Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 15.06.2011.

Pismeni ispit iz predmeta **Uvod u linearnu algebru**

1. **a)** Neka su dati skupovi $A = \{10k + 7 \,|\, k \in \mathbb{N}\}$ i $B = \{4p + 13 \,|\, p \in \mathbb{N}\}$. Dokazati da je $A \cap B \neq \emptyset$.

b) Neka je $A = \{a, b, c, d, e, f\}$. Neka je $\rho \subseteq A \times A$ zadana ovako
$\rho = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, b), (d, d), (d, e), (e, e), (e, d), (f, f)\}$.
Dokazati da je ρ (ro) relacija ekvivalencije u A.

2. **a)** Izračunati determinantu n-tog reda $\begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 1 & 1+a & a & \dots & 0 & 0 \\ 0 & 1 & 1+a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 1+a & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix}$.

b) Data je matrica $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. Provjeriti da li je $A^{-1} = \frac{1}{4}A$.

3. Diskutovati rješenja sistema u u zavisnosti od parametra t

$$\begin{aligned} 2x - y + 3z &= -7 \\ x + 2y - 6z &= t \\ tx + 5y - 15z &= 8\ . \end{aligned}$$

4. Neka je $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \,|\, x_1 - x_2 = x_2 - x_3 = x_3 - x_4 \, i \, x_5 = 0\}$. Sabiranje u V je definisano na uobičajen način
$(x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5)$ kao i množenje sa skalarom $\alpha(x_1, x_2, x_3, x_4, x_5) = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5)$. Dokažite da je V vektorski prostor te mu nađite neku bazu i odredite dimenziju.

(Rješenja su skinuta sa stranice \pf.unze.ba\nabokov
Za sve uočene greške pisati na **infoarrt@gmail.com**)

(#) Neka su dati skupovi $A=\{10k+7 \mid k\in\mathbb{N}\}$ i $B=\{4p+13 \mid p\in\mathbb{N}\}$. Dokazati da je $A\cap B\neq\emptyset$. Odgovor obrazložiti!

Rj. Pokušajmo naći broj a koji je element i skupa A i skupa B. Neka je $a=10k+7=4p+13$ za neko $k\in\mathbb{N}$ i neko $p\in\mathbb{N}$.

$$10k+7=4p+13$$

$$10k-4p=6$$

$$2(5k-2p)=6 \quad /:2$$

$$5k-2p=3$$

Odavde vidimo da za $k=1$ i $p=1$ imamo $5-2=3$ tj.

$17\in A$ i $17\in B$

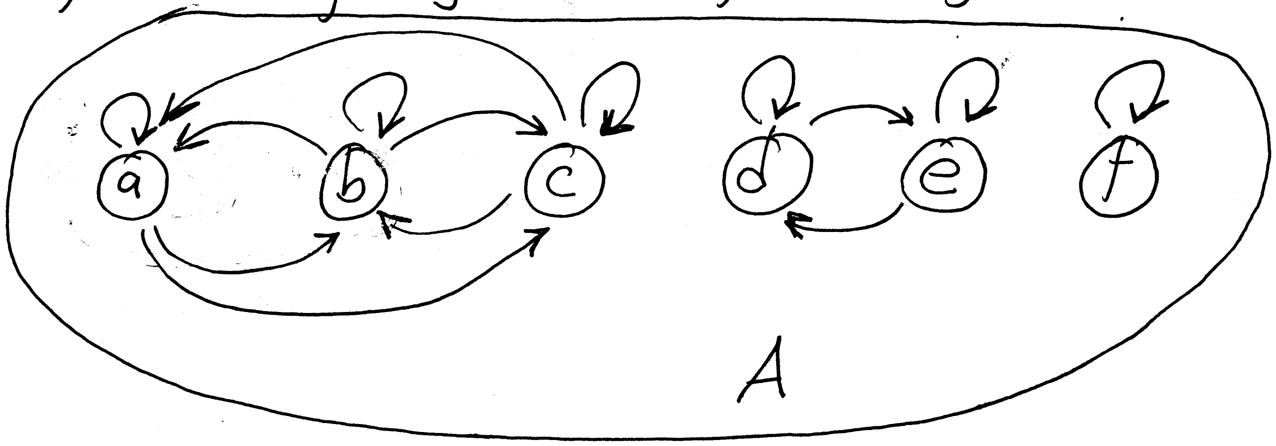
$A\cap B\neq\emptyset$

q.e.d.

