(3 \cdot \frac{1}{3}x^3\Big|_0^{\pi} -\right.$$

$$\left. - 6\pi \cdot \frac{1}{2}x^2\Big|_0^{\pi} + 2\pi^2 \cdot x\Big|_0^{\pi}\right) = \frac{1}{6\pi}(\pi^3 - 3\pi^3 + 2\pi^3) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx \xrightarrow{f^*(x) \text{ parna}} \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx =$$

$$= \frac{1}{6\pi}\left(3\int_0^\pi x^2\cos nx\,dx - 6\pi\int_0^\pi x\cos nx\,dx + 2\pi^2\int_0^\pi \cos nx\,dx\right) \overset{(*)}{=}$$

$$I_1 = \int_0^\pi x^2\cos nx\,dx = \left|\begin{matrix} u=x^2 & dv=\cos nx\,dx \\ du=2x\,dx & v=\frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x^2\sin nx\Big|_0^\pi - \frac{2}{n}\int_0^\pi x\sin nx\,dx =$$

$$= \left|\begin{matrix} u=x & dv=\sin nx\,dx \\ du=dx & v=-\frac{1}{n}\cos nx\end{matrix}\right| = \frac{1}{n}\underbrace{(\pi^2\sin n\pi - 0)}_{=0} - \frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right)$$

$$= \frac{2}{n^2}(\pi\cos n\pi - 0) - \frac{2}{n^2}\sin nx\Big|_0^\pi = (-1)^n\frac{2\pi}{n^2}$$

$$I_2 = \int_0^\pi x\cos nx\,dx = \left|\begin{matrix} u=x & dv=\cos nx\,dx \\ du=dx & v=\frac{1}{n}\sin nx\end{matrix}\right| = \frac{1}{n}x\sin nx\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx =$$

$$= \frac{1}{n}\underbrace{(\pi\sin n\pi - 0)}_{=0} - \frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^\pi = \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{1}{n^2}((-1)^n - 1)$$

$$I_3 = \int_0^\pi \cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^\pi = \frac{1}{n}(\sin n\pi - \sin 0) = 0$$

$$\overset{(*)}{=} \frac{1}{\cancel{2\pi}}(-1)^n\frac{\cancel{2\pi}}{n^2} - \frac{1}{n^2}((-1)^n - 1) = \frac{1}{n^2}((-1)^n - (-1)^n + 1) = \frac{1}{n^2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx, \quad x\in(0,\pi)$$

razvoj f-je $f(x)$ u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$).

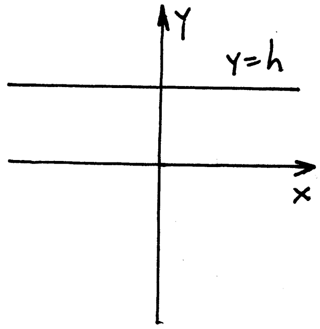
(#) Izračunati trostruki integral

$$K = \iiint_T y\,dx\,dy\,dz$$

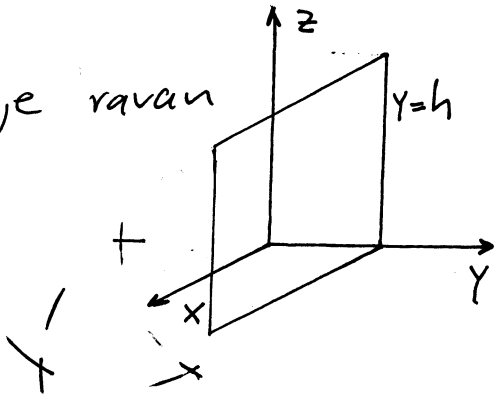
gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

Rj. Pokušajmo skicirati oblast T.

