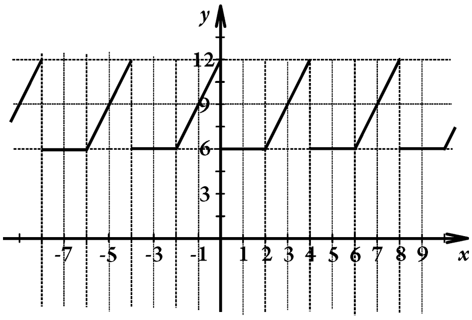
Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 29.06.2012.

Pismeni ispit iz predmeta **Analiza 3**



1. Funkciju definisanu grafikom pretvoriti u Furijerov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

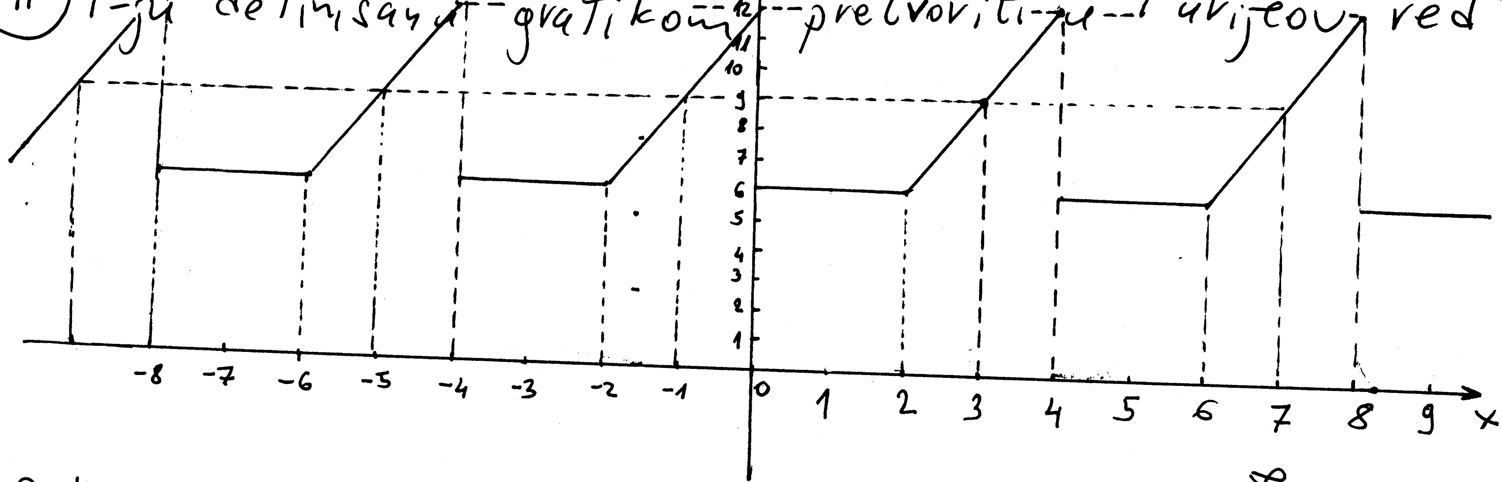
2. Izračunati zapreminu dijela kugle $x^2 + y^2 + z^2 = R^2$ koji se nalazi između dvije paralelne ravni $z = 0$ i $z = a$ $(0 < a < R)$.

3. Izračunati integral $I = \oint_c y^2\,dx$ po krivoj koja nastaje kao presjek kugle $x^2 + y^2 + z^2 = R^2$ i valjka $x^2 + y^2 = Rx$. (Mala pomoć: Da bi ste izračunali ovaj integral treba parametrizirati krivu c. Jedan od načina kako to možete postići je da krenete od parametrizacije kruga...)

4. Izračunati površinu onog dijela kupe $z^2 = x^2 + y^2$ koji se nalazi unutar valjka $x^2 + y^2 = 2x$.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) F-ja definisana grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

Rj. Prvo primjetimo da je data f-ja periodična perioda 4, što znači da je možemo pretvoriti u Furijeov red i to dovoljno ju je pretvoriti u Furijeov red na intervalu $(0,4)$.

