Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 17.02.2010.

Pismeni ispit iz predmeta Analiza 3

1. Ako je $z = e^y \varphi(ye^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna funkcija, dokazati da je $(x^2 - y^2)\dfrac{\partial z}{\partial x} + xy\dfrac{\partial z}{\partial y} = xyz.$

2. Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

3. Izračunati zapreminu tijela ograničenog ravninom $x0y$, valjkom $x^2 + y^2 = 2ax$ i čunjem $x^2 + y^2 = z^2$.

4. Izračunati krivoliniski integral $I = \int\limits_c (xy + x + y)dx + (xy + x - y)dy$ ako je $c:\ x^2 + y^2 = 3x$.

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Za uočene greške pisati na **infoarrt@gmail.com**)

(#) Ako je $z = e^y \varphi(y e^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj.

$$z = e^y \varphi(\xi), \text{ gdje je } \xi(x,y) = y e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}} \left(\frac{1}{2}x^2 y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}} \left(\frac{1}{2}x^2 \cdot (-2) y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \varphi}{\partial x} = \frac{x}{y} e^y e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^y \varphi(\xi) + e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} = e^y \varphi(\xi) + e^y e^{\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} -$$

$$- e^y \cdot \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2) \cdot \frac{x}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi} +$$

$$+ xy\left(e^y \varphi(\xi) + e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}\right)$$

$$= \underline{\frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}} \underline{\underline{- yx e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} + xy e^y \varphi(\xi) +$$

$$\underline{\underline{+ xy e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} \underline{- \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}} =$$

$$= xy e^y \varphi(\xi) = xy e^y \varphi(y e^{\frac{x^2}{2y^2}}) = xyz$$

(#) Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

Rj: Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(p_1,p_2,p_3)$ se računaju po formuli:

α: $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$

$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)} = \frac{y-p_2}{F'_y(p_1,p_2,p_3)} = \frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

$$2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$$

$$\left(\frac{x}{z}\right)'_z = (xz^{-1})'_z = (-1)x\,z^{-2}$$

$$F(x,y,z) = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8 = 0$$

$$F'_x = 2^{\frac{x}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_x(2,2,1) = 4\ln 2.$$

$$F'_y = 2^{\frac{y}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_y(2,2,1) = 4\ln 2$$

$$F'_z = 2^{\frac{x}{z}} \ln 2 \cdot \left(\frac{x}{z}\right)'_z + 2^{\frac{y}{z}} \ln 2 \left(\frac{y}{z}\right)'_z = -\frac{x}{z^2} 2^{\frac{x}{z}} \ln 2 - \frac{y}{z^2} 2^{\frac{y}{z}} \ln 2$$

$$= -\frac{1}{z^2} \ln 2 \left(x 2^{\frac{x}{z}} + y 2^{\frac{y}{z}}\right)$$

$$F'_z(2,2,1) = -\ln 2 (2\cdot 4 + 2 \cdot 4) = -16\ln 2$$

$$4\ln 2(x-2) + 4\ln 2(y-2) + (-16\ln 2)(z-1) = 0$$

$4x\ln 2 + 4y\ln 2 - 16z\ln 2 + 8\ln 2 = 0$ jednačina tangentne ravni

$$\frac{x-2}{4\ln 2} = \frac{y-2}{4\ln 2} = \frac{z-1}{-16\ln 2} \Rightarrow \frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{-4}$$

jednačina normale na površ

(#) Izračunati zapreminu tijela ograničenog ravninom XOY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint\limits_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

