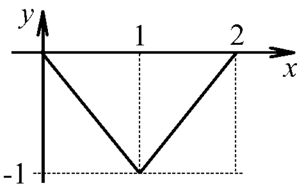
Pismeni ispit iz Analize III, 15.06.2015.
ispit pisati isključivo hemijskom olovkom



1. Dio grafika f-je $y = f(x)$ je prikazan na slici lijevo. Datu funkciju razviti u Furijer-ov red samo po sin-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

2. Izračunati $\iint\limits_D \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R}^2 : -x \le x^2+y^2 \le -2x,\ x \le 0,\ y \le 0\}$.

3. Izračunati krivoliniski integral

$$I = \int\limits_c (xy+x+y)dx + (xy+x-y)dy$$

ako je $c:\ x^2+y^2 = -3y$.

4. Izračunati površinski integral prvog tipa $I = \iint\limits_S (x^2+y^2)\mathrm{d}S$ gdje je S-površina dijela paraboloida $y^2+z^2 = 2x$ koja se nalazi "ispod" ravni $x = 1$ (dio paraboloida $y^2+z^2 = 2x$ koju odsjeca ravan $x = 1$).

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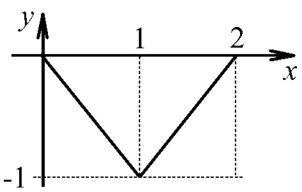

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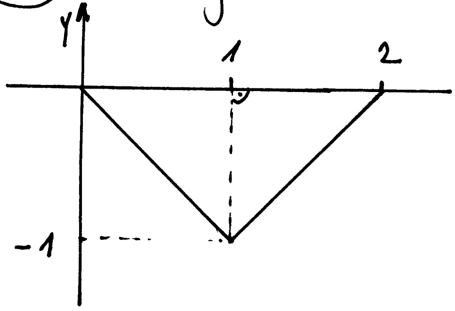
(#) Dio grafika f-je je prikazan na slici.



Datu f-ju razviti u Furijer-ov red samo po sinusima. Dobijeni rezultat iskoristiti za sumiranje reda

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

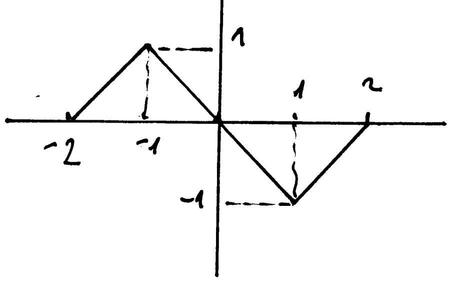
Rj. Prisjetimo se definicije Furijer-ovog reda f-je $y=f(x)$ na (a,b)

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

gdje su

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

Primjetimo da će $f(x)$ biti razvijena po sinusima akko je $a_n=0$. Kako je $\cos x$ parna f-ja $f(x)\cdot\cos x$ će biti neparna akko je $f(x)$ nepar. Tj. $a_n=0$ akko je $f(x)$ neparna f-ja i interval (a,b) je simetričan u odnosu na nula. Drugim riječima f-ju koju razvijamo u Furijer-ov red moramo razvijati na intervalu $(-2,2)$ i f-ja će izgledati ovako



$$f(x)=\begin{cases} y=x+2, & x\in(-2,-1)\\ y=-x, & x\in[-1,1)\\ y=x-2, & x\in[1,2)\end{cases}$$

a b
$(-2,2)$ $b-a=4$

$(-2,0)$, $(-1,1)$: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$

$\frac{x+2}{1}=\frac{y}{1}$

$y=x+2$

$(-1,1)$, $(1,-1)$: $\frac{x+1}{2}=\frac{y-1}{-1-1}$

$x+1=-y+1$

$y=-x$

$(1,-1)$, $(2,0)$: $\frac{x-1}{2-1}=\frac{y+1}{1}$ $y+1=x-1$

$$\frac{2}{b-a}=\frac{2}{4}=\frac{1}{2}$$

$$\frac{2n\pi x}{b-a}=\frac{n\pi x}{2}$$

Sada možemo bez problema izračunati Furijerov koeficijent b_n

$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{f(x)}_{\text{neparna f-ja (sim. u odnosu na koordinatni početak)}} \underbrace{\sin\frac{n\pi x}{2}}_{\text{neparna}} = \frac{1}{2}\cdot 2\int_{0}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \int_{0}^{1}(-x)\sin\frac{n\pi x}{2}\,dx +$$

