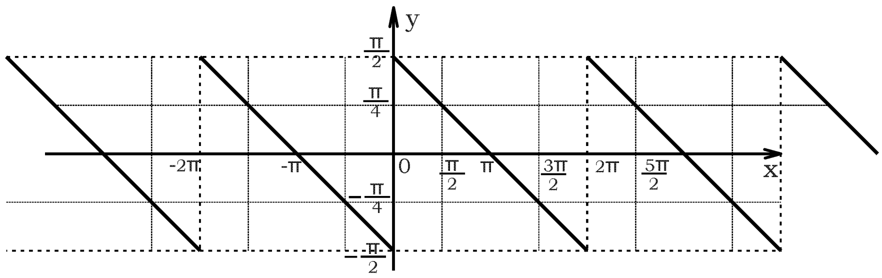
Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 16.07.2012.

Pismeni ispit iz predmeta **Analiza 3**



1. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(4n-1)(4n-3)}$.

2. Izračunati dvostruki integral $\int\limits_{0}^{\sqrt{\frac{\pi}{2}}} dx \int\limits_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}} cos(x^2+y^2)\, dy.$

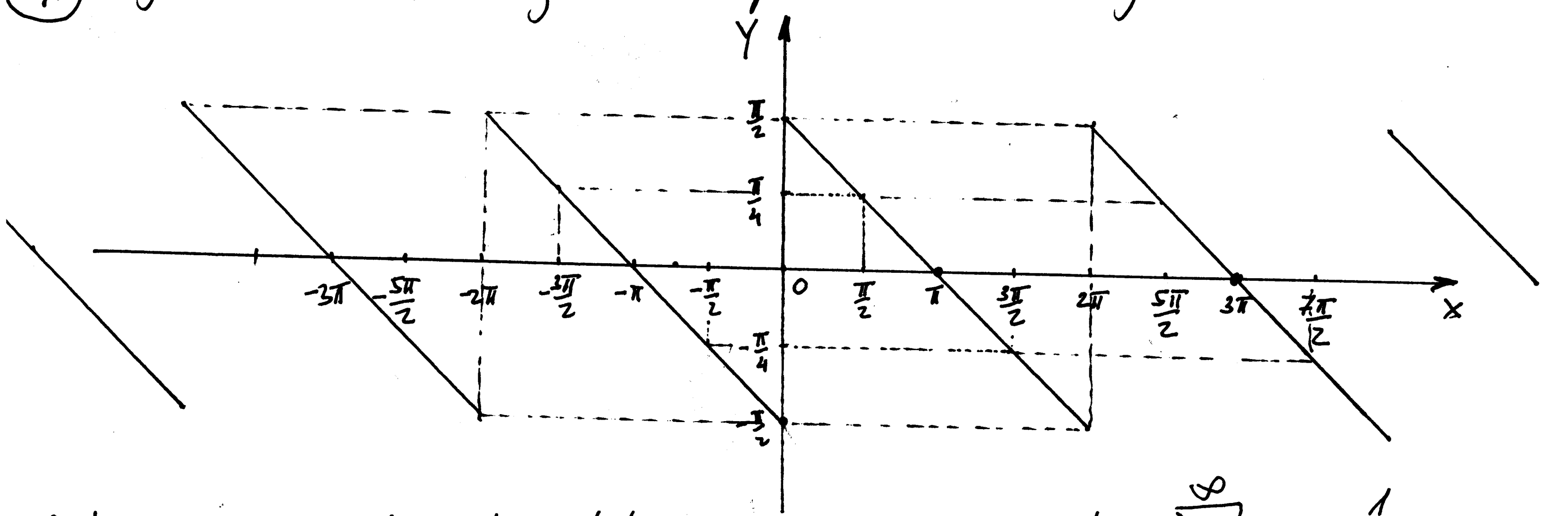
3. Izračunati krivoliniski integral prve vrste $\oint\limits_{c} (x+y)\, dS$ ako je $c : \begin{cases} x = a\, cos\varphi \sqrt{cos2\varphi} \\ y = a\, sin\varphi \sqrt{cos2\varphi} \\ -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4} \end{cases}$

(kriva c je desna latica lemniskate $\rho = a\sqrt{cos2\varphi}$).