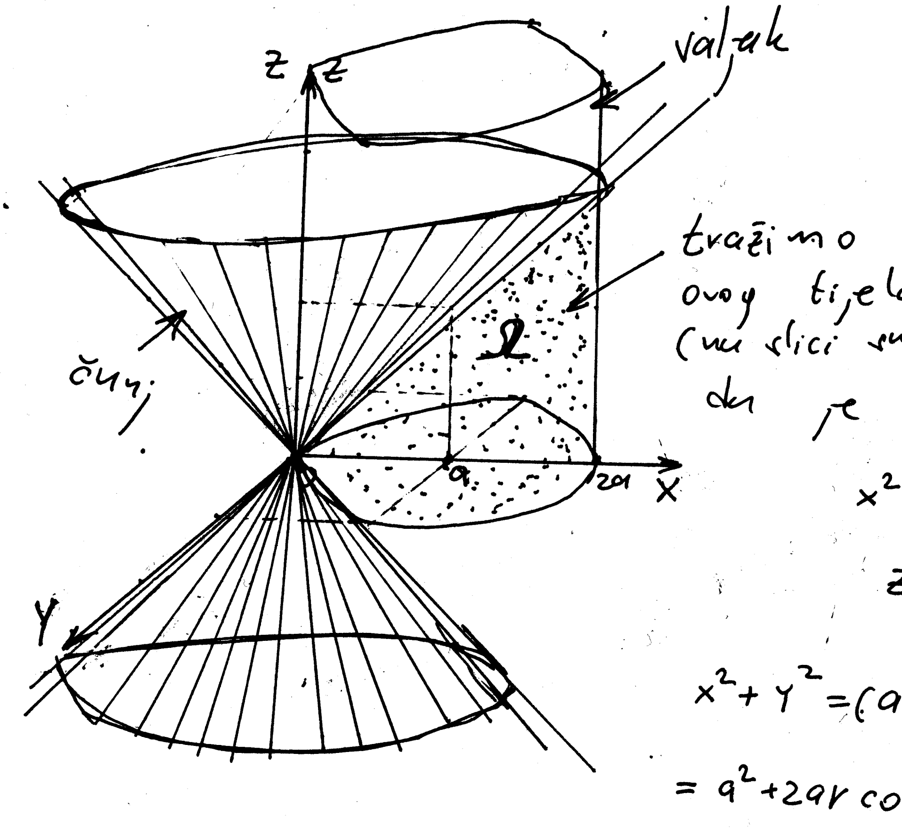
$x^2-2ax+y^2=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

valjak u presjeku sa XOY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XOY ravni je tačka, a u presjeku sa YOZ ili sa XOZ su po dvije prave



Oblast Ω je najlakše projicirati na XOY ravan.

Uvodimo cilindrične koordinate

$x=a+r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

$\Omega': \begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \end{cases}$

$x^2+y^2=z^2$

$z=\pm\sqrt{x^2+y^2}$ čunj

$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$

$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$

$=a^2+2ar\cos\varphi+r^2$

$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}} r\,dz = \ldots$ ovo je teško izračunati

Pokušajmo uvesti drugačije smjene.

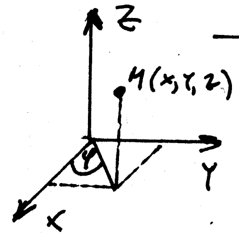
$x = r\cos\varphi$

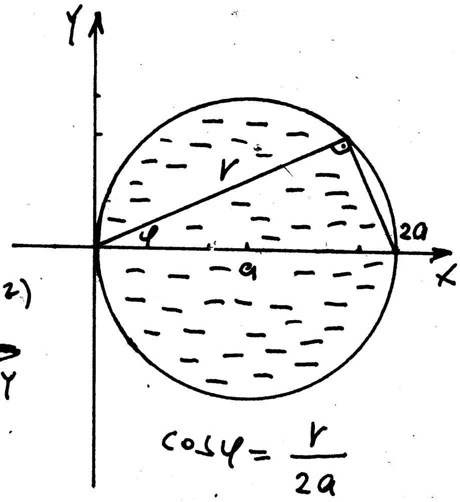
$y = r\sin\varphi$

$z = z$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$





$\cos\varphi = \dfrac{r}{2a}$

$r = 2a\cos\varphi$

$$\Omega'': \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iiint\limits_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(r z \Big|_0^r \right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^{2a\cos\varphi} \right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi \cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi (1-\sin^2\varphi)\,d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t \Big|_{-1}^{1} - \frac{1}{3} t^3 \Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \dfrac{32}{9} a^3$ tražena zapremina

Izračunati krivoliniski integral

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy \quad \text{ako je } c: x^2+y^2=3x.$$

