Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 15.02.2011.

Pismeni ispit iz predmeta **Analiza 3**

1. Razviti funkciju $f(x) = x(\frac{\pi}{2} - x)$ po sinusima višestrukih uglova u intervalu $(0, \frac{\pi}{2})$.

2. Izračunati dvostruki integral $I = \iint\limits_D xy\,dx\,dy$, gdje je D oblast ograničena linijama $xy = 1$, $x + y = \frac{5}{2}$.

3. Izračunati zapreminu tijela ograničenog dijelom površi $(x^2 + y^2 + z^2)^3 = \dfrac{a^6 z^2}{x^2 + y^2}$, $a > 0$ u I oktantu.

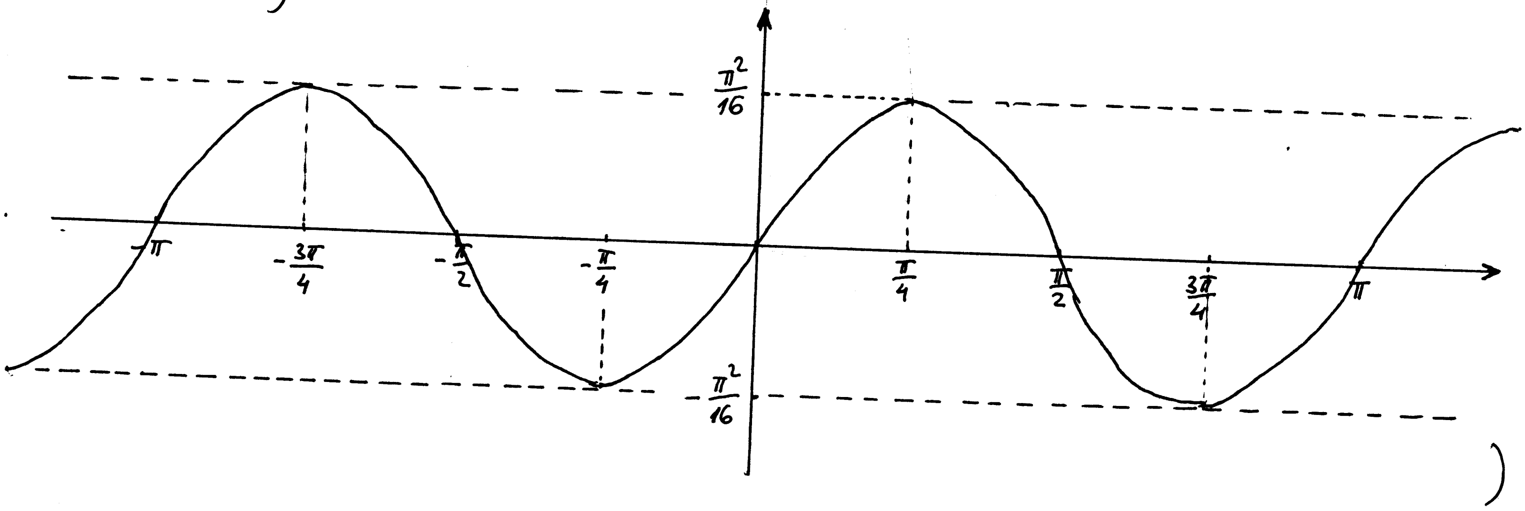
4. Izračunati površinski integral $I = \int\limits_S xyz\,dS$, ako je S dio ravni $x + y + z = 1$ u I oktantu.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Razviti f-ju $f(x) = x\left(\frac{\pi}{2} - x\right)$ po sinusima višestrukih uglova u intervalu $\left(0, \frac{\pi}{2}\right)$.

Rj.

F-ja koju razvijamo u red po sinusima grafički izgleda ovako:



Neka je $f(x)$ neka 2π periodična f-ja. Tada

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{Furijeov red f-je } f(x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{Furijeovi koeficijenti f-je } f(x)$$

Ako je $f(x)$ parna f-ja tada $f(x)\sin nx$ je neparna $\Rightarrow b_n = 0 \ \forall n \in \mathbb{Z}$

Ako je $f(x)$ neparna f-ja tada $f(x)\cos nx$ neparna $\Rightarrow a_n = 0 \ \forall n \in \mathbb{Z}$

Trebamo napraviti neparno produženje f-je $f(x)$ (novu f-ju ću nazvati $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in \left(0, \frac{\pi}{2}\right) \\ -f(-x), & x \in \left(-\frac{\pi}{2}, 0\right) \end{cases} = \begin{cases} x\left(\frac{\pi}{2} - x\right), & x \in \left(0, \frac{\pi}{2}\right) \\ x\left(\frac{\pi}{2} + x\right), & x \in \left(-\frac{\pi}{2}, 0\right) \end{cases}$$

