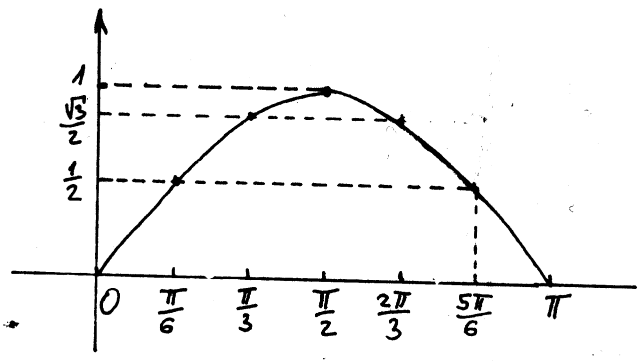
Dio grafika f-je $y=f(x)$ je prikazan na slici.



Datu f-ju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}$.

Rj. Furijerov red za f-ju $y=f(x)$ na intervalu (a,b) glasi

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right) \quad \dots(1)$$

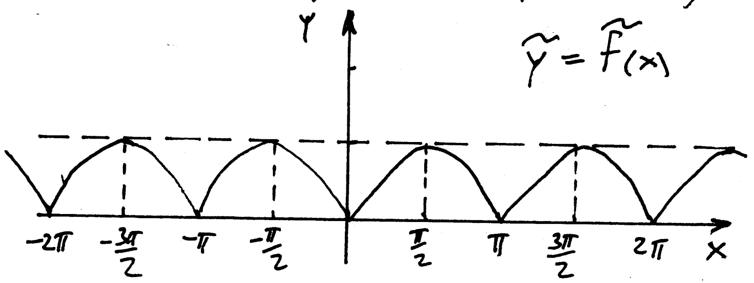
gdje se Furijer-ovi koeficijenti računaju po formuli

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx \quad \dots(2)$$

Prema formuli (1) da bi f-ju pretvorili u Furijer-ov red samo po cos-inusima, potrebno je i dovoljno imati f-ju za koju će vrijediti da je $b_n=0$. Prema formuli (2) da bi b_n bio jednak nuli, interval (a,b) mora biti simetričan u odnosu na nulu i f-ja $f(x)$ mora biti parna (zato što

$$b_n=\frac{2}{b-a}\int_a^b \underbrace{\underbrace{f(x)}_{parna}\;\underbrace{\sin\frac{2n\pi x}{b-a}}_{neparna}}_{neparna}\,dx$$).

Prema tome pravimo proširenje date f-je:



Naravno f-ja koju pretvaramo u Furijer-ov red mora biti periodična.

Prvo primjetimo da je data f-ja $y=f(x)$ u stvari f-ja $y=\sin x$ na intervalu $[0,\pi)$. Proširenje date f-je je u stvari f-ja $\tilde{y}=|\sin x|$.

$(a,b) = (-\pi, \pi)$, $\frac{2}{b-a} = \frac{2}{\pi-(-\pi)} = \frac{2}{2\pi} = \frac{1}{\pi}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{2\pi} = nx$

F-ja $\tilde{y} = |\sin x|$ je parna $\Rightarrow$ $b_n = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\tilde{f}(x)\cos nx}_{\text{parna f-ja}}\,dx = \frac{2}{\pi}\int_0^{\pi} \tilde{f}(x)\cos nx\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin(n+1)x + \sin(n-1)x\right)dx =$$

$$\boxed{\begin{aligned} &\sin(A+B) = \sin A\cos B + \sin B\cos A \\ +\,&\sin(A-B) = \sin A\cos B - \sin B\cos A \\ \hline &\sin A\cos B = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)\end{aligned}}$$

$$= \frac{1}{\pi}\left(\frac{-1}{1+n}\cos(1+n)x\Big|_0^{\pi} + \frac{-1}{1-n}\cos(1-n)x\Big|_0^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(\frac{-1}{1+n}\Big(\underbrace{\cos(1+n)\pi}_{(-1)^{n+1}} - 1\Big) + \frac{-1}{1-n}\Big(\underbrace{\cos(1-n)\pi}_{(-1)^{n-1}} - 1\Big)\right) =$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{1-n}\right) = \frac{1}{\pi}\,\frac{\left((-1)^n+1\right)(1+n+1-n)}{(1+n)(1-n)} = \frac{1}{\pi}\,\frac{2}{(1+n)(1-n)}\left((-1)^n+1\right)$$

za $n = 1, 3, 3, \ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}\left(-\cos x\Big|_0^{\pi}\right) = \frac{-2}{\pi}(-1-1) = \frac{4}{\pi}$$

$$\tilde{f}(x) \sim \frac{4}{\pi} + \sum_{n=1}^{\infty}\frac{2}{\pi}\,\frac{(-1)^n+1}{1-n^2}\cos nx = \frac{4}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$$

