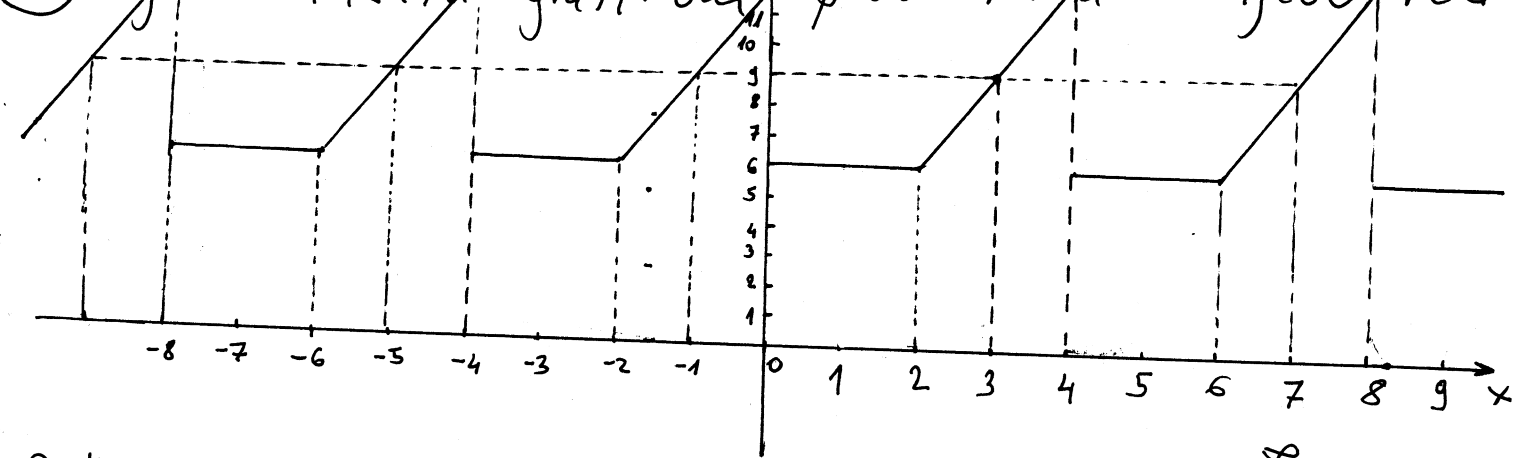
F-ja definisana grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

Rj. Prvo primjetimo da je data f-ja periodična perioda 4, što znači da je možemo pretvoriti u Furijeov red i to dovoljno ju je pretvoriti u Furijeov red na intervalu $(0,4)$.
Data f-ja na intervalu $(0,4)$ je definisana na sljedeći način
$$f(x) = \begin{cases} 6, & x \in (0,2] \\ 3x, & x \in (2,4) \end{cases}.$$

Furijerov red na proizvoljnom intervalu $[a,b]$ izgleda
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$
a Furijerovi koeficijenti računaju po formulama
$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \qquad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad n=1,2,\dots$$
$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx, \quad n=1,2,\dots$$

Što znači Furijerov red na intervalu $[0,4)$ je
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}\right)$$
Izračunajmo Furijeove koeficijente
$$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx = \frac{1}{2}\int_0^2 6\,dx + \frac{1}{2}\int_2^4 3x\,dx = 3x\Big|_0^2 + \frac{3}{2}\cdot\frac{1}{2}x^2\Big|_2^4 = 6 + \frac{3}{\cancel{4}_1}\overbrace{\cancel{12}^{3}}^{16-4} = 15$$
$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_0^2 6\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_2^4 3x\cos\frac{n\pi x}{2}\,dx = \left| \begin{matrix} u=x & dv=\cos\frac{n\pi x}{2}dx \\ du=dx & v=\frac{2}{n\pi}\sin\frac{n\pi x}{2} \end{matrix} \right|$$

$$=\underbrace{3\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2}_{0-0}+\frac{3}{2}\left[\underbrace{\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}-\frac{2}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx\right]=$$

$$=-\frac{3}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx=\frac{3}{n\pi}\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_2^4=\frac{6}{n^2\pi^2}(1-\cos n\pi),\quad n\neq 0$$