4. Odrediti brojeve a i b tako da vektorsko polje $\vec{v} = (yz + axy, xz + bx^2 + yz^2, axy + y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1;1;1)$ prema tački $B(2;2;2)$

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) F-ju definisanu grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n-3)}$.

Rj. Prvo primjetimo da je f-ja periodična što znači da se može pretvoriti u Furijeov red. Dalje, primjetimo da je period 2π što znači da možemo posmatrati npr. interval $[0,2\pi]$.

F-ja na intervalu $[0,2\pi]$ prolazi kroz sljedeće tačke $(0,\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{\pi}{4})$ $(\pi,0)$, $(\frac{3\pi}{2},-\frac{\pi}{4})$, $(2\pi,-\frac{\pi}{2})$. Jednačina prave kroz dvije tačke je

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \quad \xRightarrow{\text{ako posmatramo}} \quad (\overset{x_1}{0},\overset{y_1}{\tfrac{\pi}{2}}) \text{ i } (\overset{x_2}{\pi},\overset{y_2}{0}) \qquad \frac{x-0}{\pi}=\frac{y-\frac{\pi}{2}}{-\frac{\pi}{2}}$$

$$\Rightarrow \quad y-\frac{\pi}{2}=\frac{x}{\pi}\cdot\left(-\frac{\pi}{2}\right) \Rightarrow y-\frac{\pi}{2}=-\frac{x}{2} \Rightarrow y=\frac{\pi-x}{2}.$$

Furijeov red na intervalu $[a,b]$ je oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

$$\frac{2n\pi x}{2\pi}=nx$$

gdje se Furijeovi koeficijenti računaju po formuli

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx, \qquad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\,dx=\frac{1}{2\pi}\left(\pi x\Big|_0^{2\pi}-\frac{1}{2}x^2\Big|_0^{2\pi}\right)=\frac{1}{2\pi}(2\pi^2-2\pi^2)=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\cos nx\,dx=\left|\begin{matrix}u=\pi-x & dv=\cos nx\,dx\\ du=-dx & v=\frac{1}{n}\sin nx\end{matrix}\right|=$$

$$= \frac{1}{2\pi}\left(\frac{1}{n}(\pi - x)\sin nx\Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right) = \underbrace{-\frac{1}{2n\pi}\sin 2n\pi}_{=0} + \frac{1}{2n\pi}\cdot\frac{-1}{n}\cos nx\Big|_0^{2\pi} =$$

$$= -\frac{1}{2n^2\pi}\left(\cos 2n\pi - 1\right) = \frac{1}{2n^2\pi}\left(1 - \cos 2n\pi\right) = \frac{1}{2n^2\pi}(1-1) = 0$$

Sa grafika date f-je možemo primijetiti da je f-ja simetrična u odnosu na koordinatni početak tj. da je neparna, pa je $a_0 = 0$ i $a_n = 0$ $\forall n \in \mathbb{N}$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx = \left|\begin{matrix} u = \pi - x & dv = \sin nx\,dx \\ du = -dx & v = -\frac{1}{n}\cos nx \end{matrix}\right|$$

$$= \frac{1}{2\pi}\left(-\frac{1}{n}(\pi - x)\cos nx\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right) = -\frac{1}{2n\pi}\left(-\pi\cos 2n\pi - \pi\right) -$$

$$- \frac{1}{2n\pi}\sin nx\Big|_0^{2\pi} = \frac{1}{2n}\left(\cos 2n\pi + 1\right) = \frac{1}{2n}\cdot 2 = \frac{1}{n}$$