Prema tome $\sin x \sim \frac{4}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{1-4k^2}$, $x \in (0,\pi)$

Za $x = \frac{\pi}{2}$ imamo:

$$\sin\frac{\pi}{2} = \frac{4}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\cdot\frac{\pi}{2}}{1-kk^2} \Rightarrow 1 - \frac{4}{\pi} = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\pi}{1-4k^2} \quad /\cdot\pi$$

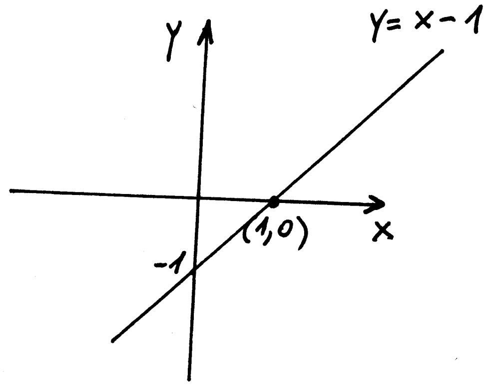
$$4\sum_{k=1}^{\infty}\frac{(-1)^k}{1-4k^2} = \pi - 4$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2} = \frac{\pi}{4} - 1 \quad \text{tražena suma}$$

Ispitati neprekidnost f-je $f(x,y) = \begin{cases} \dfrac{(x-1)^2 \ln x}{(x-1)^2+y^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$.

R: Jedina sumnjiva tačka u kojoj f-ja može imati prekid je tačka $(1,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0)$$



tj. akko $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$.

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=0$.

$$\lim_{(x,0)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+0^2} = \lim_{(x,0)\to(1,0)} \ln x = 0$$

Posmatrajmo približavanje tački $(1,0)$ preko prave $y = x-1$

$$\lim_{(x,x-1)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+(x-1)^2} = \lim_{(x,x-1)\to(1,0)} \frac{\ln x}{2} = 0$$

Odavde možemo naslutiti da je možda vrijednost ovog limesa u tački $(1,0)$ jednaka 0.

Prisjetimo se teoreme "dva policajca":

$$\forall (x,y) \in \mathbb{R}^2 \quad g(x,y) \le f(x,y) \le h(x,y) \text{ i } \lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = M$$

$$\Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = M \Leftrightarrow |f(x,y) - M| \to 0,\ (x,y)\to(a,b)$$

$(x+1)^2 \ge 0 \quad \forall x \in \mathbb{R}$

$(x-1)^2 + y^2 \ge 0 \quad \forall x,y \in \mathbb{R}$

$$(x-1)^2+y^2 \ge (x+1)^2 \Rightarrow 0 \le \frac{(x-1)^2}{(x-1)^2+y^2} \le 1 \Rightarrow$$

$$\Rightarrow 0 \le \frac{(x-1)|\ln x|}{(x-1)^2+y^2} \le |\ln x| \to 0,\ (x,y)\to(1,0) \quad \xRightarrow{\text{Teor. dva polic.}} \quad \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$$

F-ja je neprekidna.

(#) Provjeriti da li f-ja $z = \operatorname{arctg} \frac{x}{y}$, u kojoj je $x = u+v$, $y = u-v$, zadovoljava jednakost

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u-v}{v^2+u^2}.$$

Rj. Primjetimo da je f-ja z složena f-ja dvije promjenjive

$z = z(u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \frac{x}{y} = x y^{-1}$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} (-1) x y^{-2} = \frac{-y^2}{x^2+y^2} \cdot \frac{x}{y^2} = \frac{-x}{x^2+y^2}$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 1$$

$$\frac{\partial z}{\partial u} = \frac{y}{x^2+y^2} \cdot 1 + \frac{(-x)}{x^2+y^2} \cdot 1 = \frac{y-x}{x^2+y^2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{y}{x^2+y^2} \cdot 1 + \frac{-x}{x^2+y^2} \cdot (-1) = \frac{y+x}{x^2+y^2}$$