$$+\int_{1}^{2}(x-2)\sin\frac{n\pi x}{2}\,dx = I_1 + I_2$$

$$I_1 = \int_0^1 (-x)\sin\frac{n\pi x}{2}\,dx = \left|\begin{matrix} u=(-x) & dv=\sin\frac{n\pi x}{2}dx & d\left(\frac{n\pi x}{2}\right)=\frac{n\pi}{2}dx \\ du=-dx & v=-\frac{2}{n\pi}\cos\frac{n\pi x}{2} & \end{matrix}\right| = \underset{\text{lagana vježba}}{\ldots} =$$

$$= \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$I_2 = \int_1^2 (x-2)\sin\frac{n\pi x}{2}\,dx = \left|\begin{matrix} u=x-2 & dv=\sin\frac{n\pi x}{2}dx \\ du=dx & v=-\frac{2}{n\pi}\cos\frac{n\pi x}{2} \end{matrix}\right| = \underset{\text{lagana vježba}}{\ldots} =$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Prema tome $I_1 + I_2 = -\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} = b_n$

$$f(x) \sim -\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{2} = -\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin\frac{(2k-1)\pi}{2}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{2}$$

$$= -\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{2}$$

Komentar: Ovaj rezultat se može veoma lagano provjeriti ako u Geo-gebri ukucate sljedeću naredbu:

y=(-8/pi^2)*Sum[Sequence[(-1)^(n+1)*(sin((2*n-1)*pi*x/2))/(2*n-1)^2, n, 1, 100]] Pitanje: šta ćemo dobiti ako umjesto -8 stavimo 8?

Na kraju za $x=1$ imamo

$$f(1) = -1 = -\frac{8}{\pi^2}\sum\frac{(-1)^{k+1}}{(2k-1)^2}(-1)^{k+1} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Izračunati integral $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ gdje je

$$D = \{(x,y) \in \mathbb{R}^2 \mid -x \le x^2+y^2 \le -2x,\ x \le 0,\ y \le 0\}.$$

Rj. Skicirajmo oblast D:

$(-1, 1)$

$1 \le \underbrace{(-1)^2 + 1^2}_{=2} \le (-2)(-1)$

$x^2+y^2=-2x$

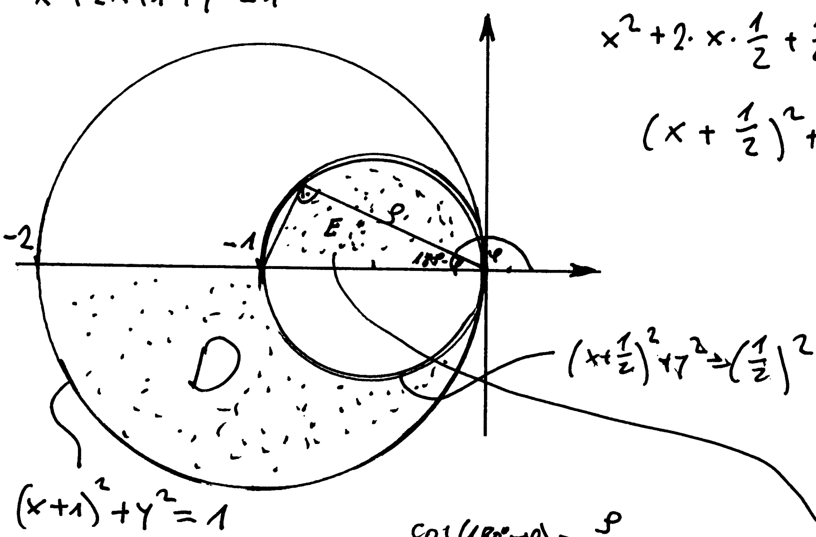
$x^2+2x+1+y^2=1$

$x^2+y^2=-x$

$x^2+x+y^2=0$

$x^2+2\cdot x\cdot \frac{1}{2}+\frac{1}{4}+y^2=\frac{1}{4}$

$(x+\frac{1}{2})^2+y^2=(\frac{1}{2})^2$



Uvedimo polarne koordinate

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$ $\quad dx\,dy = \rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transf.}} D'$