Prema tome

$$f(x) \sim \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi}{2}n = \sum_{k=1}^{\infty}\Big(\frac{1}{4k-1}\sin(4k-1)\frac{\pi}{2} + \frac{1}{4k-2}\underbrace{\sin(4k-2)\frac{\pi}{2}}_{\sin(2k-1)\pi = 0} +$$

$$+ \frac{1}{4k-3}\sin(4k-3)\frac{\pi}{2} + \frac{1}{4k}\underbrace{\sin 4k\frac{\pi}{2}}_{=\sin 2k\pi = 0}\Big) =$$

$$= \sum_{k=1}^{\infty}\frac{1}{4k-1}\sin\underbrace{(4k-1)\frac{\pi}{2}}_{=3,7,11,15} + \sum_{k=1}^{\infty}\frac{1}{4k-3}\sin\underbrace{(4k-3)\frac{\pi}{2}}_{=1,5,9,13}$$

$$= -1 \qquad\qquad = 1$$

$$= \sum_{k=1}^{\infty}\left(-\frac{1}{4k-1} + \frac{1}{4k-3}\right) = \sum_{k=1}^{\infty}\frac{4k-1-4k+3}{(4k-1)(4k-3)} = \sum_{k=1}^{\infty}\frac{2}{(4k-1)(4k-3)}$$

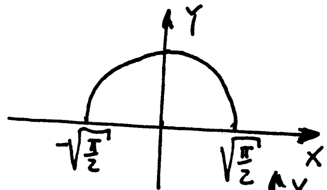
Kako je $f(\frac{\pi}{2}) = \frac{\pi}{4}$ bo je $\sum_{n=1}^{\infty}\frac{2}{(4n-1)(4n-3)} = \frac{\pi}{4}$.

\# Izračunati dvastruki integral

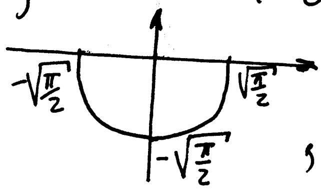
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy.$$

Rj. Oblast integracije D je $D:\begin{cases}0\le x\le\sqrt{\frac{\pi}{2}}\\ -\sqrt{\frac{\pi}{2}-x^2}\le y\le\sqrt{\frac{\pi}{2}-x^2}\end{cases}$

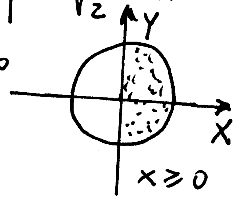
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda



, sve zajedno $\Rightarrow$



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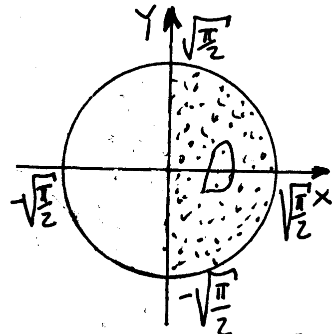
Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

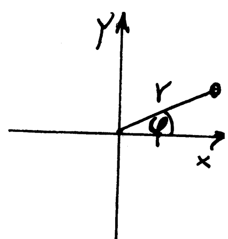
$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$x^2+y^2=\ldots=r^2$

$D\xrightarrow{\text{transform.}}D':\begin{cases}0\le r\le\sqrt{\frac{\pi}{2}}\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$