Izračunajmo Furijeove koeficijente

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f^*(x)\sin nx\,dx \ \underset{\text{f-ja}}{\overset{f^*(x)\sin nx \text{ parna}}{=\!=\!=}} \ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2} - x\right)\sin nx\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2}x - x^2\right)\sin nx\,dx = \frac{2}{\pi}\left(\frac{\pi}{2}\int_0^{\pi/2} x\sin nx\,dx - \int_0^{\pi/2} x^2 \sin nx\,dx\right) \overset{(*)}{=\!=}$$

$$I_1 = \int_0^{\pi/2} x \sin nx \, dx = \left| \begin{matrix} u = x & dv = \sin nx \, dx \\ du = dx & v = -\frac{1}{n}\cos nx \end{matrix} \right| = -\frac{1}{n} x \cos nx \Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\pi/2} \cos nx \, dx =$$

$$= -\frac{1}{n}\left(\frac{\pi}{2}\cos n \cdot \frac{\pi}{2} - 0\right) + \frac{1}{n}\cdot\frac{1}{n}\sin nx \Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}$$

($\sin\frac{n\pi}{2}$ i $\cos\frac{n\pi}{2}$ mogu uzimati tri vrijednosti $0, 1, -1$)

$$I_2 = \int_0^{\pi/2} x^2 \sin nx \, dx = \left| \begin{matrix} u = x^2 & dv = \sin nx \, dx \\ du = 2x \, dx & v = -\frac{1}{n}\cos nx \end{matrix} \right| = -\frac{1}{n} x^2 \cos nx \Big|_0^{\frac{\pi}{2}} +$$

$$+ \frac{2}{n}\int_0^{\pi/2} x \cos nx \, dx = \left| \begin{matrix} u = x & dv = \cos nx \, dx \\ du = dx & v = \frac{1}{n}\sin nx \end{matrix} \right| = -\frac{\pi^2}{4n}\cos\frac{n\pi}{2} +$$

$$+ \frac{2}{n}\Bigg(\underbrace{\frac{1}{n} x \sin nx \Big|_0^{\frac{\pi}{2}}}_{\frac{\pi}{2}\sin\frac{n\pi}{2}} - \frac{1}{n}\underbrace{\int_0^{\pi/2} \sin nx \, dx}_{\left(-\frac{1}{n}\right)\cos nx \Big|_0^{\frac{\pi}{2}}} \Bigg) = \underline{\frac{-\pi^2}{4n}\cos\frac{n\pi}{2}} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} + \underline{\frac{2}{n^3}\left(\cos\frac{n\pi}{2} - 1\right)}$$

$$= \left(\frac{2}{n^3} - \frac{\pi^2}{4n}\right)\cos\frac{n\pi}{2} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} - \frac{2}{n^3}$$

Primjetimo

$$I_1 = \begin{cases} -\frac{\pi}{2n}(-1)^k, & n = 2k \\ \frac{1}{n^2}(-1)^k, & n = 2k+1 \end{cases} \quad k = 0, 1, 2, \ldots \qquad I_2 = \begin{cases} \left(\frac{2}{n^3} - \frac{\pi^2}{4n}\right)(-1)^k - \frac{2}{n^3}, & n = 2k \\ \frac{\pi}{n^2}(-1)^k - \frac{2}{n^3}, & n = 2k+1 \end{cases} \quad k = 0, 1, 2, \ldots$$

$$\underline{\underline{(*)}} \; -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{2}{\pi}\left(\left(\frac{2}{n^3} - \frac{\pi^2}{4n}\right)\cos\frac{n\pi}{2} + \frac{\pi}{n^2}\sin\frac{n\pi}{2} - \frac{2}{n^3}\right) =$$

$$= \underline{-\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \underline{\underline{\frac{1}{n^2}\sin\frac{n\pi}{2}}} + \left(\frac{-4}{\pi n^3} + \underline{\frac{\pi}{2n}\right)\cos\frac{n\pi}{2}} - \underline{\underline{\frac{2}{n^2}\sin\frac{n\pi}{2}}} + \frac{4}{\pi n^3} =$$

$$= -\frac{4}{\pi n^3}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{4}{\pi n^3} = \begin{cases} -\frac{4}{\pi n^3}(-1)^k + \frac{4}{\pi n^3}, & n = 2k \\ -\frac{1}{n^2}(-1)^k + \frac{4}{\pi n^3}, & n = 2k+1 \end{cases} \quad k = 0, 1, 2, 3, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx =$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\left(\frac{(-1)^{k+1} + 1}{(2k)^3}\right)\sin nx + \sum_{k=0}^{\infty}\left(\frac{(-1)^{k+1}}{(2k+1)^2} + \frac{4}{\pi(2k+1)^3}\right)\sin nx$$

razvoj f-je po sinusima

(#) Izračunati dvostruki integral $I = \iint\limits_D xy \, dx \, dy$, gdje je D oblast ograničena linijama $xy = 1$, $x + y = \frac{5}{2}$.