Dio unutrašnjost manjeg kruga gornjeg dijela kruga možemo opisati sa

$E: \begin{cases} 0 \le \rho \le -\cos\varphi \\ \frac{\pi}{2} \le \varphi \le \pi \end{cases}$

$\cos(180°-\varphi) = \frac{\rho}{1}$

$\cos(180°-\varphi) = \underbrace{\cos 180°}_{-1}\cdot\cos\varphi + \underbrace{\sin 180°}_{=0}\sin\varphi$

$-\cos\varphi = \frac{\rho}{1}$

pa je donji dio ovog kruga

$\begin{cases} 0 \le \rho \le -\cos\varphi \\ \pi \le \varphi \le \frac{3\pi}{2} \end{cases}$

Sad nije teško zaključiti da je

$D': \begin{cases} -\cos\varphi \le \rho \le -2\cos\varphi \\ \pi \le \varphi \le \frac{3\pi}{2} \end{cases}$

Kako je $x^2+y^2=\rho^2$ imamo

$\frac{1}{3}(-8\cos^3\varphi + \cos^3\varphi)$

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \left|\begin{matrix}\text{uvedemo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right| = \iint_{D'} \rho\,\rho\,d\rho\,d\varphi = \int_{\pi}^{\frac{3\pi}{2}} d\varphi \int_{-\cos\varphi}^{-2\cos\varphi} \rho^2\,d\rho = -\frac{7}{3}\int_{\pi}^{3\pi/2} \cos^3\varphi\,d\varphi$$

$$= -\frac{7}{3}\int_{\pi}^{\frac{3\pi}{2}} (1-\sin^2\varphi)\,d(\sin\varphi) = \dots = \left(-\frac{7}{3}\right)\left(-\frac{2}{3}\right) = \frac{14}{9}$$ traženo rješenje

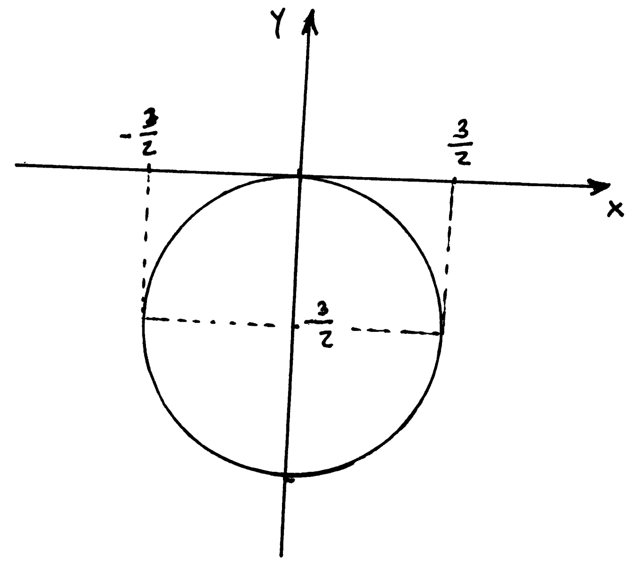
Izračunati krivolinijski integral

$$I = \oint_C (xy + x + y)\,dx + (xy + x - y)\,dy$$

ako je $C: x^2 + y^2 = -3y$.

Rj. Skicirajmo datu krivu C.

$$x^2 + y^2 = -3y$$
$$x^2 + y^2 + 3y = 0$$
$$x^2 + y^2 + 2 \cdot y \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$
$$x^2 + \left(y + \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$
$$C\left(0, -\frac{3}{2}\right), \quad r = \frac{3}{2}$$



Zadatak je najjednostavnije rješiti upotrebom Greenove formule.

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad D\text{-površ koju ograničava kriva } C$$

$$\left.\begin{aligned} Q = (xy + x - y) &\Rightarrow \frac{\partial Q}{\partial x} = y + 1 \\ P = xy + x + y &\Rightarrow \frac{\partial P}{\partial y} = x + 1 \end{aligned}\right\} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - x$$

