Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
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Pismeni ispit iz predmeta **Analiza 3**

1. Naći ekstreme funkcije $z = x + y + 4 + 4\, sin\, x\, sin\, y$.

2. Izračunati trostruki integral $I = \iiint\limits_{D} \frac{dx\, dy\, dz}{(x+y+z+1)^3}$ ako je Ω oblast omeđena koordinatnim ravnima i sa ravni $x + y + z = 1$.

3. Izračunati pomoću krivoliniskog integrala druge vrste površinu ravne figure ograničene konturom

$$\text{c} : \begin{cases} x = a(t - sin\, t) \\ y = a(1 - cos\, t) \\ \quad 0 \leq t \leq 2\pi \end{cases}$$

4. Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2 + z^2)\vec{i} + 2y(x^2 + z^2)\vec{j} + 2z(x^2 + y^2)\vec{k}.$$

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Naći ekstreme f-je $z = x + y + 4 + 4\sin x \sin y$.

Rj. $\frac{\partial z}{\partial x} = 1 + 4\cdot\cos x \sin y$

$\frac{\partial z}{\partial y} = 1 + 4\sin x \cos y$

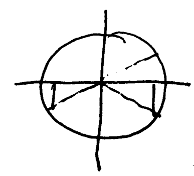
$$1 + 4\cos x \sin y = 0$$
$$1 + 4\sin x \cos y = 0$$

$\sin y \cos x = -\frac{1}{4}$ (a)

$\sin x \cos y = -\frac{1}{4}$ (b)

(a)+(b): $\sin y \cos x + \sin x \cos y = -\frac{1}{2}$

$\sin(y+x) = -\frac{1}{2}$



$y + x = \frac{7\pi}{6}$ ili $y + x = \frac{11\pi}{6}$

(a)-(b):

$\sin y \cos x - \sin x \cos y = 0$

$\sin(y-x) = 0$

$y - x = 0$ ili $y - x = \pi$

1° $x + y = \frac{7\pi}{6}$
$+\ -x + y = 0$

$2y = \frac{7\pi}{6}$

$y = \frac{7\pi}{12} \Rightarrow x = \frac{7\pi}{12}$

2° $x + y = \frac{7\pi}{6}$
$-x + y = \pi$

$2y = \frac{13\pi}{6}$

$y = \frac{13\pi}{12} \Rightarrow x = \frac{\pi}{12}$.

3° $x + y = \frac{11\pi}{6}$
$-x + y = 0$

$y = \frac{11\pi}{12} \Rightarrow$
$x = \frac{11\pi}{12}$

4° $y + x = \frac{11\pi}{6}$
$+\ y - x = \pi$

$y = \frac{17\pi}{12} \Rightarrow x = \frac{5\pi}{12}$

Stacionarne tačke su
$M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$, $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$,
$M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$, $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$\frac{\partial^2 z}{\partial x^2} = -4\sin x \sin y$

$\frac{\partial^2 z}{\partial x \partial y} = 4\cos x \cos y$

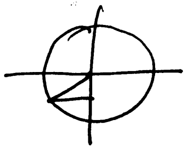
$\frac{\partial^2 z}{\partial y^2} = -4\sin x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$ (I)
$\cos(x-y) = \cos x \cos y + \sin x \sin y$ (II)

(I) + (II): $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

(II) − (I): $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

• Za $M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$



$A = -4\cdot\frac{1}{2}\left(\cos 0 - \cos\frac{7\pi}{6}\right) = -2\left(1+\frac{\sqrt{3}}{2}\right) = -2-\sqrt{3}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos 0\right) = 2\left(-\frac{\sqrt{3}}{2}+1\right) = -\sqrt{3}+2$

$C = -2-\sqrt{3}$

$D = AC - B^2 = (2+\sqrt{3})^2 - (2-\sqrt{3})^2 > 0 \Rightarrow$ f-ja ima ekstrem

$A < 0$ f-ja u tački M_1 ima maksimum.