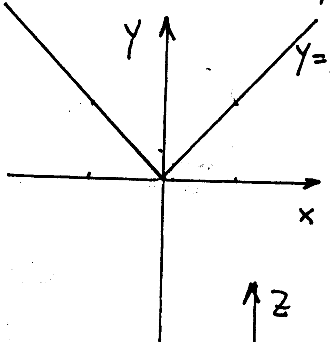
U xOy-ravni $y=h$ je prava.



U prostoru $y=h$ je ravan

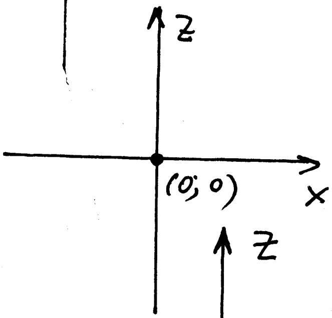


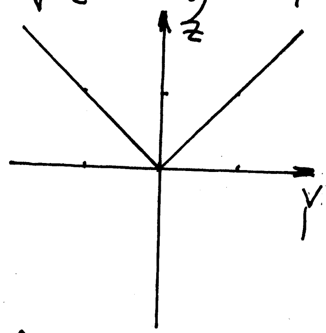
U xOy-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{x^2}$



U xOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $0=\sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{z^2}$ tj. $y=|z|$.

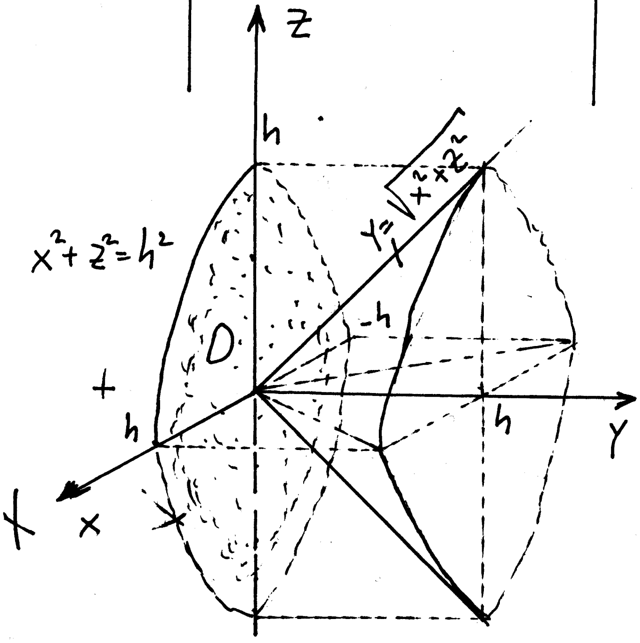




Ako napravimo presjek površina $y=\sqrt{x^2+z^2}$ i $y=h$ dobićemo $h=\sqrt{x^2+z^2}$ tj.

$$x^2+z^2=h^2$$

(krug poluprečnika h)



Oblast T (pola čunja) je prikazan na slici lijevo. Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T : \begin{cases} -h \le x \le h \\ -\sqrt{h^2-z^2} \le z \le \sqrt{h^2-z^2} \\ \sqrt{x^2+z^2} \le y \le h \end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$$\begin{aligned} x &= r\cos\varphi \\ z &= r\sin\varphi \\ y &= y \\ dx\,dy\,dz &= r\,dr\,d\varphi\,dy \end{aligned} \qquad T \xrightarrow{\text{transformiše}} T' : \begin{cases} 0 \le r \le h \\ 0 \le \varphi \le 2\pi \\ r \le y \le h \end{cases}$$