Data f-ja na intervalu $(0,4)$ je definisana na sljedeći način

$$f(x) = \begin{cases} 6, & x \in (0,2] \\ 3x, & x \in (2,4) \end{cases}.$$

Furijerov red na proizvoljnom intervalu $[a,b]$ izgleda

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a}\right)$$

a Furijerovi koeficijenti računaju po formulama

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx, \qquad a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}\,dx, \quad n = 1,2,\ldots$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}\,dx, \quad n = 1,2,\ldots$$

Što znači Furijerov red na intervalu $[0,4)$ je

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2}\right)$$

Izračunajmo Furijeove koeficijente

$$a_0 = \frac{1}{2} \int_0^4 f(x)\,dx = \frac{1}{2} \int_0^2 6\,dx + \frac{1}{2} \int_2^4 3x\,dx = 3x\Big|_0^2 + \frac{3}{2} \cdot \frac{1}{2} \overbrace{x^2\Big|_2^4}^{16-4} = 6 + \frac{3}{\cancel{4}_1}\cancel{12}^{3} = 15$$

$$a_n = \frac{1}{2} \int_0^4 f(x) \cos \frac{n\pi x}{2}\,dx = \frac{1}{2} \int_0^2 6 \cos \frac{n\pi x}{2}\,dx + \frac{1}{2} \int_2^4 3x \cos \frac{n\pi x}{2}\,dx = \left| \begin{matrix} u = x & dv = \cos \frac{n\pi x}{2}\,dx \\ du = dx & v = \frac{2}{n\pi} \sin \frac{n\pi x}{2} \end{matrix} \right.$$

$$=\underbrace{3\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2}_{0-0}+\frac{3}{2}\left[\underbrace{\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}-\frac{2}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx\right]=$$

$$=-\frac{3}{n\pi}\int_2^4\sin\frac{n\pi x}{2}dx=\frac{3}{n\pi}\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_2^4=\frac{6}{n^2\pi^2}(1-\cos n\pi),\quad n\neq 0$$

Odavde vidimo $a_n=\begin{cases}0, & n \text{ parno}\\ \frac{12}{n^2\pi^2}, & n \text{ neparno}\end{cases}$ $n\in\mathbb{N}$

$$b_n=\frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}dx=\frac{1}{2}\int_0^2 6\sin\frac{n\pi x}{2}dx+\frac{1}{2}\int_2^4 3x\sin\frac{n\pi x}{2}dx=\left|\begin{matrix}u=x & dv=\sin\frac{n\pi x}{2}dx\\ du=dx & v=-\frac{2}{n\pi}\cdot\cos\frac{n\pi x}{2}\end{matrix}\right|$$

$$=3\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2+\frac{3}{2}\left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_2^4+\frac{2}{n\pi}\int_2^4\cos\frac{n\pi x}{2}dx\right]=$$

$$=\left(-\frac{6}{n\pi}\right)(\cos n\pi-1)+\frac{3}{2}\left[\left(-\frac{2}{n\pi}\right)(4\cos 2n\pi-2\cos n\pi)+\underbrace{\frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}\right]$$

$$=\frac{6}{n\pi}(1-\cos n\pi)-\frac{3}{n\pi}(4-2\cos n\pi)=\frac{6}{n\pi}(1-\cos n\pi)-\frac{6}{n\pi}(2-\cos n\pi)$$

$$=\frac{6}{n\pi}(1\underline{-\cos n\pi}-2\underline{+\cos n\pi})=-\frac{6}{n\pi}$$

Prema tome $f(x)\sim\frac{15}{2}+\sum_{n=1}^{\infty}\left(\frac{6}{n^2\pi^2}(1-\cos n\pi)\cos\frac{n\pi x}{2}+\left(-\frac{6}{n\pi}\right)\sin\frac{n\pi x}{2}\right)$

$$=\frac{15}{2}+\sum_{k=1}^{\infty}\left(\frac{12}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}-\frac{6}{k\pi}\sin\frac{k\pi x}{2}\right)$$

$$f(x)\sim\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\frac{\pi}{2}x}{(2k-1)^2}-\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\frac{\pi}{2}x}{k}$$