Neka je $A=\{a,b,c,d,e,f\}$. Neka je $\rho \subseteq A\times A$ zadana ovako $\rho=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d),(d,e),(e,e),(e,d),(f,f)\}$. Dokazati da je ρ (ro) relacija ekvivalencije u A.

Rj.

Nacrtajmo relaciju ρ kao orjentisan graf



REFLEKSIVNOST

$\forall x\in A \quad (x,x)\in\rho$ svaki element skupa A je u relaciji sam sa sobom
jest refleksivno

SIMETRIČNOST

$\forall x,y\in A \quad (x,y)\in\rho \Rightarrow (y,x)\in\rho$
svaki element x skupa A koji je u vezi sa y imamo da je i y u vezi sa x.
ρ jest simetrično

TRANZITIVNOST

$\forall x,y,z\in A \quad (x,y)\in\rho \wedge (y,z)\in\rho \Rightarrow (x,z)\in\rho$
ako je x u vezi sa y i y u vezi sa z tada je i x u vezi sa z.
ρ jest tranzitivno

ρ jest relacija ekvivalencije q.e.d.

(#) Izračunati determinantu n-tog reda

$$\begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 1 & 1+a & a & \dots & 0 & 0 \\ 0 & 1 & 1+a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 1+a & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix}.$$

Rj. Izračunajmo prvo determinante drugog, trećeg i četvrtog tipa

$$\begin{vmatrix} 1 & a \\ 1 & 1+a \end{vmatrix} = 1+a-a=1$$

$$\begin{vmatrix} 1 & a & 0 \\ 1 & 1+a & a \\ 0 & 1 & 1+a \end{vmatrix} \xlongequal{II_v - I_v} \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 1 & 1+a \end{vmatrix} \xlongequal{III_v - II_v} \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & a & 0 & 0 \\ 1 & 1+a & a & 0 \\ 0 & 1 & 1+a & a \\ 0 & 0 & 1 & 1+a \end{vmatrix} \xlongequal{II_v - I_v} \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 1 & 1+a & a \\ 0 & 0 & 1 & 1+a \end{vmatrix} \xlongequal{III_v - II_v} \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 1 & 1+a \end{vmatrix} \xlongequal{IV_v - III_v}$$

$$= \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 1 & 1+a \end{vmatrix} \xlongequal{IV_v - III_v} \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 1 & 1+a & a & \dots & 0 & 0 \\ 0 & 1 & 1+a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+a & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix} \xlongequal{II_v - I_v} \begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 0 & 1 & a & \dots & 0 & 0 \\ 0 & 1 & 1+a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+a & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix} \xlongequal{III_v - II_v} \begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 0 & 1 & a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+a & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix}$$

$$\overset{\substack{IV_v - III_v \\ V_v - IV_v \\ \vdots \\ n-1\text{-vrsta minus } n-2 \text{ vrsta}}}{=\dots} = \begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 0 & 1 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a \\ 0 & 0 & 0 & \dots & 1 & 1+a \end{vmatrix} \xlongequal[\text{minus } n-1 \text{ vrsta}]{n\text{-ta vrsta}} \begin{vmatrix} 1 & a & 0 & \dots & 0 & 0 \\ 0 & 1 & a & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$= 1$ ← tražena vrijednost determinante

II način: MATEMATIČKOM INDUKCIJOM

\# Data je matrica $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Provjeriti da li je $A^{-1} = \frac{1}{4} A$.

