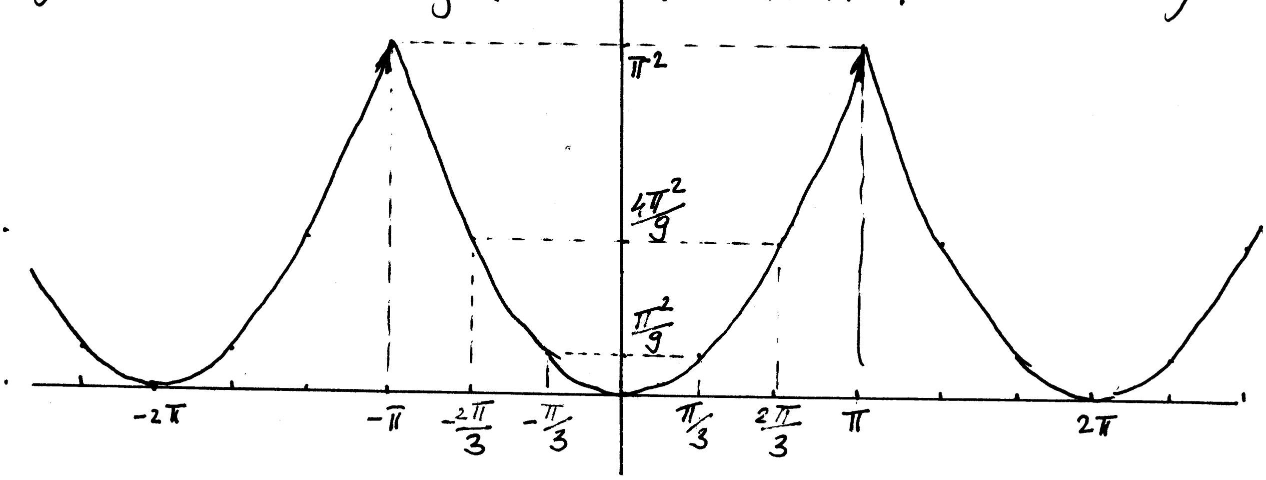
(#) F-ju definisanu grafikom razviti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje redova

a) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$; b) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$.

Rj. Primjetimo da je f-ja periodična perioda 2π, pa je možemo razviti u Furijeov red.

Posmatrajmo f-ju na intervalu $(-\pi,\pi)$.

$f(-\pi)=\pi^2$ $\quad f(0)=0$

$f(-\frac{2\pi}{3})=\frac{4\pi^2}{9}$ $\quad f(\frac{\pi}{3})=\frac{\pi^2}{}$

$f(-\frac{\pi}{3})=\frac{\pi^2}{9}$

Primjetimo da je $f(x)=x^2$, za $x\in(-\pi,\pi)$.

Dovoljno ju je pretvoriti u Furijer-ov red na ovom intervalu.

Prisjetimo se: Trigonometrijski red oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right),\quad x\in[a,b]$$

zovemo Furijer-ov red f-je $f(x)$ na intervalu $[a,b]$, gdje su $a_0=\frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$

i $b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$ brojevi koje zovem Furijer-ovi koeficijenti.

Posmatramo interval $[-\pi, \pi]$, $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx=\frac{1}{\pi}\cdot\frac{1}{3}x^3\Big|_{-\pi}^{\pi}=\frac{2\pi^3}{3\pi}=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\ \underset{\text{parna}}{\cos nx}}_{\text{parna}}\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx=\begin{vmatrix}u=x^2 & dv=\cos nx\,dx\\ du=2x & v=\frac{1}{n}\sin nx\end{vmatrix}$$

$$=\frac{2}{\pi}\left[\underbrace{\frac{1}{n}x^2\sin nx\Big|_0^{\pi}}_{=0-0}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\right]=\begin{vmatrix}u=x & dv=\sin nx\,dx\\ du=dx & v=\frac{-1}{n}\cos nx\end{vmatrix}=$$

$$=-\frac{4}{n\pi}\left[-\frac{1}{n}x\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]=\frac{4}{n^2\pi}(\pi\cos n\pi-0)+\underbrace{\frac{1}{n^2}\sin nx\Big|_0^{\pi}}_{=0-0}$$

$$=(-1)^n\frac{4}{n^2},\quad n\neq 0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\ \underset{\text{neparna}}{\sin nx}}_{=\text{neparna}}\,dx=0$$

traženi Furijer-ov red

Prema tome $$x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$$

Za $x=0$ imamo

$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{3}+4\left(\frac{-1}{1}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\right)$$

$$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}$$

Za $x=\pi$ imamo

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots=\frac{\pi^2}{6}$$

(#) Ako je $f(x) = \arcsin \frac{x}{y}$ gdje je $y = \sqrt{x^2+1}$

provjeriti da li je $\frac{df}{dx} = \frac{1}{x^2+1}$.