Prema tome

$$K = \iiint_T y\,dx\,dy\,dz = \left| \begin{matrix}\text{uvodimo} \\ \text{cilindrične} \\ \text{koordinate}\end{matrix} \right| = \iiint_{T'} y\,r\,dr\,d\varphi\,dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^h r\,dr \int_r^h y\,dy = \int_0^{2\pi} d\varphi \int_0^h r \underbrace{\frac{1}{2} y^2 \Big|_r^h}_{h^2-r^2} dr =$$

$$= \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^h (rh^2 - r^3)\,dr = \frac{1}{2} \int_0^{2\pi} \Big(\underbrace{\frac{1}{2} r^2 h^2 \Big|_0^h}_{\frac{1}{2} r^4} \underbrace{- \frac{1}{4} r^4 \Big|_0^h}_{-\frac{1}{4} h^4} \Big)\,d\varphi$$

$$= \frac{1}{2} \cdot \frac{1}{4} h^4 \int_0^{2\pi} d\varphi = \frac{1}{8} h^4 \varphi \Big|_0^{2\pi} = \frac{h^4 \pi}{4}$$

traženo rješenje

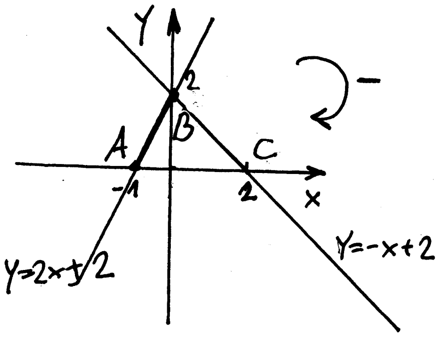
#) Izračunati krivoliniske integrale

a) $\displaystyle\oint_{-l} 2x\,dx-(x+2y)\,dy$

b) $\displaystyle\oint_{+l} y\cos x\,dx+\sin x\,dy$

gdje je l kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj. a)
Nacrtajmo trougao ΔABC.



Provucimo pravu kroz tačke $B(\overset{x_1}{0};\overset{y_1}{2})$ i $C(\overset{x_2}{2};\overset{y_2}{0})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad y=-x+2$$

$$\frac{x}{2}=\frac{y-2}{-2} \quad /\cdot 2$$

$$x=-y+2$$

Provucimo pravu kroz $A(\overset{x_1}{-1};\overset{y_1}{0})$ i $B(\overset{x_2}{0};\overset{y_2}{2})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad \frac{x+1}{1}=\frac{y}{2}$$

$$y=2x+2$$

$$\oint_{-l} 2x\,dx-(x+2y)\,dy=\int_{B(0;2)}^{C(2;0)} 2x\,dx-(x+2y)\,dy+\int_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy+\int_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy$$

$$\int_{(0;2)}^{(2;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=-x+2\\ dy=-dx\end{matrix}\right|=\int_{(0;2)}^{(2;0)}\left[2x-\left(x+2(-x+2)\right)(-1)\right]dx=$$

$$=\int_{(0;2)}^{(2;0)}\left[2x+x-2x+4\right]dx=\int_{(0;2)}^{(2;0)}(x+4)\,dx=\left(\frac{1}{2}x^2+4x\right)\Big|_0^2=2+8=10$$

$$\int_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=0\\ dy=0\end{matrix}\right|=\int_2^{-1} 2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=(1-4)=-3$$

$$\int_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=2x+2\\ dy=2\,dx\end{matrix}\right|=\int_{-1}^{0}\left[2x-\left(x+2(2x+2)\right)2\right]dx=$$

$$= \int_{-1} (2x - 2x - 8x - 8)\,dx = (-8)\int_{-1}(x+1)\,dx = (-8)\Big[\underbrace{\tfrac{1}{2}x^2\Big|_{-1}^{0}}_{0-1} + \underbrace{x\Big|_{-1}^{0}}_{0+1}\Big] =$$

$$= (-8)\left(-\frac{1}{2} + 1\right) = -4$$

Prema tome $\displaystyle\oint_{\Delta ABC} 2x\,dx - (x+2y)\,dy = 10 - 3 - 4 = 3$

b) $$\oint_{+l} y\cos x\,dx + \sin x\,dy = \int_{AC} y\cos x\,dx + \sin x\,dy + \int_{CB} y\cos x\,dx + \sin x\,dy + \int_{BA} y\cos x\,dx + \sin x\,dy$$