Rj. Skiciramo oblast D

$xy = 1$

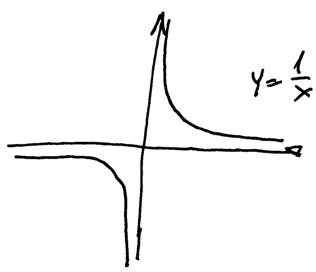
$y = \frac{1}{x}$ $\mathcal{D}: x \in \mathbb{R} \setminus \{0\}$

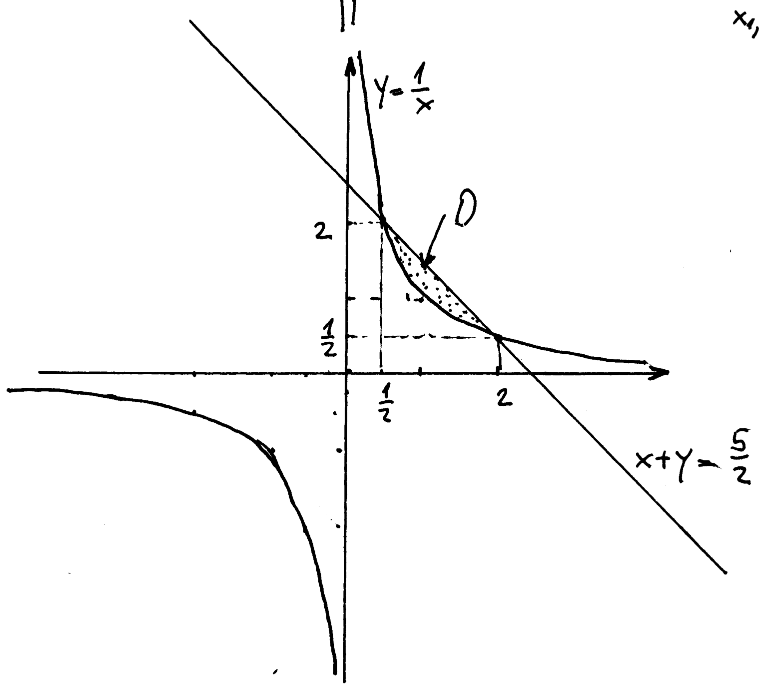
f-ja je neparna (simetrična u odnosu na 0)

ne sijeće y-osu, ne sijeće x-osu

$\lim\limits_{x \to 0+} \frac{1}{x} = +\infty$

$\lim\limits_{x \to \infty} \frac{1}{x} = 0$





Nađimo presječne tačke krive $xy = 1$ i prave $x + y = \frac{5}{2}$.

$$\begin{array}{l} xy = 1 \\ x + y = \frac{5}{2} \\ \hline y = \frac{1}{x} \\ x + y = \frac{5}{2} \\ \hline x + \frac{1}{x} = \frac{5}{2} \quad / \cdot x \end{array}$$

$x^2 - \frac{5}{2}x + 1 = 0 \quad / \cdot 2$

$2x^2 - 5x + 2 = 0$

$D = 25 - 16 = 9$

$x_{1,2} = \frac{5 \pm 3}{4} \qquad x_1 = \frac{2}{4} = \frac{1}{2}$

$x_2 = 2$

$x_1 = \frac{1}{2} \Rightarrow y_1 = 2$

$x_2 = 2 \Rightarrow y = \frac{1}{2}$

$$D: \begin{cases} \frac{1}{2} < x < 2 \\ \frac{1}{x} < y < \frac{5}{2} - x \end{cases}$$

$$\iint\limits_D xy \, dx \, dy = \int_{\frac{1}{2}}^{2} x \, dx \int_{\frac{1}{x}}^{\frac{5}{2} - x} y \, dy =$$

$$= \int_{\frac{1}{2}}^{2} x \frac{1}{2} y^2 \Big|_{\frac{1}{x}}^{\frac{5}{2} - x} dx = \frac{1}{2} \int_{\frac{1}{2}}^{2} x \left(\left(\frac{5}{2} - x\right)^2 - \frac{1}{x^2} \right) dx = \frac{1}{2} \int_{\frac{1}{2}}^{2} x \left(\frac{25}{4} - 5x + x^2 - \frac{1}{x^2} \right) dx$$

$$\frac{1}{2} \int_{\frac{1}{2}}^{2} \left(\frac{25}{4} x - 5x^2 + x^3 - \frac{1}{x} \right) dx = \int_{\frac{1}{2}}^{2} \left(\frac{1}{2} x^3 - \frac{5}{2} x^2 + \frac{25}{8} x - \frac{1}{2} \cdot \frac{1}{x} \right) dx = \ldots = \frac{165}{128} - \ln 2$$

rješenje

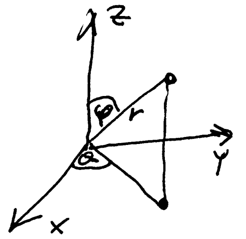
(#) Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$, $a>0$ u I oktantu.

Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V=\iiint\limits_{\Omega} dx\,dy\,dz$.

Datu površ $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$ ne možemo skicirati.
Uvedimo sferne koordinate

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$



$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$\Omega \xrightarrow{\text{transformacije}} \Omega'$

pa pokušajmo naći granice na osnovu date formule.

$$x^2+y^2+z^2 = \underline{r^2\sin^2\varphi}\cos^2\theta + \underline{r^2\sin^2\varphi}\sin^2\theta + r^2\cos^2\varphi = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2$$

$$(x^2+y^2+z^2)^3 = (r^2)^3 = r^6$$
$$z^2 = r^2\cos^2\varphi$$
$$x^2+y^2 = r^2\sin^2\varphi$$

$$(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$$

sad postaje $r^6 = \frac{a^6 r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj. $r^6 = a^6\,\mathrm{ctg}^2\varphi$
$r = \sqrt[6]{a^6\,\mathrm{ctg}^2\varphi}$
$r = a\sqrt[3]{\mathrm{ctg}\varphi}$.

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

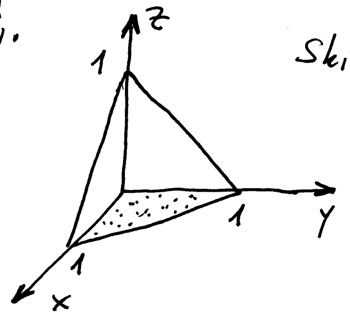
$$\Omega': \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le a\sqrt[3]{\mathrm{ctg}\varphi} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$V = \iiint\limits_{\Omega} r^2\sin\varphi\,dr\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi\,d\varphi \int_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} r^2\,dr = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi \left.\frac{r^3}{3}\right|_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} d\varphi$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi = \frac{a^3}{3}\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\varphi\,d\varphi = \frac{a^3}{3}\cdot\theta\Big|_0^{\pi/2}\cdot\sin\varphi\Big|_0^{\pi/2} = \frac{a^3\pi}{6}$$ tražena zapremina.

(#) Izračunati površinski integral $I = \int\limits_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

Rj.



Skiciramo $x+y+z=1$.

Ako je D projekcija površi S, koja je opisana $z=z(x,y)$, na xOy ravan tada

$$\int\limits_S f(x,y,z)\,dS = \int\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

U našem slučaju S je opisa formulom $z=1-x-y$.

Ako S projiciramo na xOy ravan dobićemo $D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$

$$\int\limits_S xyz\,dS = \iint\limits_D xy(1-x-y)\,dx\,dy = \int_0^1 dx \int_0^{1-x} (xy - x^2y - xy^2)\,dy =$$

$$\frac{\partial z}{\partial x} = -1 \qquad \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

$$\frac{\partial z}{\partial y} = -1$$

$$= \int_0^1 \left(\frac{1}{2}xy^2\Big|_0^{1-x} - \frac{1}{2}x^2y^2\Big|_0^{1-x} - \frac{1}{3}xy^3\Big|_0^{1-x}\right)dx =$$

$$= \int_0^1 (1-x)^2\Big(\frac{1}{2}x - \frac{1}{2}x^2 \underbrace{- \frac{1}{3}x(1-x)}_{-\frac{1}{3}x+\frac{1}{3}x^2}\Big)dx = \int_0^1 (1-x)^2\left(-\frac{1}{6}x^2 + \frac{1}{6}x\right)dx$$

$$= -\frac{1}{6}\int_0^1 (1-x)^2 \cdot \underbrace{(x^2-x)}_{x(x-1)}\,dx = \frac{1}{6}\int_0^1 (1-x)^3 x\,dx = \left|\begin{matrix} 1-x=t & x\Big|_0^1 \to \\ -dx=dt & y\Big|_1^0 \\ dx=-dt & \\ x=1-t & \end{matrix}\right|$$

$$= \frac{1}{6}\int_0^1 t^3(1-t)\,dt = \frac{1}{6}\int_0^1 (t^3-t^4)\,dt = \frac{1}{6}\cdot\frac{1}{4}t^4\Big|_0^1 - \frac{1}{6}\cdot\frac{1}{5}t^5\Big|_0^1$$

$$= \frac{1}{24} - \frac{1}{30} = \frac{30-24}{24\cdot 30} = \frac{6}{24\cdot 30} = \frac{1}{24\cdot 5} = \frac{1}{120}$$ traženi rezultat