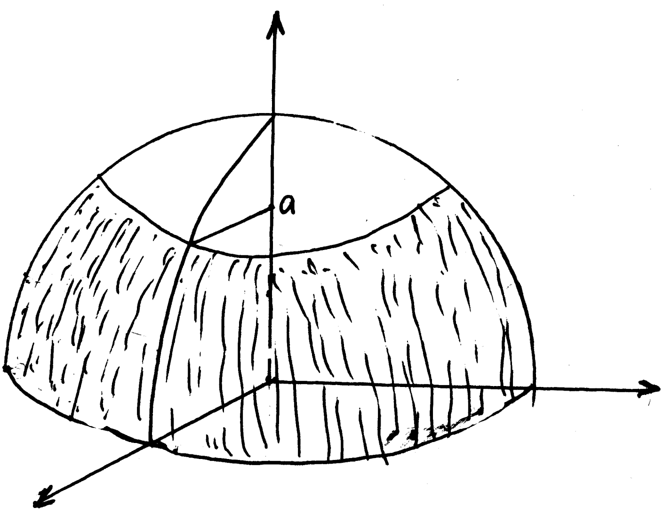
Za $x=2$ imamo

$$f(2)=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2}-\underbrace{\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi}{k}}_{=0}$$

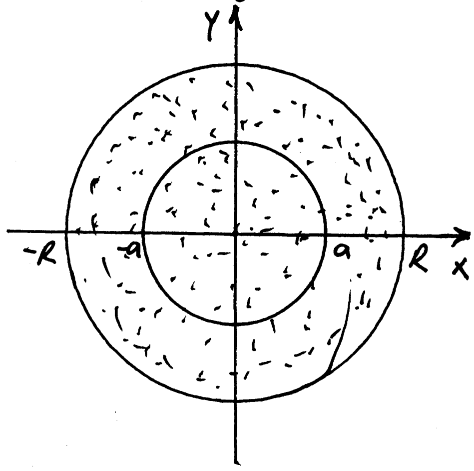
$$6=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)}{(2k-1)^2}\implies-\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=-\frac{3}{2}$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(-\frac{3}{2}\right)\left(-\frac{\pi^2}{12}\right)=\frac{\pi^2}{2\cdot 4}\qquad\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}\ \text{tražena suma}$$

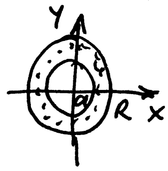
Izračunati zapreminu dijela kugle $x^2+y^2+z^2=R^2$ koji se nalazi između dvije paralelne ravni $z=0$ i $z=a$ $(0<a<R)$.



Ortogonalna projekcija figure na xOy ravan izgleda



Sa slike vidimo, da bi odredili zapreminu V pronaći ćemo dvije zapremine V_1 i V_2 i to za figure Ω_1 i Ω_2 čije će ortogonalne projekcije na xOy ravan biti:



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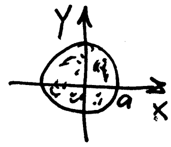
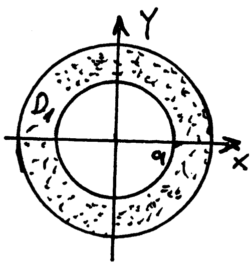


. Na kraju

$$V = V_1 + V_2$$



$$V_1 = \iint\limits_{D_1} f_1(x,y)\,dx\,dy =$$

u ovom slučaju
$z^2 = R^2 - x^2 - y^2$
$z = \pm\sqrt{R^2-x^2-y^2}$
nama treba $+\sqrt{}$

polarne koordinate
$x = r\cos\varphi$
$y = r\sin\varphi$
$dx\,dy = r\,dr\,d\varphi$

$$D_1 \xrightarrow{\text{transf.}} D_1' : \begin{cases} a \le r \le R \\ 0 \le \varphi < 2\pi \end{cases}$$

$$= \int_0^{2\pi} d\varphi \int_a^R \sqrt{R^2-r^2}\,dr \overset{(*)}{=}$$

$$\int \sqrt{R^2-x^2}\,dx = \left| \begin{matrix} u = \sqrt{R^2-x^2} & dv = dx \\ du = \dfrac{-x}{\sqrt{R^2-x^2}} & v = x \end{matrix} \right| = x\sqrt{R^2-x^2} + \int \frac{\overbrace{x^2 - R^2 + R^2}^{-(R^2-x^2)+R^2}}{\sqrt{R^2-x^2}}\,dx =$$