Rj.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} A_{kof}^{T}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \underset{\mathrm{IV}_v - \mathrm{I}_v}{\overset{\mathrm{II}_v - \mathrm{I}_v,\ \mathrm{III}_v - \mathrm{I}_v}{=\!=\!=}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{vmatrix} \overset{\mathrm{II}_k - \mathrm{III}_k}{=\!=\!=}$$

$$= \begin{vmatrix} 0 & 0 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = (-2) \begin{vmatrix} -2 & 2 \\ -2 & -2 \end{vmatrix} = (-2)(4+4) = -16$$

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} \overset{\mathrm{I}_v + \mathrm{II}_v}{=\!=\!=} \begin{vmatrix} 0 & 0 & -2 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = (-2) \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = (-2)(1+1) = -4$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{\mathrm{I}_k + \mathrm{III}_k}{=\!=\!=} - \begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = (-2) \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} = (-2)(1+1) = -4$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{\mathrm{I}_k + \mathrm{III}_k}{=\!=\!=} \begin{vmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = 2(-1-1) = -4$$

$$A_{14} = (-1)^5 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} \overset{\mathrm{I}_v + \mathrm{II}_v}{=\!=\!=} - \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = (-2) \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = (-2)(1+1) = -4$$

$$A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} \overset{\mathrm{I}_v + \mathrm{II}_v}{=\!=\!=} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} = 2(-1-1) = -4$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{\mathrm{II}_v + \mathrm{III}_v}{=\!=\!=} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = (-2)(1+1) = -4$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \overset{\mathrm{I}_v + \mathrm{II}_v}{=\!=\!=} - \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (-2) \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} = (-2)(-1-1) = 4$$

$$A_{24}=(-1)^6\begin{vmatrix}1&1&1\\1&-1&1\\1&-1&-1\end{vmatrix}\overset{I_v+III_v}{=\!=\!=}\begin{vmatrix}2&0&0\\1&-1&1\\1&-1&-1\end{vmatrix}=2\begin{vmatrix}-1&1\\-1&-1\end{vmatrix}=2(1+1)=4 \qquad \boxed{A=\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}}$$

$$A_{31}=(-1)^4\begin{vmatrix}1&1&1\\1&-1&-1\\-1&-1&1\end{vmatrix}\overset{I_v+II_v}{=\!=\!=}\begin{vmatrix}2&0&0\\1&-1&-1\\-1&-1&1\end{vmatrix}=2\begin{vmatrix}-1&-1\\-1&1\end{vmatrix}=2(-1-1)=-4$$

$$A_{32}=(-1)^5\begin{vmatrix}1&1&1\\1&-1&-1\\1&-1&1\end{vmatrix}\overset{I_v+II_v}{=\!=\!=}-\begin{vmatrix}2&0&0\\1&-1&-1\\1&-1&1\end{vmatrix}=-2\begin{vmatrix}-1&-1\\-1&1\end{vmatrix}=(-2)(-1-1)=4$$

$$A_{33}=(-1)^6\begin{vmatrix}1&1&1\\1&1&-1\\1&-1&1\end{vmatrix}\overset{II_v+III_v}{=\!=\!=}\begin{vmatrix}1&1&1\\2&0&0\\1&-1&1\end{vmatrix}=(-2)\begin{vmatrix}1&1\\-1&1\end{vmatrix}=(-2)\cdot(1+1)=-4$$

$$A_{34}=(-1)^7\begin{vmatrix}1&1&1\\1&1&-1\\1&-1&-1\end{vmatrix}\overset{I_v+III_v}{=\!=\!=}-\begin{vmatrix}2&0&0\\1&1&-1\\1&-1&-1\end{vmatrix}=(-2)\begin{vmatrix}1&-1\\-1&-1\end{vmatrix}=(-2)(-1-1)=4$$

$$A_{41}=(-1)^5\begin{vmatrix}1&1&1\\1&-1&-1\\-1&1&-1\end{vmatrix}\overset{I_v+II_v}{=\!=\!=}-\begin{vmatrix}2&0&0\\1&-1&-1\\-1&1&-1\end{vmatrix}=(-2)(1+1)=-4$$