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{y-x+y+x}{x^2+y^2} = \frac{2y}{x^2+y^2} = \frac{2(u-v)}{u^2\underline{+2uv}+v^2+u^2\underline{-2uv}+v^2} =$$

$$= \frac{2(u-v)}{2(u^2+v^2)} = \frac{u-v}{u^2+v^2}$$

vrijedi data jednakost

Dati dvostruki integral $\int\limits_{\frac{R}{2}}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\, dx$

iz pravouganih koordinata transformisati na polarne koordinate.

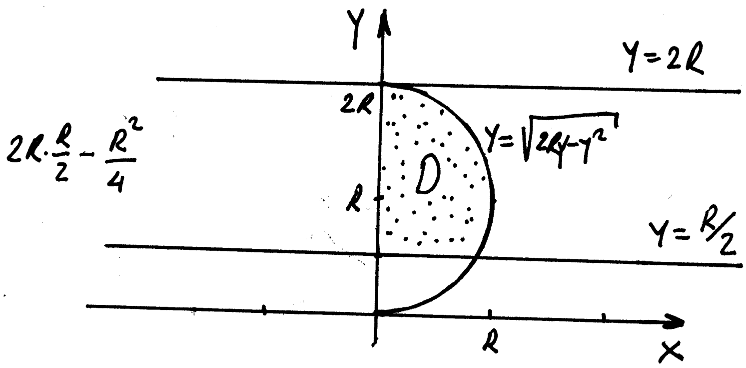
Rj: Skicirajmo oblast integracije.

Iz postavke vidimo da je x ograničen sa pravom $x=0$ i krivom $x=\sqrt{2Ry-y^2}$

$$D: \begin{cases} 0 \le x \le \sqrt{2Ry-y^2} \\ 2R \le y \le R/2 \end{cases}$$

$$x^2 = 2Ry - y^2$$
$$x^2 + y^2 - 2\cdot y\cdot R + R^2 - R^2 = 0$$
$$x^2 + (y-R)^2 = R^2$$

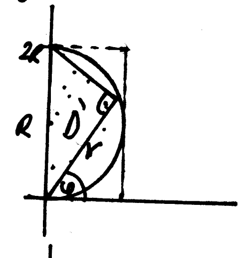
krug sa centrom u tački $(0,R)$ poluprečnika R.



Polarne koordinate glase

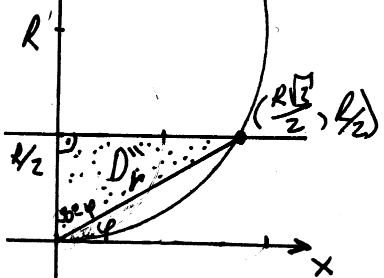
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

Da bi došli do ideje kako opisati oblast D posmatrajmo sljedeće "jednostavnije" oblasti D' i D'':



$$\cos(90^\circ - \varphi) = \frac{r}{2R} \Rightarrow r = 2R\sin\varphi$$
$$\cos(90^\circ - \varphi) = \sin\varphi$$

$$D': \begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$



$$\cos(90^\circ - \varphi) = \frac{R/2}{r}$$
$$\sin\varphi = \frac{R}{2r}$$
$$2r = \frac{R}{\sin\varphi} \Rightarrow r = \frac{R}{2\sin\varphi}$$

$$D'': \begin{cases} 0 \le r \le \frac{R}{2\sin\varphi} \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases}$$

Sad nije teško vidjeti da će oblast D opisana pomoću polarnih koordinata postati

$$D : \begin{cases} \dfrac{R}{2\sin\varphi} \leq r \leq 2R\sin\varphi \\ \dfrac{\pi}{6} \leq \varphi \leq \dfrac{\pi}{2} \end{cases}$$

Prema tome

$$\int_{R/2}^{2R} dy \int_{0}^{\sqrt{2Ry-y^2}} f(x,y)\, dx = \int_{\pi/6}^{\pi/2} d\varphi \int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(r\cos\varphi, r\sin\varphi)\, r\, dr$$