$$= x\sqrt{R^2-x^2} - \int \frac{R^2-x^2}{\sqrt{R^2-x^2}} + R^2 \int \frac{dx}{\sqrt{R^2-x^2}} = x\sqrt{R^2-x^2} - \int \sqrt{R^2-x^2} + R^2 \arcsin\frac{R}{a} + C_1$$

$$\Rightarrow \int \sqrt{R^2-x^2}\,dx = \frac{1}{2}x\sqrt{R^2-x^2} + \frac{1}{2}R^2 \arcsin\frac{x}{R} + C$$

$$\overset{(*)}{=} \int_0^{2\pi} \left(\frac{1}{2}x\sqrt{R^2-x^2}\Big|_a^R + \frac{1}{2}R^2 \arcsin\frac{x}{R}\Big|_a^R \right) d\varphi =$$

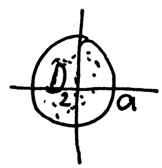
$$= \left(-\frac{1}{2}a\sqrt{R^2-a^2} + \frac{1}{2}R^2\left(\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \arcsin\frac{a}{R} \right) \right) 2\pi$$

$$= -a\sqrt{R^2-a^2}\,\pi + \frac{\pi^2 R^2}{2} - R^2 \arcsin\frac{a}{R}\,\pi$$

$$V_2 = \iint_{D_2} f_2(x,y)\,dx\,dy =$$

u ovom slučaju

$f_2(x,y) = z = a$



polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$$D_2 \xrightarrow{\text{transf.}} D_2' : \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$= \int_0^{2\pi} d\varphi \int_0^a r\,dr\,d\varphi = \frac{1}{2}r^2\Big|_0^a \cdot \varphi\Big|_0^{2\pi} = a^2\pi$$

$$V = V_1 + V_2 = a^2\pi - a\sqrt{R^2-a^2}\,\pi + \frac{\pi^2 R^2}{2} - \pi R^2 \arcsin\frac{a}{R}$$

tražena zapremina

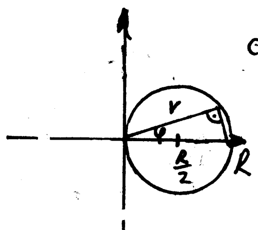
Izračunati integral $I = \oint_C y^2 dx$

po krivoj koja nastaje kao presjek kugle i valjka
$$x^2+y^2+z^2=R^2, \quad x^2+y^2=Rx.$$

Rj.:

$$C: \begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases}$$

Posmatrajmo xOy ravan.
Prvo napišimo krug $x^2+y^2=Rx$ u parametarskom obliku.



$\cos\varphi = \dfrac{r}{R}$

$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$x^2+y^2=Rx$$
$$x^2-2\cdot x\cdot\frac{R}{2}+\frac{R^2}{4}-\frac{R^2}{4}+y^2=0$$
$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$

Prisjetimo se polarnih koordinata
$x = r\cos\varphi$
$y = r\sin\varphi$

U našem slučaju za krug $x^2+y^2=Rx$ za r ćemo uzeti $r = R\cos\varphi$
Parametarski oblik kruga $x^2+y^2=Rx$ je $x = R\cos\varphi\cos\varphi = R\cos^2\varphi$
$y = R\cos\varphi\sin\varphi.$

Uvrstimo ove vrijednosti u kuglu
$$x^2+y^2+z^2=R^2$$
$$\underline{R^2\cos^2\varphi}\cos^2\varphi+\underline{R^2\cos^2\varphi}\sin^2\varphi+z^2=R^2$$
$$R^2\cos^2\varphi+z^2=R^2$$
$$z^2=R^2-R^2\cos^2\varphi$$
$$z^2=R^2(1-\cos^2\varphi)$$
$$z^2=R^2\sin^2\varphi$$

Parametarski oblik date krive je:
$x = R\cos^2\varphi$
$y = R\cos\varphi\sin\varphi$
$z = R\sin\varphi$
$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$I=\oint_C y^2\,dx = \left| \begin{array}{l} x=R\cos^2\varphi \\ dx = R\,2\cos\varphi(-\sin\varphi)\,d\varphi \\ y = R\cos\varphi\sin\varphi \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2} \end{array} \right| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2\cos^2\varphi\sin^2\varphi\cdot(-2)R\sin\varphi\cos\varphi\,d\varphi$$