$$A_{42}=(-1)^6\begin{vmatrix}1&1&1\\1&-1&-1\\1&1&-1\end{vmatrix}\overset{I_v+II_v}{=\!=\!=}\begin{vmatrix}2&0&0\\1&-1&-1\\1&1&-1\end{vmatrix}=2\begin{vmatrix}-1&-1\\1&-1\end{vmatrix}=2(1+1)=4$$

$$A_{43}=(-1)^7\begin{vmatrix}1&1&1\\1&1&-1\\1&-1&-1\end{vmatrix}\overset{I_k+III_k}{=\!=\!=}-\begin{vmatrix}2&1&1\\0&1&-1\\0&-1&-1\end{vmatrix}=(-2)\begin{vmatrix}1&-1\\-1&-1\end{vmatrix}=(-2)(-1-1)=4$$

$$A_{44}=(-1)^8\begin{vmatrix}1&1&1\\1&1&-1\\1&-1&1\end{vmatrix}\overset{II_v+III_v}{=\!=\!=}\begin{vmatrix}1&1&1\\2&0&0\\1&-1&1\end{vmatrix}=(-2)\begin{vmatrix}1&1\\-1&1\end{vmatrix}=(-2)(1+1)=-4$$

$$A_{kof}=\begin{bmatrix}-4&-4&-4&-4\\-4&-4&4&4\\-4&4&-4&4\\-4&4&4&-4\end{bmatrix}$$

Možemo primjetiti da je ova matrica simetrična pa je $A_{kof}=A_{kof}^T$

$$A^{-1}=\frac{-1}{16}\begin{bmatrix}-4&-4&-4&-4\\-4&-4&4&4\\-4&4&-4&4\\-4&4&4&-4\end{bmatrix}=\frac{-1}{16}(-4)\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}$$

Prema tome jednakost $A^{-1}=\frac{1}{4}A$ važi.

(#) Diskutovati rješenja sistema u zavisnosti od parametra t:

$$2x - y + 3z = -7$$
$$x + 2y - 6z = t$$
$$tx + 5y - 15z = 8.$$

Rj. Riješimo sistem Kramerovom metodom

$$D = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -6 \\ t & 5 & -15 \end{vmatrix} \overset{II_v + I_v \cdot 2}{=\!=\!=} \begin{vmatrix} 2 & -1 & 3 \\ 5 & 0 & 0 \\ t & 5 & -15 \end{vmatrix} = (-5) \begin{vmatrix} -1 & 3 \\ 5 & -15 \end{vmatrix} = (-5)(15-15) = 0$$

$$D_x = \begin{vmatrix} -7 & -1 & 3 \\ t & 2 & -6 \\ 8 & 5 & -15 \end{vmatrix} \overset{III_v + I_v \cdot 5}{=\!=\!=} \begin{vmatrix} -7 & -1 & 3 \\ t & 2 & -6 \\ -27 & 0 & 0 \end{vmatrix} = (-27) \underbrace{\begin{vmatrix} -1 & 3 \\ 2 & -6 \end{vmatrix}}_{6-6} = 0$$

$$D_y = \begin{vmatrix} 2 & -7 & 3 \\ 1 & t & -6 \\ t & 8 & -15 \end{vmatrix} \overset{II_v + I_v \cdot 2}{\underset{III_v + I_v \cdot 5}{=\!=\!=}} \begin{vmatrix} 2 & -7 & 3 \\ 5 & t-14 & 0 \\ t+10 & -27 & 0 \end{vmatrix} = 3 \begin{vmatrix} 5 & t-14 \\ t+10 & -27 \end{vmatrix} = 3(-135 - \overbrace{(t-14)(t+10)}^{t^2-4t-140})$$

$$= 3(-135 - t^2 + 4t + 140) = (-3)(t^2 - 4t - 5) = (-3)(t-5)(t+1)$$

$D = 16 + 20 = 36 \quad t_{1,2} = \frac{4 \pm 6}{2} \quad t_1 = -1 \quad t_2 = 5$