Odavde vidimo $a_n=\begin{cases}0, & n \text{ parno}\\ \frac{12}{n^2\pi^2}, & n \text{ neparno}\end{cases}$ $n\in\mathbb{N}$

$$b_n=\frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}dx=\frac{1}{2}\int_0^2 6\sin\frac{n\pi x}{2}dx+\frac{1}{2}\int_2^4 3x\sin\frac{n\pi x}{2}dx=\begin{vmatrix}u=x & dv=\sin\frac{n\pi x}{2}dx\\ du=dx & v=-\frac{2}{n\pi}\cdot\cos\frac{n\pi x}{2}\end{vmatrix}$$

$$=3\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2+\frac{3}{2}\left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_2^4+\frac{2}{n\pi}\int_2^4\cos\frac{n\pi x}{2}dx\right]=$$

$$=\left(-\frac{6}{n\pi}\right)(\cos n\pi-1)+\frac{3}{2}\left[\left(-\frac{2}{n\pi}\right)(4\cos 2n\pi-2\cos n\pi)+\underbrace{\frac{2}{n\pi}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}\right]$$

$$=\frac{6}{n\pi}(1-\cos n\pi)-\frac{3}{n\pi}(4-2\cos n\pi)=\frac{6}{n\pi}(1-\cos n\pi)-\frac{6}{n\pi}(2-\cos n\pi)$$

$$=\frac{6}{n\pi}(\underline{1-\cos n\pi}-2\underline{+\cos n\pi})=-\frac{6}{n\pi}$$

Prema tome $f(x)\sim\frac{15}{2}+\sum_{n=1}^{\infty}\left(\frac{6}{n^2\pi^2}(1-\cos n\pi)\cos\frac{n\pi x}{2}+\left(-\frac{6}{n\pi}\right)\sin\frac{n\pi x}{2}\right)$

$$=\frac{15}{2}+\sum_{k=1}^{\infty}\left(\frac{12}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}-\frac{6}{k\pi}\sin\frac{k\pi x}{2}\right)$$

$$f(x)\sim\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\frac{\pi}{2}x}{(2k-1)^2}-\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\frac{\pi}{2}x}{k}$$

Za $x=2$ imamo

$$f(2)=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2}-\underbrace{\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi}{k}}_{=0}$$

$$6=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)}{(2k-1)^2}\Rightarrow-\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=-\frac{3}{2}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(-\frac{3}{2}\right)\left(-\frac{\pi^2}{12}\right)=\frac{\pi^2}{2\cdot 4}\qquad\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}\ \text{tražena suma}$$

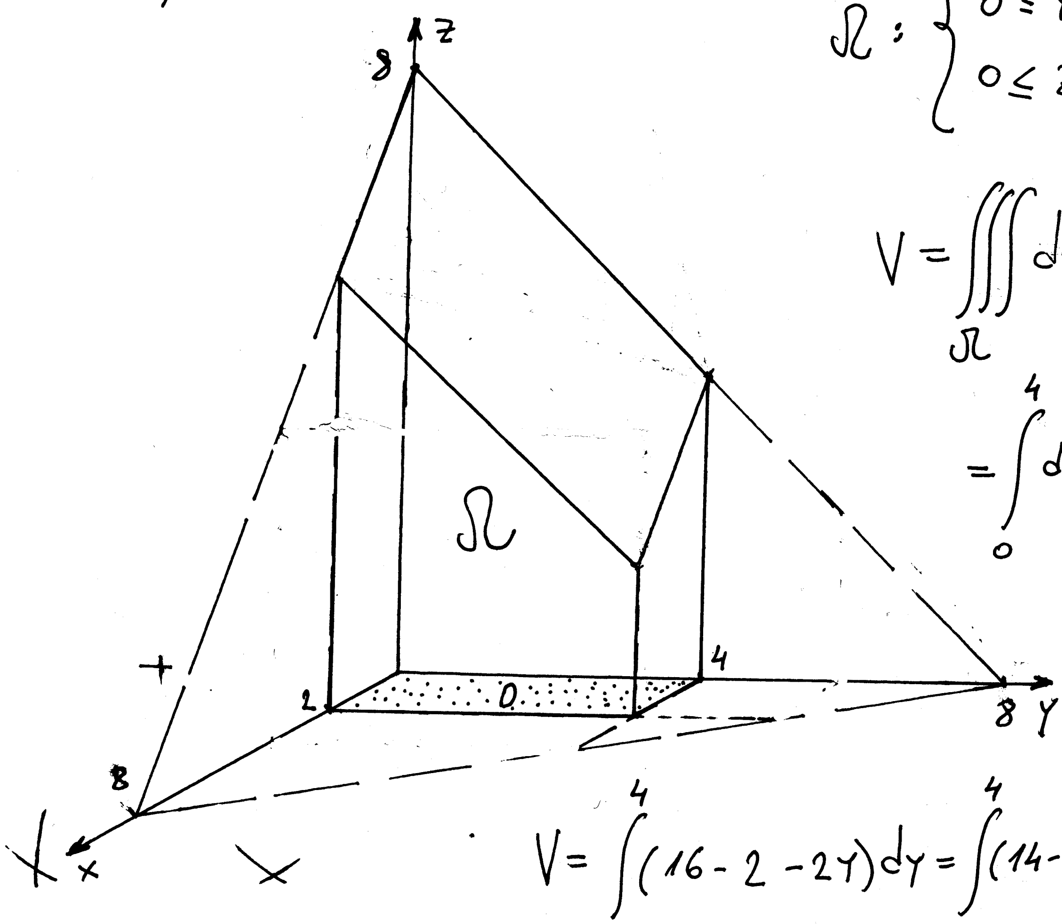
(#) Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj. Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T=\frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz, \quad y_T=\frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz, \quad z_T=\frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega : \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 4 \\ 0 \le z \le 8-x-y \end{cases}$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{D} (8-x-y)\,dx\,dy$$