Unutrašnjost datog kruga možemo opisati upotrebom polarnih koordinata

$$x = \rho\cos\varphi$$
$$y = -\frac{3}{2} + \rho\sin\varphi \qquad D \xrightarrow{\text{transformiš}} D': \begin{cases} 0 \le \rho \le \frac{3}{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$
$$dx\,dy = \rho\,d\rho\,d\varphi$$

$$I = \oint_C (xy+x+y)\,dx + (xy+x-y)\,dy = \left| \begin{matrix} \text{Greenova} \\ \text{formula} \end{matrix} \right| = \iint_D (y-x)\,dx\,dy =$$

$$= \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = \iint_{D'} \left(-\frac{3}{2} + \rho\sin\varphi - \rho\cos\varphi\right)\rho\,d\rho\,d\varphi =$$

$$= \iint_{D'} -\frac{3}{2}\rho\,d\rho\,d\varphi + \iint_{D'} \rho^2\sin\varphi\,d\rho\,d\varphi + \iint_{D'} \rho^2\cos\varphi\,d\rho\,d\varphi = I_1 + I_2 + I_3$$

$$I_1 = \iint_{D'} -\frac{3}{2}\rho\,d\rho\,d\varphi = \left(-\frac{3}{2}\right)\int_0^{2\pi} d\varphi \int_0^{\frac{3}{2}} \rho\,d\rho = \left(-\frac{3}{2}\right)\cdot \varphi\Big|_0^{2\pi} \cdot \frac{1}{2}\rho^2\Big|_0^{\frac{3}{2}}$$

$$= \left(-\frac{3}{2}\right)\cdot 2\pi \cdot \frac{1}{2}\cdot\frac{9}{4} = -\frac{27}{8}\pi$$

$$I_2 = \iint_{D'} \rho^2\sin\varphi\,d\rho\,d\varphi = \int_0^{2\pi}\sin\varphi\,d\varphi \int_0^{\frac{3}{2}}\rho^2\,d\rho = \frac{1}{3}\rho^3\Big|_0^{\frac{3}{2}} \cdot \underbrace{(-\cos\varphi)\Big|_0^{2\pi}}_{=0} = 0$$

Slično $I_3 = \iint_{D'} \rho^2\cos\varphi\,d\rho\,d\varphi = 0$

Prema tome

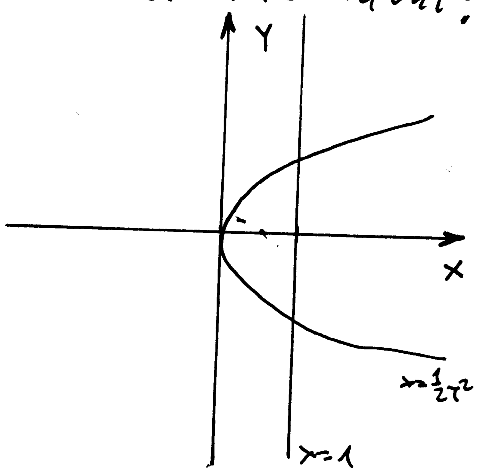
$$\oint_C (xy+x+y)\,dx + (xy+x-y)\,dy = -\frac{27}{8}\pi$$

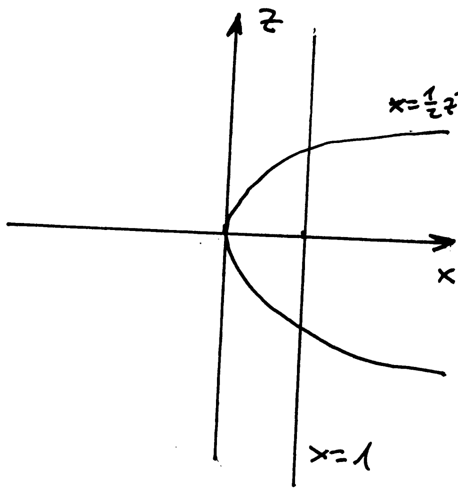
Izračunati površinski integral prvog tipa

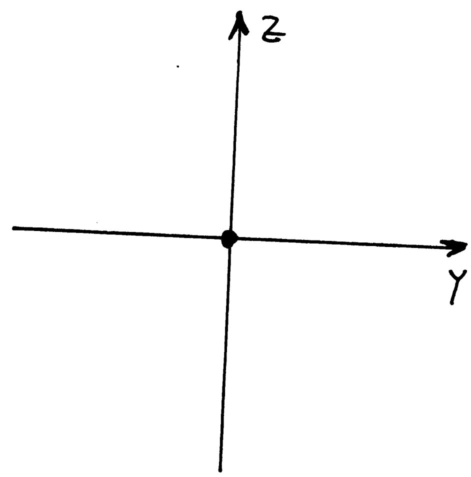
$$\iint_S (y^2+z^2)\,dS$$

gdje je S - površina dijela paraboloida $y^2+z^2=2x$ koja se nalazi "ispod" ravni $x=1$ (dio paraboloida $y^2+z^2=2x$ koju odsjeca ravan $x=1$).