$$D_z = \begin{vmatrix} 2 & -1 & -7 \\ 1 & 2 & t \\ t & 5 & 8 \end{vmatrix} \overset{II_v + I_v \cdot 2}{\underset{III_v + I_v \cdot 5}{=\!=\!=}} \begin{vmatrix} 2 & -1 & -7 \\ 5 & 0 & t-14 \\ t+10 & 0 & -27 \end{vmatrix} = \begin{vmatrix} 5 & t-14 \\ t+10 & -27 \end{vmatrix} = -(t-5)(t+1)$$

Diskusija

1° $t \neq 5$; $t \neq -1$ sistem nema rješenja ($D = 0$ ali $D_y \neq 0$; $D_z \neq 0$)

2° $t = 5$

$D = D_x = D_y = D_z = 0$. sistem ćemo riješiti Gausovom metodom.

Sistem postaje

$$2x - y + 3z = -7 \quad (1)$$
$$x + 2y - 6z = 5 \quad (2)$$
$$5x + 5y - 15z = 8 \quad (3)$$

$(2)+(1)\cdot 2: \quad 5x = -9 \Rightarrow x = -\frac{9}{5}$

$(2) \Rightarrow \quad -\frac{9}{5} + 2y - 6z = 5 \quad / \cdot 5$

$-9 + 5y - 15z = 8$

$10y - 30z = 34$

$5y - 15z = 17$

$z = s$

$y = \frac{15s + 17}{5}$

Rješenje sistema je $\left(-\frac{9}{5}, \frac{15s+17}{5}, s\right)$, $s \in \mathbb{R}$.

3° $t = -1$, $D = D_x = D_y = D_z$ ~~sistem posbeyt~~

$$2x - y + 3z = -7 \quad (I)$$
$$x + 2y - 6z = -1 \quad (II)$$
$$-x + 5y - 15z = 8 \quad (III)$$

$(II) + (I)\cdot 2: \quad 5x = -15$

$x = -3$

$(I) \Rightarrow -y + 3z = -1$

Rješenje sisbe je $(-3, 3u+1, u)$, $u \in \mathbb{R}$

(#) Neka je $V=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 \mid x_1-x_2=x_2-x_3=x_3-x_4 \text{ i } x_5=0\}$.
Dokažite da je V vektorski prostor te nađite mu neku bazu i dimenziju.

Rj. Sabiranje u V je definisano na uobičajen način

$$(x_1,x_2,x_3,x_4,x_5)+(y_1,y_2,y_3,y_4,y_5)=(x_1+y_1,\ x_2+y_2,\ x_3+y_3,\ x_4+y_4,\ x_5+y_5)$$

kao i množenje sa skalarom α

$$\alpha(x_1,x_2,x_3,x_4,x_5)=(\alpha x_1,\alpha x_2,\alpha x_3,\alpha x_4,\alpha x_5).$$

Vektorski prostor (ili linearni prostor) je uređena četvorka $(X,+,\cdot,F)$ gdje je F polje, a $\cdot$ je f-ja sa $F\times X$ u X, čija vrijednost (α,x) označavamo sa αx tako da za $\alpha,\beta\in F$ i $x,y\in X$ vrijedi

i) $(X,+)$ je Abelova grupa

ii) $\alpha(\beta x)=(\alpha\beta)x$

iii) $1\cdot x=x$ gdje je 1 multiplikativna jedinica od F

iv) $\alpha(x+y)=\alpha x+\alpha y$ i $(\alpha+\beta)x=\alpha x+\beta x$.

Članove od X zovemo vektori, a članove iz F zovemo skalari. Operaciju $\cdot$ zovemo skalarno množenje. Zbog kratkoće vektorski prostor $(X,+,\cdot,F)$ često označavamo sa X i kažemo da je X vektorski prostor nad poljem F.