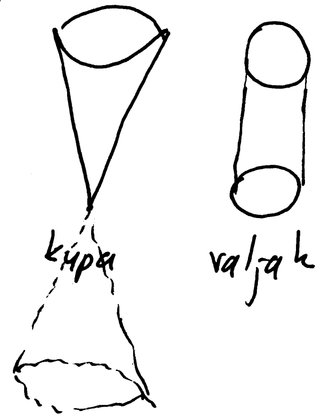
$$=(-2)R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\varphi\cos^3\varphi\,d\varphi = (-2)R^3\cdot\frac{1}{8}\int_{-\pi/2}^{\pi/2}(2\sin\varphi\cos\varphi)^3\,d\varphi = -\frac{1}{4}R^3\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin2\varphi)^3\,d(2\varphi)$$

$$= -\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\pi/2} \sin^2 2\varphi \cdot \sin 2\varphi \, d(2\varphi) = -\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\pi/2} (1-\cos^2 2\varphi) \sin 2\varphi \, d(2\varphi) =$$

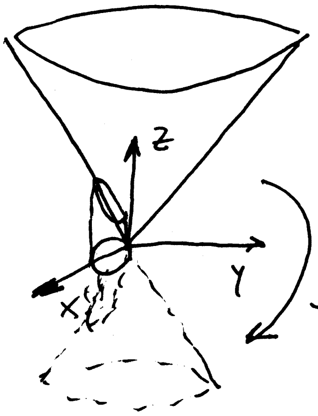
$$= +\frac{1}{8}R^3 \int_{-\frac{\pi}{2}}^{\pi/2} (1-\cos^2 2\varphi) \cdot d(\cos 2\varphi) = \frac{1}{8}R^3 \Big(\underbrace{\cos 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} - \underbrace{\frac{1}{3}\cos^3 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} \Big) = 0$$

Izračunati površinu onog dijela kupe $z^2=x^2+y^2$ koji se nalazi unutar valjka $x^2+y^2=2x$.

Rj.



Prema zadatku dio kupe se nalazi unutar valjka



$$x^2+y^2=2x$$
$$x^2-2\cdot x\cdot 1+1+y^2=1$$
$$(x-1)^2+y^2=1$$

sličnu figuru ćemo imati i sa druge strane xOy ravni

$$P=\iint_S dS$$ gdje je S dio kupe koji se nalazi unutar valjka

$$P=\iint_D \sqrt{1+{z'_x}^2+{z'_y}^2}\,dx\,dy$$

$$z=\pm\sqrt{x^2+y^2}$$

Ako za z uzmemo $z=\sqrt{x^2+y^2}$ dobićemo površinu dijela kupe iznad xOy ravni.

$$z=\sqrt{x^2+y^2}$$

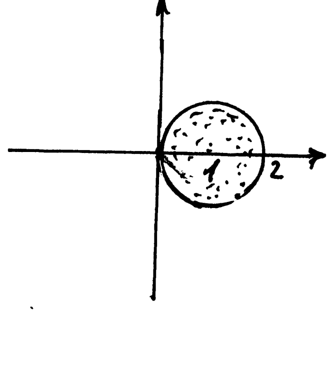
$$z'_x=\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}},\quad z'_y=\frac{y}{\sqrt{x^2+y^2}},\quad 1+{z'_x}^2+{z'_y}^2=1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=1+1=2$$

uvedimo polarne koordinate

$x=r\cos\varphi+1$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$



D: unutrašnjost kruga $x^2+y^2=2x$

$$D \xrightarrow{\text{transf.}} D':\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\end{cases}$$

$$\frac{1}{2}P=\iint_D \sqrt{2}\,dx\,dy=\sqrt{2}\iint_{D'} r\,dr\,d\varphi=\sqrt{2}\int_0^{2\pi}d\varphi\int_0^1 r\,dr=\sqrt{2}\cdot\frac{1}{2}r^2\Big|_0^1\,\varphi\Big|_0^{2\pi}=\sqrt{2}\,\pi$$

$$P=2\sqrt{2}\,\pi$$