Rj. Kako izgledaju presjeke paraboloida $y^2+z^2=2x$ i ravni $x=1$ sa tri koordinatne ravni?







$x=1$

$y^2=2x$

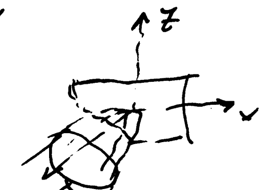
$x=\frac{1}{2}y^2$

$x=1$

$z^2=2x$

$x=\frac{1}{2}z^2$

Na osnovu ova tri presjeka možemo zaključiti kako tijelo izgleda u prostoru



Primjećujemo da projekciju tijela moramo napraviti na yOz ravan tj. koristimo sljedeću formulu za računanje površinskog integrala prve vrste

$$\iint_S f(x,y,z)\,dS = \iint_D f(\eta(x),y,z)\sqrt{1+\left(\frac{\partial\eta}{\partial z}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2}\,dy\,dz$$

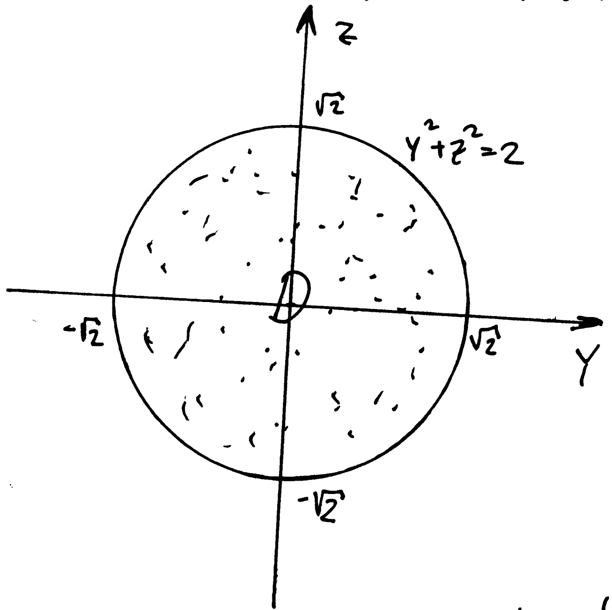
gdje je $x=\eta(y,z)$ a D je projekcija površi na yOz ravan.

Kako dobiti ortogonalnu projekciju površi S na yOz ravan?
Postoji više načina za ovo a jedan od njih je da se na neki način iz sistema

$$\begin{aligned} y^2+z^2&=2x \\ x&=1 \end{aligned}$$

riješimo x-a.

Time dobijamo da je $y^2+z^2=2$ ortogonalna projekcija površi S na yOz ravan.



$$\iint_S (y^2+z^2)\,dS = \left| \begin{array}{l} x=\frac{1}{2}(y^2+z^2) \\ \frac{\partial x}{\partial y}=y \quad \frac{\partial x}{\partial z}=z \end{array} \right|$$

$$= \iint_D (y^2+z^2)\sqrt{1+y^2+z^2}\,dx\,dy =$$

$$= \left| \begin{array}{l} \text{uvodimo polarne koordinate} \\ z=\rho\cos\varphi \\ y=\rho\sin\varphi \\ dy\,dz=\rho\,d\rho\,d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D' : \begin{cases} 0\le\rho\le\sqrt{2} \\ 0\le\varphi\le 2\pi \end{cases} \right| =$$

$$= \iint_{D'} \rho^2\sqrt{1+\rho^2}\,\rho\,d\rho\,d\varphi = \int_0^{\sqrt{2}} \rho^2\sqrt{1+\rho^2}\,\rho\,d\rho \int_0^{2\pi} d\varphi = 2\pi\int_0^{\sqrt{2}} \rho^2\sqrt{1+\rho^2}\,\rho\,d\rho =$$

$$= \left| \begin{array}{l} \text{uvodimo smjenu} \\ 1+\rho^2=t^2 \\ 2\rho\,d\rho=2t\,dt \end{array} \quad \ldots \right| = \underset{\ldots}{\text{lagana vježba}} = \frac{(24\sqrt{3}+4)\pi}{15}$$

traženo rješenje