(1) $(V,+)$ je Abelova grupa

zatvorenost $(\forall\vec{a},\vec{b}\in V\ \ \vec{a}+\vec{b}\in V)$

$$\underbrace{(x_1,x_2,x_3,x_4,x_5)}_{\vec{a}},\ \underbrace{(y_1,y_2,y_3,y_4,y_5)}_{\vec{b}}\in\mathbb{R}^5 \qquad \vec{a}+\vec{b}=(x_1+y_1,x_2+y_2,x_3+y_3,x_4+y_4,x_5+y_5)\in\mathbb{R}^5.$$

Kako je $x_1-x_2=x_2-x_3=x_3-x_4$ i $x_5=0$ i $y_1-y_2=y_2-y_3=y_3-y_4$ i $y_5=0$ to

$(x_1+y_1)-(x_2+y_2)=(x_2+y_2)-(x_3+y_3)=(x_3+y_3)-(x_4+y_4)$ i $x_5+y_5=0$ to $\vec{a}+\vec{b}\in V$

Operacija $+$ je zatvorena u V

asocijativnost $(\forall\vec{a},\vec{b},\vec{c}\in V,\ \ (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}))$

$$(\vec{a}+\vec{b})+\vec{c}=(x_1+y_1,x_2+y_2,x_3+y_3,x_4+y_4,x_5+y_5)+(z_1,z_2,z_3,z_4,z_5)=((x_1+y_1)+z_1,(x_2+y_2)+z_2,(x_3+y_3)+z_3,$$
$$(x_4+y_4)+z_4,(x_5+y_5)+z_5)=(x_1+(y_1+z_1),x_2+(y_2+z_2),x_3+(y_3+z_3),x_4+(y_4+z_4),x_5+(y_5+z_5))=$$
$$=(x_1,x_2,x_3,x_4,x_5)+(y_1+z_1,y_2+z_2,y_3+z_3,y_4+z_4,y_5+z_5)=\vec{a}+(\vec{b}+\vec{c})$$ $+$ je asocijativna u V

neutralni element ($\forall \vec{a} \in V \ \exists \vec{0} \in V$ t.d. $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$)

$\vec{0}=(0,0,0,0,0)$ $\vec{0} \in V$ zato što je $0-0=0-0=0-0$ i $x_5=0$.

$\vec{0}$ jest neutralni element za $+$ u V

inverzni element ($\forall \vec{a} \in V \ \exists -\vec{a} \in V$ t.d. $\vec{a}+(-\vec{a})=(-\vec{a})+\vec{a}=\vec{0}$)

Inverzni element: elementa (x_1,x_2,x_3,x_4,x_5) je $(-x_1,-x_2,-x_3,-x_4,-x_5)$

(kako je $x_1-x_2=x_2-x_3=x_3-x_4$ to je $-x_1+x_2=-x_2+x_3=-x_3+x_4$)

element $(-x_1,-x_2,-x_3,-x_4,-x_5)$ jest inverzni element za $\forall (x_1,x_2,x_3,x_4,x_5) \in V$.

komutativnost ($\forall \vec{a}, \vec{b} \in V$ $\vec{a}+\vec{b}=\vec{b}+\vec{a}$)

$\vec{a}+\vec{b}=(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)=(y_1+x_1, y_2+x_2, y_3+x_3, y_4+x_4, y_5+x_5)=\vec{b}+\vec{a}$

$+$ jest komutativno u V $(V,+)$ jest Abelova grupa q.e.d.

(II) $\forall \alpha,\beta \in \mathbb{R}$ $\forall \vec{a} \in V$ $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$

$\alpha(\beta\vec{a})=\alpha\cdot(\beta x_1, \beta x_2, \beta x_3, \beta x_4, \beta x_5)=(\alpha(\beta x_1), \alpha(\beta x_2), \alpha(\beta x_3), \alpha(\beta x_4), \alpha(\beta x_5))$

$=((\alpha\beta)x_1, (\alpha\beta)x_2, (\alpha\beta)x_3, (\alpha\beta)x_4, (\alpha\beta)x_5)=(\alpha\beta)\vec{a}$ tj. $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$ q.e.d.