$Z_{max}\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right) = \frac{7\pi}{12} + \frac{7\pi}{12} + 4 + 2 + \sqrt{3} = 6+\sqrt{3} + \frac{7\pi}{6}$ traženi ekstrem

• Za $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$

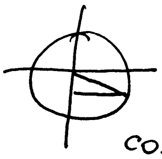
$A = -4\cdot\frac{1}{2}\left(\cos(-\pi) - \cos\frac{7\pi}{6}\right) = -2\left(-1+\frac{\sqrt{3}}{2}\right) = 2-\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos(-\pi)\right) = 2\left(-\frac{\sqrt{3}}{2}-1\right) = -\sqrt{3}-2$

$C = 2-\sqrt{3}$

$D = AC - B^2 = (2-\sqrt{3})^2 - (2+\sqrt{3})^2 > 0 \Rightarrow$ f-ja u tački M_2 nema ekstrema

• Za $M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$



$A = -4\cdot\frac{1}{2}\left(\cos 0 - \cos\frac{11\pi}{6}\right) = -2\left(1-\frac{\sqrt{3}}{2}\right) = -2+\sqrt{3}$

$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos 0\right) = 2\left(\frac{\sqrt{3}}{2}+1\right) = \sqrt{3}+2$

$C = \sqrt{3}-2$

$D = (\sqrt{3}-2)^2 - (\sqrt{3}+2)^2 < 0 \Rightarrow$ f-ja u tački M_3 nema ekstrem

• Za $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$A = -4\cdot\frac{1}{2}\left(\cos(-\pi) - \cos\frac{11\pi}{6}\right) = -2\left(-1-\frac{\sqrt{3}}{2}\right) = 2+\sqrt{3}$

$B = 4\cdot\frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos(-\pi)\right) = 2\left(\frac{\sqrt{3}}{2}-1\right) = \sqrt{3}-2$

$C = 2+\sqrt{3}$, $D = AC - B^2 = (2+\sqrt{3})^2 - (\sqrt{3}-2)^2 > 0 \Rightarrow$ f-ja u tački M_4 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

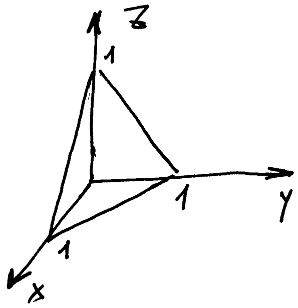
$Z_{min}\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right) = \frac{5\pi}{12} + \frac{17\pi}{12} + 4 + (-2-\sqrt{3}) = 2-\sqrt{3} + \frac{11\pi}{6}$ traženi ekstrem

Izračunati trostruki integral $I=\iiint\limits_{\Omega}\frac{dx\,dy\,dz}{(x+y+z+1)^3}$,

ako je Ω oblast omeđena koordinatnim ravnima i ravni $x+y+z=1$.

Rj. $x+y+z=1$ je ravan koja na koordinatnim osama prolazi kroz tačke $(1,0,0)$, $(0,1,0)$ i $(0,0,1)$

$$\Omega:\begin{cases}0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le 1-x-y\end{cases}$$



$$I=\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{dz}{(x+y+z+1)^3}\overset{(*)}{=}$$

$$\int\frac{dz}{(x+y+z+1)^3}=\left|\begin{matrix}x+y+z+1=t\\ dz=dt\end{matrix}\right|=\int\frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-2}}{-2}+C=$$

$$=\frac{-1}{2(x+y+z+1)^2}+C$$

$$\overset{(*)}{=}\int_0^1 dx\int_0^{1-x}\frac{-1}{2(x+y+z+1)^2}\Bigg|_0^{1-x-y}dy=\int_0^1 dx\int_0^{1-x}\Bigg(\frac{-1}{2(x+y+1-x-y+1)^2}-$$

$$-\frac{-1}{2(x+y+0+1)^2}\Bigg)dy=\int_0^1 dx\int_0^{1-x}\left(-\frac{1}{8}+\frac{1}{2(x+y+1)^2}\right)dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}\left(\frac{1}{4}-\frac{1}{(x+y+1)^2}\right)dy\overset{(**)}{=}-\frac{1}{2}\int_0^1\left(\frac{1}{4}y\Big|_0^{1-x}+\frac{1}{x+y+1}\Big|_0^{1-x}\right)dx$$