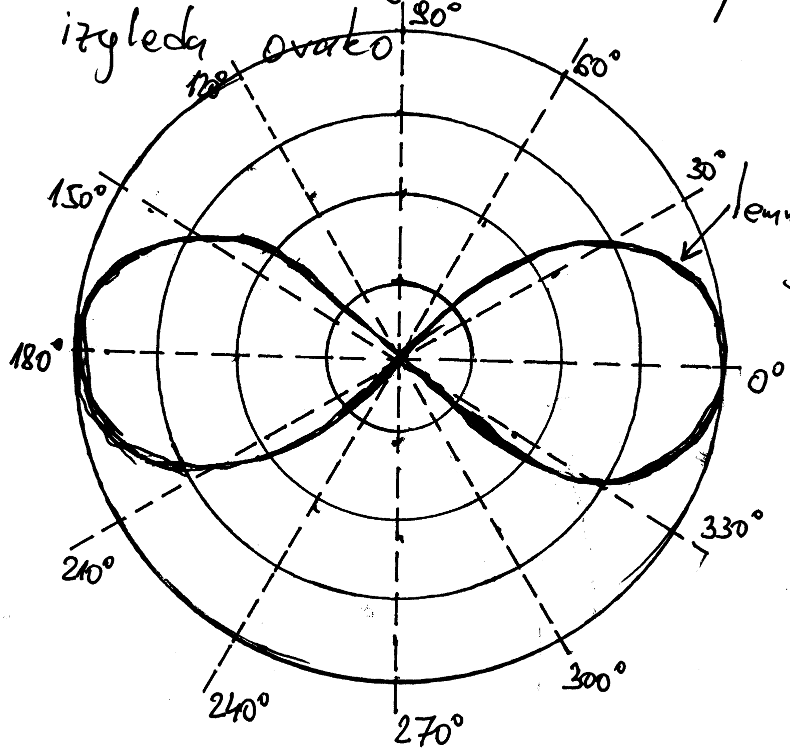
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy=\iint_D\cos(x^2+y^2)\,dx\,dy=\iint_{D'}\cos(r^2)\,r\,dr\,d\varphi=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{\sqrt{\frac{\pi}{2}}}r\cos(r^2)\,dr=\left|\begin{array}{l}r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\\ r\Big|_0^{\sqrt{\frac{\pi}{2}}}\Rightarrow t\Big|_0^{\frac{\pi}{2}}\end{array}\right|=\frac{1}{2}\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi/2}\cos t\,dt=\frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\cdot\pi\cdot1=\frac{\pi}{2}$$ traženo rješenje

(#) Izračunati krivoliniski integral prve vrste $\int_c (x+y)\,dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

Rj: Lemniskata $\rho = a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



lemniskata $\rho = a\sqrt{\cos 2\varphi}$

Data kriva je prikazana u polarnim koordinatama

$$c : \begin{cases} \rho = a\sqrt{\cos 2\varphi} \\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \end{cases}$$

Prisjetimo se,

$$\int_c (x+y)\,dS = \int_{t_1}^{t_2} (\eta(t) + \mu(t)) \sqrt{(\eta'(t))^2 + (\mu'(t))^2}\,dt$$

ako je c data u obliku

$$c : \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}.$$

$$\int_{c'} (x+y)\,dS = \left| \begin{array}{l} \text{kao pomoć uvedimo polarne koordinate} \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ \text{za } \rho \text{ dewo uzeti} \\ \rho = a\sqrt{\cos 2\varphi} \end{array} \right.$$

Prema tome

desna latica lemniskate

$$c' : \begin{cases} x = a\cos\varphi\sqrt{\cos 2\varphi} \\ y = a\sin\varphi\sqrt{\cos 2\varphi} \\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} \end{cases}$$

zato što posmatramo desnu stranu lemniskate

$$x' = \left(a(-\sin\varphi)\sqrt{\cos 2\varphi} + a\cos\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi) \cdot 2\right) d\varphi$$
$$= a\left(-\sin\varphi\sqrt{\cos 2\varphi} - \cos\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right) d\varphi \overset{\text{adic. teor.}}{=} \ldots = -a\frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$y' = \left(a\cos\varphi\sqrt{\cos 2\varphi} + a\sin\varphi \cdot \frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}} (-\sin 2\varphi) \cdot 2\right) d\varphi$$
$$= \left(a\cos\varphi\sqrt{\cos 2\varphi} - a\sin\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right) d\varphi \overset{\text{adiciona teor.}}{=} \ldots = a\frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$x'^2 + y'^2 = a^2\frac{\sin^2 3\varphi}{\cos 2\varphi}\,d\varphi^2 + a^2\frac{\cos^2 3\varphi}{\cos 2\varphi}\,d\varphi^2 = a^2\frac{1}{\cos 2\varphi}\,d\varphi^2 \quad \Bigg| =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cancel{\sqrt{\cos 2\varphi}}\, a(\cos\varphi + \sin\varphi) \cdot a\frac{1}{\cancel{\sqrt{\cos 2\varphi}}}\,d\varphi = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\varphi + \sin\varphi)\,d\varphi =$$