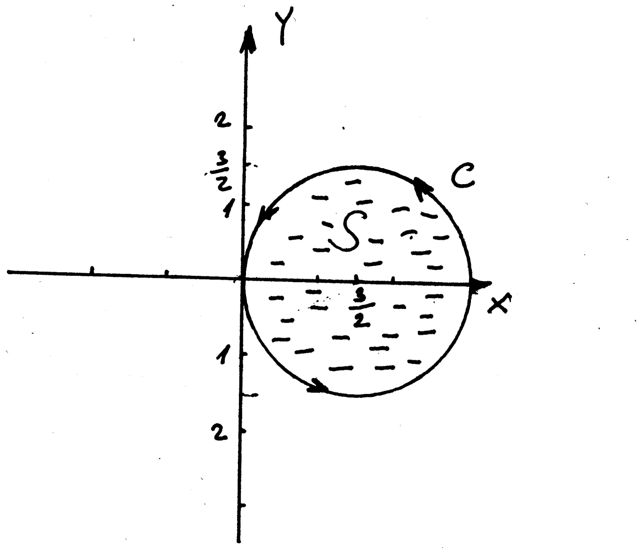
Rj. $x^2+y^2=3x$

$x^2-3x+y^2=0$

$x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+y^2=0$

$(x-\frac{3}{2})^2+y^2=\frac{9}{4}$

c: Krug sa centrom u tački $(\frac{3}{2},0)$ poluprečnika $r=\frac{3}{2}$



I način: Greenova formula za ravan

$$\int_c P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P=xy+x+y$, $\frac{\partial P}{\partial y}=x+1$, $Q=xy+x-y$, $\frac{\partial Q}{\partial x}=y+1$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y+1-(x+1)=y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$x=\frac{3}{2}+r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$$D:\begin{cases}0\le r\le\frac{3}{2}\\0\le\varphi\le2\pi\end{cases}$$

$$I=\int_c(xy+x+y)\,dx+(xy+x-y)\,dy=\iint_S(y-x)\,dx\,dy=\iint_D\left(r\sin\varphi-\left(\frac{3}{2}+r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$=\int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi-\frac{3}{2}r-r^2\cos\varphi\right)d\varphi\right]dr=\int_0^{3/2}\left(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0}-\frac{3r}{2}\varphi\Big|_0^{2\pi}-\underbrace{r\sin\varphi\Big|_0^{2\pi}}_{=0}\right)dr$$

$$=\int_0^{3/2}-3\pi r\,dr=-3\pi\frac{r^2}{2}\Big|_0^{3/2}=-\frac{3}{2}\pi\cdot\frac{9}{4}=-\frac{27}{8}\pi$$

II način: Klasičan način

C kriva u ravni opisana jednačinom $y=\eta(x)$, $a \le x \le b$

$$\int_C P(x,y)dx + Q(x,y)dy = \int_a^b [P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1 \le t \le t_2$ tada

$$\int_C P(x,y)dx + Q(x,y)dy = \int_{t_1}^{t_2} [P(\mu(t),\eta(t))\mu'(t) + Q(\mu(t),\eta(t))\eta'(t)]dt$$

U našem slučaju c je kružnica. Parametarizirajmo kružnicu

$x = \frac{3}{2} + r\cos\varphi$

$y = r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$x = \frac{3}{2} + \frac{3}{2}\cos t$

$y = \frac{3}{2}\sin t$

gdje $0 \le t \le 2\pi$

$\frac{\partial x}{\partial t} = -\frac{3}{2}\sin t$

$\frac{\partial y}{\partial t} = \frac{3}{2}\cos t$

$$I = \int_C (xy+x+y)dx + (xy+x-y)dy = \int_0^{2\pi} \Big[\Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big) + \big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big) + \big(\tfrac{3}{2}\sin t\big)\Big)\big(-\tfrac{3}{2}\sin t\big) + \Big(\big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big)\big(\tfrac{3}{2}\sin t\big) + \big(\tfrac{3}{2}+\tfrac{3}{2}\cos t\big) - \big(\tfrac{3}{2}\sin t\big)\Big)\tfrac{3}{2}\cos t\Big]dt = \ldots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I = -\frac{27}{8}\pi$$