$$\boxed{\int\frac{dy}{(x+y+1)^2}=\left|\begin{matrix}x+y+1=t\\ dy=dt\end{matrix}\right|=\int\frac{dt}{t^2}=\frac{t^{-1}}{-1}+C=\frac{-1}{t}+C=\frac{-1}{x+y+1}+C\quad\ldots(**)}$$

$$=-\frac{1}{2}\int_0^1\left(\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{x+1}\right)dx=-\frac{1}{2}\Bigg(\overbrace{\frac{1}{4}x\Big|_0^1-\frac{1}{4}\cdot\frac{x^2}{2}\Big|_0^1}^{\frac{1}{8}}+\frac{1}{2}x\Big|_0^1-\underbrace{\ln|x+1|\Big|_0^1}_{\ln 2-\ln 1}\Bigg)=\frac{1}{2}\ln 2-\frac{5}{16}$$

Izračunati pomoću krivolinijskog integrala II vrste površinu ravne figure ograničene konturom

$$c: \begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \\ 0 \leq t \leq 2\pi . \end{cases}$$

Rj. Površina figure ograničenu zatvorenom linijom c računamo po formuli $P = \frac{1}{2}\int_c x\,dy - y\,dx$.

$x = a(t-\sin t)$ $\qquad$ $y = a(1-\cos t)$

$dx = a(1-\cos t)$ $\qquad$ $dy = a\sin t$

$$x\,dy - y\,dx = a(t-\sin t)\cdot a\sin t - a(1-\cos t)\cdot a(1-\cos t)$$
$$= a^2 t\sin t - a^2\sin^2 t - a^2(1-\cos t)^2$$
$$= a^2(t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t)$$
$$= a^2(t\sin t + 2\cos t - 2)$$

$$P = \frac{1}{2}\int_c x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a^2(t\sin t + 2\cos t - 2))\,dt =$$

$$= \frac{a^2}{2}\left(\int_0^{2\pi} t\sin t\,dt + 2\int_0^{2\pi}\cos t\,dt - 2\int_0^{2\pi} dt\right) = \ldots = \frac{a^2}{2}(-2\pi + 0 - 4\pi) = 3a^2\pi$$

(#) Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\,\vec{v} = \vec{0}$.
Rotor vektorskog polja $rot\,\vec{v}$ se računa ..

$$rot\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x = 2x(y^2+z^2)$ $V_y = 2y(x^2+z^2)$ $V_z = 2z(x^2+y^2)$

$\frac{\partial V_x}{\partial y} = 4xy$ $\frac{\partial V_y}{\partial x} = 4xy$ $\frac{\partial V_z}{\partial x} = 4xz$

$\frac{\partial V_x}{\partial z} = 4xz$ $\frac{\partial V_y}{\partial z} = 4yz$ $\frac{\partial V_z}{\partial y} = 4yz$

$$rot\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{i}(4yz-4yz) - \vec{j}(4xz-4xz) + \vec{k}(4xy-4xy)$$

$$= (0,0,0) = \vec{0}$$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v} = grad\,u$.

$$grad\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x(y^2+z^2) \\ \frac{\partial u}{\partial y} &= 2y(x^2+z^2) \\ \frac{\partial u}{\partial z} &= 2z(x^2+y^2) \end{aligned}\right\} \ldots (1)$$

$u = u(x,y,z)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$u = \int 2x(y^2+z^2)\,dx + \varphi(y,z)$$

$u = x^2(y^2+z^2) + \varphi(y,z)$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= 2yx^2 + \varphi'_y \\ \frac{\partial u}{\partial z} &= 2x^2z + \varphi'_z \end{aligned}\right\} \ldots (2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y = 2yz^2$, $\varphi'_z = 2zy^2$ ... (*)

Odredimo f-ju φ: $\varphi = \int 2yz^2\,dy + \psi(z)$

$\varphi = y^2z^2 + \psi(z)$

$\varphi' = 2y^2z + \psi'$ ... (**)

(*) i (**) $\Rightarrow$ $\psi'_z = 0 \Rightarrow \psi(z) = C$

$\Rightarrow \varphi = y^2z^2 + C \Rightarrow u = x^2y^2 + x^2z^2 + y^2z^2 + C$

Potencijal vektorskog polja je $u = x^2y^2 + x^2z^2 + y^2z^2 + C$