$$= \int_0^4 dy \int_0^2 (8-x-y)\,dx =$$

$$= \int_0^4 \left(8x\Big|_0^2 - \frac{1}{2}x^2\Big|_0^2 - yx\Big|_0^2\right)d$$

$$V=\int_0^4 (16-2-2y)\,dy = \int_0^4 (14-2y)\,dy = 14y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V=14\cdot 4-16=4(14-4)=40$$

$$V=40$$

$$\iiint\limits_{\Omega} x\,dx\,dy\,dz = \int_0^4 dy\int_0^2 x\,dx \int_0^{8-x-y} dz = \int_0^4 dy \int_0^2 (8x-x^2-yx)\,dx = \int_0^4 \left(4x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2 - y\frac{1}{2}x^2\Big|_0^2\right)dy$$

$$= \int_0^4 \left(16-\frac{8}{3}-2y\right)dy = \int_0^4 \left(\frac{40}{3}-2y\right)dy = \frac{40}{3}y\Big|_0^4 - 2\cdot\frac{1}{2}y^2\Big|_0^4 = \frac{160}{3}-16=\frac{112}{3}$$

$$\iiint_\Omega y\,dx\,dy\,dz = \int_0^2 dx \int_0^4 y\,dy \int_0^{8-x-y} dz = \int_0^2 dx \int_0^4 y(8-x-y)\,dy = \int_0^2 dx \int_0^4 (8y - xy - y^2)\,dy =$$

$$= \int_0^2 \left(8\frac{1}{2}y^2\Big|_0^4 - x\frac{1}{2}y^2\Big|_0^4 - \frac{1}{3}y^3\Big|_0^4\right)dx = \int_0^2 \left(64 - 8x - \frac{64}{3}\right)dx = \int_0^2 \left(\frac{128}{3} - 8x\right)dx =$$

$$= \frac{128}{3}x\Big|_0^2 - 8\cdot\frac{1}{2}x^2\Big|_0^2 = \frac{256}{3} - 16 = \frac{208}{3}$$

$$\iiint_\Omega z\,dx\,dy\,dz = \overset{\text{završiti za vježbu}}{\ldots} = \frac{320}{3}$$

Prema tome, $x_T = \frac{1}{V}\iiint_\Omega x\,dx\,dy\,dz = \frac{1}{\cancelto{5}{\cancel{\cancel{40}}}} \cdot \frac{\cancelto{14}{\cancel{\cancel{112}}}}{3} = \frac{14}{15}$

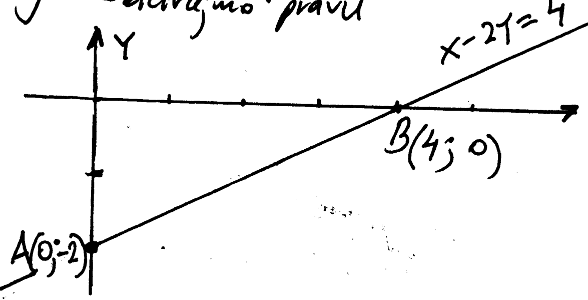
$$y_T = \frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz = \frac{1}{\cancelto{5}{40}} \cdot \frac{\cancelto{25}{208}}{3} = \frac{25}{15}$$

$$z_T = \frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz = \frac{1}{\cancelto{1}{40}} \cdot \frac{\cancelto{8}{320}}{3} = \frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15}, \frac{25}{15}, \frac{8}{3}\right)$.