$$\int_{A(-1;0)}^{C(2;0)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = 0 \\ dy = 0 \end{vmatrix} = \int_{-1}^{2} 0\,dx = 0$$

$$\int_{C(2;0)}^{B(0;2)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = -x+2 \\ dy = -dx \end{vmatrix} = \int_{2}^{0} \left[(-x+2)\cos x - \sin x\right] dx$$

$$= \begin{vmatrix} u = -x+2 & dv = \cos x \\ du = -1 & v = \sin x \end{vmatrix} = (-x+2)\sin x\Big|_{2}^{0} + \int_{2}^{0} \sin x\,dx - \int_{2}^{0} \sin x\,dx = 0$$

$$\int_{B(0;2)}^{A(-1;0)} y\cos x\,dx + \sin x\,dy = \begin{vmatrix} y = 2x+2 \\ dy = 2\,dx \end{vmatrix} = \int_{0}^{-1} \left[(2x+2)\cos x + 2\sin x\right] dx =$$

$$= 2\int_{0}^{-1} \left[(x+1)\cos x + \sin x\right] dx = \begin{vmatrix} u = x+1 & dv = \cos x \\ du = dx & v = \sin x \end{vmatrix} = 2(x+1)\sin x\Big|_{0}^{-1} - 2\int_{0}^{-1} \sin x\,dx$$

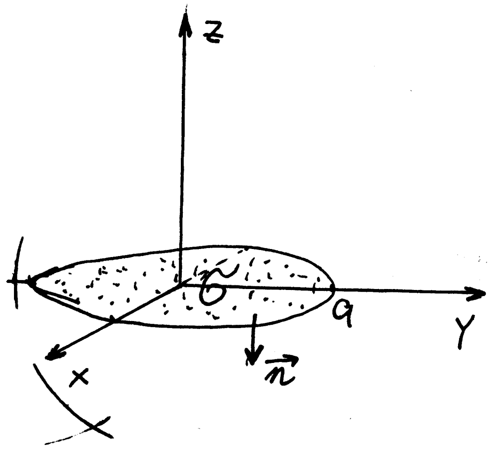
$$+ 2\int_{0}^{-1} \sin x\,dx = 0$$ Prema tome

$$\oint_{+l} y\cos x\,dx + \sin x\,dy = 0$$

(#) Izračunati površinski integral drugog tipa (po koordinatama) $I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy$ gdje je

σ - donja strana kruga $x^2+y^2\le a^2$.

Rj: Skicirajmo datu površinu



Prisjetimo se, kako se računa površinski integral drugog tipa, npr.

$$\iint\limits_S R(x,y,z)\,dx\,dy$$

posmatramo vektor normale $\vec{n}$ na površ S

ako je $\cos\gamma<0$ gdje je γ ugao između $\vec{n}$ i z-ose naš integral postaje

$$\iint\limits_S R(x,y,z)\,dx\,dy=-\iint\limits_D R(x,y,z(x,y))\,dx\,dy$$

gdje je D ortogonalna projekcija površi S a $z=z(x,y)$ jednačina površi S

U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ. Ugao γ je $\gamma=\pi$ (donja strana kruga) tj. $\cos\pi<0$.

$$I=\iint\limits_{\sigma}\sqrt[4]{x^2+y^2}\,dx\,dy=-\iint\limits_D\sqrt[4]{x^2+y^2}\,dx\,dy=$$

uvodimo polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

$$D\xrightarrow{\text{transf.}}D':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}$$

$$=-\iint\limits_{D'}\sqrt[4]{r^2}\,r\,dr\,d\varphi=-\int_0^{2\pi}d\varphi\int_0^a r^{\frac{3}{2}}\,dr=-\int_0^{2\pi}\frac{2}{5}r^{\frac{5}{2}}\Big|_0^a\,d\varphi=-\frac{2}{5}a^{\frac{5}{2}}\varphi\Big|_0^{2\pi}.$$

$$I=-\frac{4}{5}\pi\sqrt{a^5}$$ traženo rješenje