Rj.

Primjetimo da je $f(x)$ složena f-ja jedne promjenjive.
Koristimo formulu

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} = \frac{1}{y\sqrt{\frac{y^2-x^2}{y^2}}} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2} = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{df}{dx} = \frac{1}{\sqrt{y^2-x^2}} + \frac{-x^2}{\underset{\substack{\| \\ \sqrt{x^2+1}}}{y}\sqrt{y^2-x^2}\sqrt{x^2+1}} = \frac{1}{\sqrt{y^2-x^2}} - \frac{x^2}{(x^2+1)\sqrt{y^2-x^2}}$$

$$= \frac{1}{\underbrace{\sqrt{y^2-x^2}}_{=1}}\left(1 - \frac{x^2}{x^2+1}\right) = \frac{x^2+1-x^2}{x^2+1} = \frac{1}{x^2+1}$$

$y^2 - x^2 = x^2+1-x^2 = 1$

data jednakost je tačna

(#) Odrediti ekstreme f-je $f(x,y) = x e^{y + x \sin y}$.

Rj: Odredimo parcijalne izvode

$$\frac{\partial f}{\partial x} = e^{y+x\sin y} + x e^{y+x\sin y} \cdot \sin y = e^{y+x\sin y}(1 + x \sin y)$$

$$\frac{\partial f}{\partial y} = x e^{y+x\sin y} \cdot (1 + x\cos y)$$

Da bi odredili stacionarne tačke trebamo riješiti sljedeći sistem

$$e^{y+x\sin y}(1+x\sin y) = 0$$
$$x e^{y+x\sin y}(1+x\cos y) = 0$$

primjetimo da je $e^{y+x\sin y} > 0 \ \forall x,y$

$/ e^{y+x\sin y}$ obe jednačine

$$1 + x\sin y = 0$$
$$x(1 + x\cos y) = 0$$

primjetimo da x ne smije biti nula (u suprotnom iz prve jednakosti bi dobili $1=0$ # kontrad.

$$1 + x\sin y = 0$$
$$1 + x\cos y = 0$$

$$\sin y = \cos y$$

$y = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$

ili $y = \frac{5\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$

Za $y = \frac{\pi}{4} + 2k\pi$ imamo

$$1 + x\frac{\sqrt{2}}{2} = 0$$

$$x = -\sqrt{2}$$

Za $y = \frac{5\pi}{4} + 2k\pi$. imamo

$$1 + x\left(-\frac{\sqrt{2}}{2}\right) = 0$$

$$x = \sqrt{2}$$

Stacionarne tačke su

$M_k\left(-\sqrt{2}; \frac{\pi}{4} + 2k\pi\right)$

i $N_k\left(\sqrt{2}; \frac{5\pi}{4} + 2k\pi\right)$.

Odredimo druge parcijalne izvode

$$\frac{\partial^2 f}{\partial x^2} = e^{y+x\sin y}\cdot \sin y + e^{y+x\sin y}\sin y = 2e^{y+x\sin y}\sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{y+x\sin y}\cdot(1+x\cos y)(1+x\sin y) + e^{y+x\sin y}x\cos y$$

$$\frac{\partial^2 f}{\partial y^2} = x e^{y+x\sin y}\cdot(1+x\cos y)(1+x\cos y) + x e^{y+x\sin y}(-x\sin y)$$

Za tačke $M_k(-\sqrt{2}, \frac{\pi}{4}+2k\pi)$ imamo $1+x\sin y=0$ i $1+x\cos y=0$

pa je $A=2e^{\frac{\pi}{4}-\sqrt{2}\cdot\frac{\sqrt{2}}{2}}\cdot\frac{\sqrt{2}}{2} = \sqrt{2}\,e^{\frac{\pi}{4}-1}$, $B=-e^{\frac{\pi}{4}-1}$, $C=-\sqrt{2}\,e^{\frac{\pi}{4}-1}$

$$D=\begin{vmatrix} A & B \\ B & C\end{vmatrix} = AC-B^2 = -2e^{\frac{\pi}{2}-2} - e^{\frac{\pi}{2}-2} = -3e^{\frac{\pi}{2}-2} < 0$$

U tačkama $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ f-ja nema ekstrem.

Za tačke $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ imamo

$$A=-\sqrt{2}\,e^{\frac{5\pi}{4}-1},\quad B=-e^{\frac{5\pi}{4}-1},\quad C=\sqrt{2}\,e^{\frac{5\pi}{4}-1}$$

$$D=\begin{vmatrix} A & B \\ B & C\end{vmatrix} = AC-B^2 = -2e^{\frac{5\pi}{2}-2} - e^{\frac{5\pi}{2}-2} < 0$$

U tačkama $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ f-ja nema ekstrem.

(Tačke $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ i $N_k(\sqrt{2}; \frac{5\pi}{4}+2k\pi)$ su sedlaste tačke).