(#) Izračunati krivolinijski integral $\displaystyle\int_{AB} \frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

Rj. Skiciramo datu pravu



Prisjetimo se kako se računa krivolinijski integral prvog tipa, ako je kriva integracije u ravni opisana formulom $y=\eta(x)$, $a\le x\le b$

$$\int_C f(x,y)\,dl = \int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

I način:

$x-2y=4$

$2y=x-4$

$y=\frac{1}{2}x-2$

$y'=\frac{1}{2}$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_0^4 \frac{\sqrt{1+\frac{1}{4}}}{\sqrt{x^2+(\frac{1}{2}x-2)^2}}\,dx = \frac{\sqrt{5}}{2}\int_0^4 \frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$= \frac{\sqrt{5}}{2}\cdot\frac{1}{\sqrt{\frac{5}{4}}}\int_0^4 \frac{dx}{\sqrt{x^2-\frac{8}{5}x+\frac{16}{5}}} = $$

$$\left| \begin{aligned} & x^2-\frac{8}{5}x+\frac{16}{5} = \\ & = x^2-2\cdot x\cdot\frac{4}{5}+\frac{16}{25}-\frac{16}{25}+\frac{16\cdot 5}{5\cdot 5} \\ & = \left(x-\frac{4}{5}\right)^2+\frac{64}{25} \end{aligned} \right|$$

(80)

$$= \int_0^4 \frac{d(x-\frac{4}{5})}{\sqrt{(x-\frac{4}{5})^2+\frac{64}{25}}} = \ln\left|x-\frac{4}{5}+\sqrt{\left(x-\frac{4}{5}\right)^2+\frac{64}{25}}\right|\Bigg|_0^4 = \ln\left(\frac{16}{5}+\sqrt{\frac{16(16+4)}{25}}\right) -$$

$4\cdot 5$

$$- \ln\left(-\frac{4}{5}+\sqrt{\frac{16+64}{25}}\right) = \ln\left(\frac{16}{5}+\frac{8\sqrt{5}}{5}\right) - \ln\left(-\frac{4}{5}+\frac{4\sqrt{5}}{5}\right)$$

$\sqrt{16(1+4)}$

$$= \ln\frac{\frac{16+8\sqrt{5}}{\cancel{5}}}{\frac{-4+4\sqrt{5}}{\cancel{5}}} = \ln\frac{4+2\sqrt{5}}{\sqrt{5}-1}\,\frac{\cdot(\sqrt{5}+1)}{\cdot(\sqrt{5}+1)} = \ln\frac{4+6\sqrt{5}+10}{5-1} = \ln\frac{7+3\sqrt{5}}{2}$$ traž rješe

II način

$x-2y=4$

$x=2y+4$

$\frac{\partial x}{\partial y}=2$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_{-2}^{0}\frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}}\,dy = \sqrt{5}\int_{-2}^{0}\frac{dy}{\sqrt{5y^2+16y+16}} = \ldots$$