$$= a^2\Big(\underbrace{\sin\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}^{\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)} - \underbrace{\cos\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}^{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}\Big) = a^2\sqrt{2} \quad \text{traženo rješenje.}$$

Odrediti brojeve a i b tako da vektorsko polje $\vec{v} = (yz + axy,\ xz + bx^2 + yz^2,\ axy + y^2 z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.

Rj: Za vektorsko polje $\underbrace{\vec{v}}_{(\vec{v}=(v_x, v_y, v_z))}$ kažemo da je potencijalno ako je $\operatorname{rot}\vec{v} = \vec{0}$. Znamo da

$$\operatorname{rot}\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + axy & xz + bx^2 + yz^2 & axy + y^2 z \end{vmatrix} =$$

$$= \left(ax + \cancel{2yz} - x - \cancel{2yz},\ -(ay - y),\ \cancel{z} + 2bx - \cancel{z} - ax \right)$$

$$= (ax - x,\ y - ay,\ 2bx - ax).$$

$$\operatorname{rot}\vec{v} = \vec{0} \implies \begin{aligned} ax - x &= 0 \\ y - ay &= 0 \\ 2bx - ax &= 0 \end{aligned} \qquad \begin{aligned} a &= 1 \\ b &= \tfrac{1}{2} \end{aligned}$$

Za $a = 1$ i $b = \frac{1}{2}$ vektorsko polje $\vec{v}$ je potencijalno polje.

Cirkulaciju vektorskog polja $\vec{v}$ duž krive c tražimo po formuli

$$C = \int_c \vec{v}\, d\vec{r} = \int_c v_x\, dx + v_y\, dy + v_z\, dz$$

Kriva c je dio prave od tačke $A(1,1,1)$ (x_1, y_1, z_1) do tačke $B(2,2,2)$ (x_2, y_2, z_2).

Kako glasi jednačina prave u prostoru kroz dvije tačke?

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \implies \frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1} \ (= t) \qquad \begin{aligned} x - 1 &= t \\ y - 1 &= t \\ z - 1 &= t \end{aligned}$$

Krivulja c u parametarskom obliku

$$c: \begin{cases} x = t+1 & dx = dt \\ y = t+1 & dy = dt \\ z = t+1 & dz = dt \\ 0 \le t \le 1 \end{cases}$$

U našem slučaju

$$C = \int_c (yz + xy)\,dx + (xz + \tfrac{1}{2}x^2 + yz^2)\,dy + (xy + y^2 z)\,dz =$$

$$= \int_0^1 \left[(t+1)^2 + (t+1)^2 + (t+1)^2 + \tfrac{1}{2}(t+1)^2 + (t+1)^3 + (t+1)^2 + (t+1)^3\right] dt =$$

$$= \left| d(t+1) = dt \right| = \int_0^1 \left[\tfrac{9}{2}(t+1)^2 + 2(t+1)^3\right] d(t+1) =$$

$$= \frac{9}{2}\,\frac{(t+1)^3}{3}\Big|_0^1 + 2\cdot\frac{(t+1)^4}{4}\Big|_0^1 = \frac{9}{6}\,(8-1) + \frac{1}{2}\,(16-1)$$

$$= \frac{63}{6} + \frac{15\cdot 3}{2\cdot 3} = \frac{108}{6} = 18$$ traženo rješenje