(III) $\forall \vec{a} \in V$ $1\cdot\vec{a}=\vec{a}$ $(1 \in \mathbb{R})$

trivijalno $1\cdot(x_1,x_2,x_3,x_4,x_5)=(x_1,x_2,x_3,x_4,x_5)$ tj. $1\cdot\vec{a}=\vec{a}$ q.e.d.

(IV) $\forall \alpha \in \mathbb{R}$ $\forall \vec{a}, \vec{b} \in V$ $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$ i $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$

$\alpha(\vec{a}+\vec{b})=\alpha(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)=(\alpha x_1+\alpha y_1, \alpha x_2+\alpha y_2, \alpha x_3+\alpha y_3,$

$\alpha x_4+\alpha y_4, \alpha x_5+\alpha y_5)=(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5)+(\alpha y_1, \alpha y_2, \alpha y_3, \alpha y_4, \alpha y_5)$

$=\alpha\vec{a}+\alpha\vec{b}$, dobili smo $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$ q.e.d.

ZA VJEŽBU POKAZAT DA JE $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$.

Prema tome V je vektorski prostor.

II način: Možemo pokazati da je V vektorski potprostor vektorskog prostora $\mathbb{R}^5$.

Nađimo sad bazu vektorskog prostora V.

Skup vektora $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n)$ koji su linearno nezavisni i koji generišu vektorski prostor V zovemo bazu.

Baza za vektorski prostor $\mathbb{R}^5$ je $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$ $(0,0,0,0,1)$. Iskoristimo ovu bazu i formirajmo bazu za naš prostor V.

Prema pretpostavci $x_1 - x_2 = x_2 - x_3 = x_3 - x_4$ i $x_5 = 0$.

Ako uzmemo $x_1 = 0,\ x_2 = 1 \Rightarrow x_3 = 2$ i $x_4 = 3$.

Ako uzmemo $x_2 = 0,\ x_3 = 1 \Rightarrow x_4 = 2$ i $x_1 = -1$

Ako uzmemo $x_3 = 0,\ x_4 = 1 \Rightarrow x_2 = -1$ i $x_1 = -2$

Moguća baza za V je $\{(0,1,2,3,0), (-1,0,1,2,0), (-2,-1,0,1,0)\}$

Provjerimo da li je ovaj sistem linearno zavisan.

$$\alpha(0,1,2,3,0) + \beta(-1,0,1,2,0) + \gamma(-2,-1,0,1,0) = (0,0,0,0,0)$$

$$\begin{aligned} -\beta - 2\gamma &= 0 \quad (a) \\ \alpha - \gamma &= 0 \quad (b) \\ 2\alpha + \beta &= 0 \quad (c) \\ 3\alpha + 2\beta + \gamma &= 0 \quad (d) \end{aligned}$$

Riješimo se γ.

$(a)+(b)(2)$: $-2\alpha - \beta = 0$

(c): $2\alpha + \beta = 0$

$(d)+(b)$: $4\alpha + 2\beta = 0$

$2\alpha + \beta = 0$

$\alpha = 1 \Rightarrow \beta = -2$

$\gamma = 1$

Kako smo dobili $\alpha \neq 0$ (i $\beta \neq 0$ i $\gamma \neq 0$) sistem nije linearno nezavisan pa nije i baza. Izbacimo jedan element iz baze.

Nova moguća baza za V je $\{(0,1,2,3,0), (-2,-1,0,1,0)\}$

Ispitajmo linearnu zavisnost.

$$\alpha(0,1,2,3,0) + \beta(-2,-1,0,1,0) = 0$$

$$\begin{aligned} -2\beta &= 0 \\ \alpha - \beta &= 0 \\ 2\alpha &= 0 \\ 3\alpha + \beta &= 0 \end{aligned}$$

$\alpha = \beta = 0$

Sistem $\{(0,1,2,3,0), (-2,-1,0,1,0)\}$ je linearno nezavisan i on je baza za vektorski prostor V.

Dimenzija vektorskog prostora V je 2.