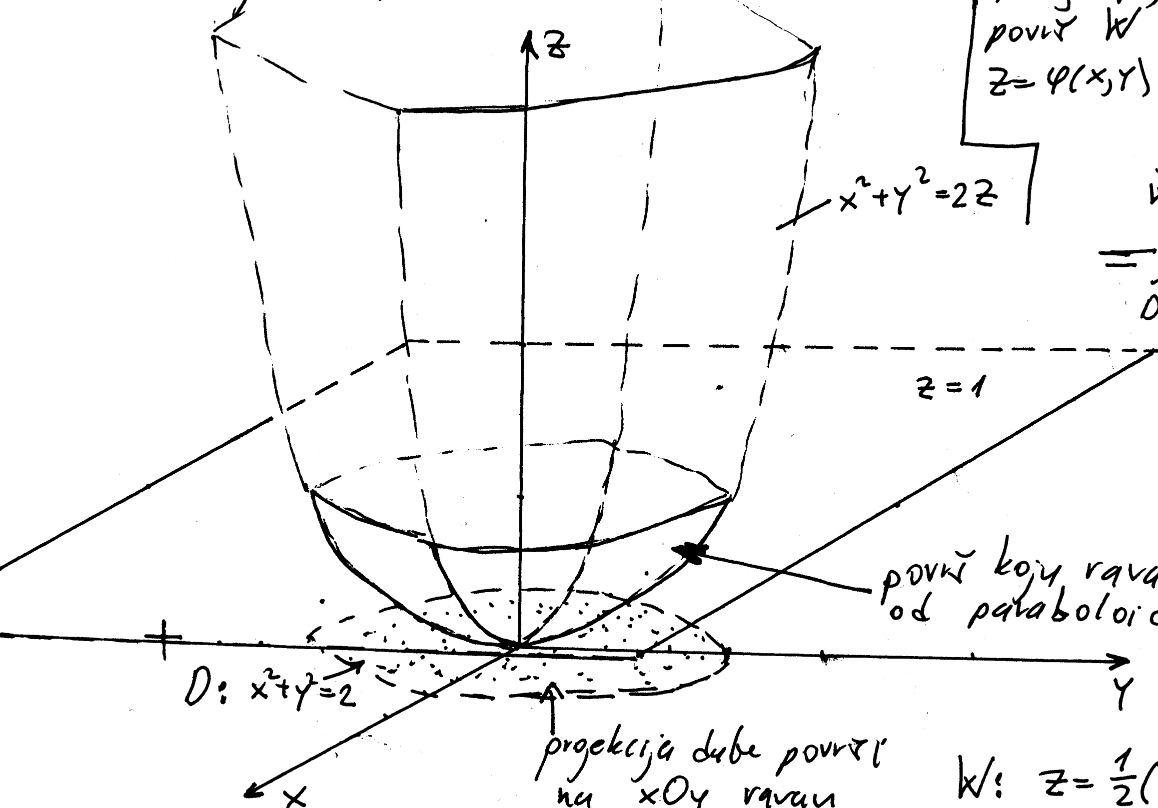
ZAVRŠITI ZA VJEŽBU

(#) Izračunati površinski integral prvog tipa

$\iint\limits_W (x^2+y^2)\,dS$, gdje je W – površina dijela paraboloida

$x^2+y^2=2z$ koju odsjeca ravan $z=1$ (dio paraboloida ispod date ravni).

Rj. Skicirajmo paraboloid i ravan



Prisjetimo se kako se računa površinski integral prvog tipa, ako je data površ W opisana formulom $z=\varphi(x,y)$

$$\iint\limits_W f(x,y,z)\,dS = \iint\limits_D f(x,y,\varphi(x,y))\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$$

gdje je D projekcija površi W na xOy ravan

$W:\ z=\frac{1}{2}(x^2+y^2)$

$\frac{\partial z}{\partial x}=x,\quad \frac{\partial z}{\partial y}=y$

$$\iint\limits_W (x^2+y^2)\,dS = \iint\limits_D (x^2+y^2)\sqrt{1+x^2+y^2}\,dxdy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ y=r\sin\varphi \\ dxdy=r\,dr\,d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D': \begin{cases} 0\le r\le\sqrt{2} \\ 0\le\varphi\le 2\pi \end{cases} \quad x^2+y^2=r^2 \right|$$

$$= \iint\limits_{D'} r^2\sqrt{1+r^2}\,r\,dr\,d\varphi = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,r\,dr = \left| \begin{array}{ll} 1+r^2=t^2 & r^2=t^2-1 \\ 2r\,dr=2t\,dt & \\ r\,dr=t\,dt & r\Big|_0^{\sqrt{2}} \Rightarrow t\Big|_1^{\sqrt{3}} \end{array} \right|$$

$$= \int_0^{2\pi} d\varphi \int_1^{\sqrt{3}} (t^2-1)\sqrt{t^2}\,t\,dt = \int_0^{2\pi} d\varphi \int_1^{\sqrt{3}} (t^4-t^2)\,dt = \varphi\Big|_0^{2\pi}\left(\frac{1}{5}t^5\Big|_1^{\sqrt{3}} - \frac{1}{3}t^3\Big|_1^{\sqrt{3}}\right) =$$

$$= 2\pi\left(\frac{9\sqrt{3}-1}{5} - \frac{3\sqrt{3}-1}{3}\right) = 2\pi\,\frac{27\sqrt{3}-3-15\sqrt{3}+5}{15} = 2\pi\,\frac{12\sqrt{3}+2}{15} = \frac{(24\sqrt{3}+4)\pi}{15}$$ traženo rješenje