Izračunati dvostruki integral

$$\int_0^{2\pi} d\varphi \int_0^a \rho^2 \sin^2\varphi \, d\rho$$

Rj.

$$\int_0^{2\pi} d\varphi \int_0^a \rho^2 \sin^2\varphi \, d\rho = \int_0^{2\pi} \sin^2\varphi \, d\varphi \int_0^a \rho^2 d\rho = \left| \begin{array}{c} 1 = \sin^2\varphi + \cos^2\varphi \\ \cos 2\varphi = \cos^2\varphi - \sin^2\varphi \\ \hline 1 - \cos 2\varphi = 2\sin^2\varphi \\ \sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi) \end{array} \right|$$

$$= \int_0^{2\pi} \frac{1}{2}(1-\cos 2\varphi) \frac{1}{3}\rho^3 \Big|_0^a d\varphi =$$

$$= \frac{a^3}{6} \int_0^{2\pi} (1-\cos 2\varphi) \, d\varphi = \frac{a^3}{6}\left(\varphi \Big|_0^{2\pi} - \frac{1}{2}\sin 2\varphi \Big|_0^{2\pi} \right)$$

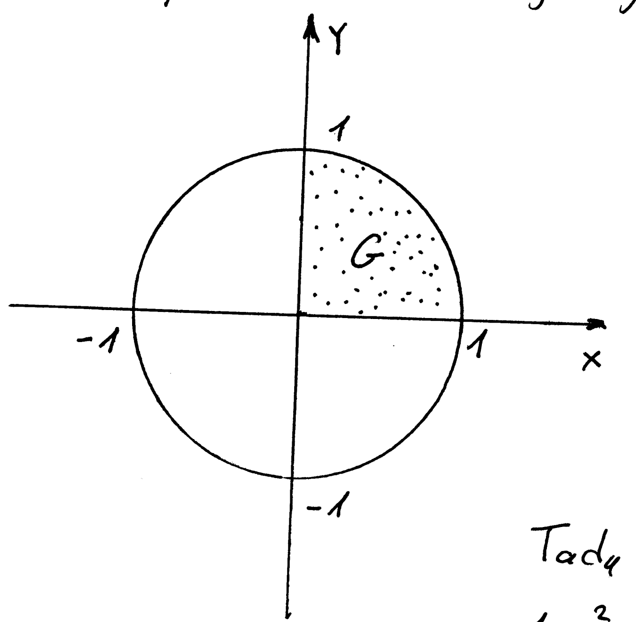
$$= \frac{a^3 \pi}{3}$$ traženo rješenje

(#) Izračunati dvojni integral $I = \iint\limits_G \frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}\,dx\,dy$

gdje je $G = \{(x,y) : x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

Rj.

Skicirajmo oblast integracije G



Ako uvedemo polarne koordinate

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$

$$G \xrightarrow{\text{transformiše}} G' : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

Tada

$$1-x^2-y^2 = 1-(x^2+y^2) = 1-\rho^2$$

$$xy = \rho^2\sin\varphi\cos\varphi$$

$$2x^2+y^2 = 2\rho^2\cos^2\varphi + \rho^2\sin^2\varphi =$$
$$= \rho^2(\underbrace{2\cos^2\varphi}_{\cos^2\varphi+\cos^2\varphi} + \sin^2\varphi) = \rho^2(\cos^2\varphi+1)$$

$$\iint\limits_G \frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}\,dx\,dy = \left|\begin{matrix}\text{uvedimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right| = \iint\limits_{G'} \frac{\cancel{\rho^2}\sin\varphi\cos\varphi\sqrt{1-\rho^2}}{\cancel{\rho^2}(\cos^2\varphi+1)}\,\rho\,d\rho\,d\varphi$$

$$= \int_0^{\pi/2} \frac{\sin\varphi\cos\varphi}{\cos^2\varphi+1}\,d\varphi \int_0^1 \rho\sqrt{1-\rho^2}\,d\rho = \left|\begin{matrix} d(\cos^2\varphi+1) = 2\cos\varphi(-\sin\varphi)\,d\varphi \\ \cos\varphi\sin\varphi\,d\varphi = -\frac{1}{2}d(\cos^2\varphi+1) \\ d(1-\rho^2) = -2\rho\,d\rho \\ \rho\,d\rho = -\frac{1}{2}d(1-\rho^2) \end{matrix}\right|$$

$$= \int_0^{\pi/2} \frac{-\frac{1}{2}d(\cos^2\varphi+1)}{\cos^2\varphi+1} \int_0^1 \left(-\frac{1}{2}\right)(1-\rho^2)^{\frac{1}{2}}d(1-\rho^2) = \frac{1}{4}\ln|\cos^2\varphi+1|\Big|_0^{\frac{\pi}{2}} \cdot \frac{2}{3}(1-\rho^2)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{6}\ln 2$$