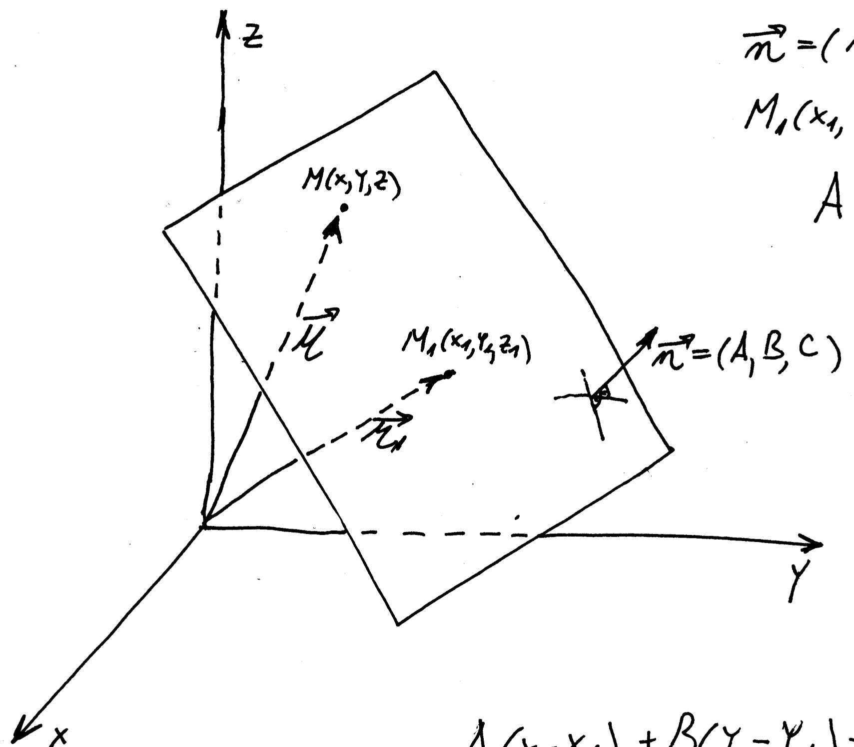
Ravan



$\vec{n}=(A,B,C)$ vektor normale

$M_1(x_1,y_1,z_1)$ tačka u ravni

$$Ax+By+Cz+D=0$$

opšti oblik jednačine ravni

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

segmentni oblik jednačine ravni

($(a,0,0)$, $(0,b,0)$ i $(0,0,c)$ su tačke preseka ravni sa x, y i z-osom)

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

skalarni oblik jednačine ravni kroz tačku $M_1(x_1,y_1,z_1)$

$(\vec{r}-\vec{r_1})\cdot\vec{n}=0$ vektorski oblik jednačine ravni

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

rastojanje tačke $M_1(x_1,y_1,z_1)$ od ravni $Ax+By+Cz+D=0$.

Ako su date dvije ravni

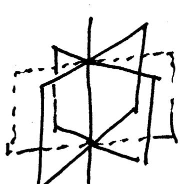
$$A_1x+B_1y+C_1z+D_1=0$$
$$A_2x+B_2y+C_2z+D_2=0$$

$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ uslov paralelnosti dvije ravni ($\vec{n_1}$ i $\vec{n_2}$ kolinearni)

$A_1A_2+B_1B_2+C_1C_2=0 \Rightarrow$ ravni međusobno normalne

$\cos\varphi=\pm\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$ ugao između dvije ravni

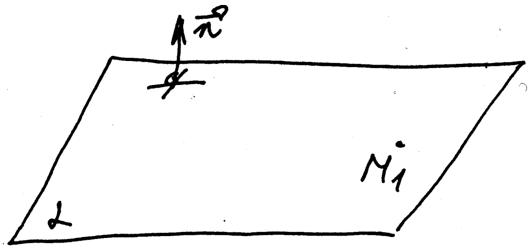
$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$ pramen ravni (skup svih ravni koje prolaze kroz istu pravu)



λ LAMBDA

(#) Napisati jednačinu ravni koja sadrži tačku $M_1(-2,1,3)$ i normalna je na vektor $\vec{n}=(1,2,7)$.

Rj.



α: ?

$M_1(-2,1,3)$

$\vec{n}=(1,2,7)$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina tražene ravni

$A=1,\ B=2,\ C=7$

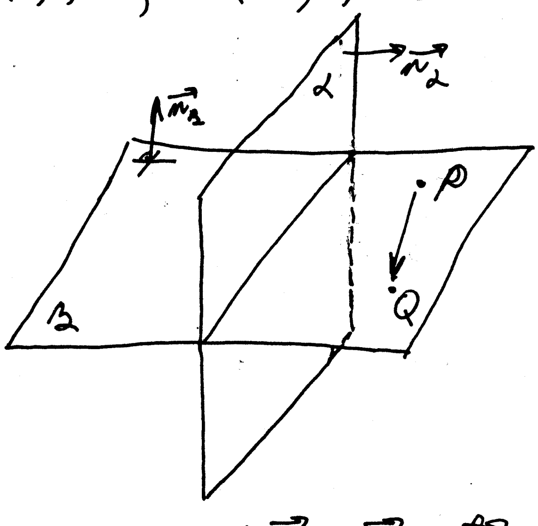
$$1(x+2)+2(y-1)+7(z-3)=0$$

$$x+2+2y-2+7z-21=0$$

$$x+2y+7z-21=0$$ jednačina tražene ravni

(#) Napisati jednačinu ravni koja prolazi kroz tačke $P(1,1,1)$, $Q(0,1,-1)$ i normalna je na ravan $\alpha: x+y+z-1=0$.

Rj.



β: ?

$$\left.\begin{array}{l}\vec{n_\beta}\perp\vec{PQ}\\ \vec{n_\beta}\perp\vec{n_\alpha}\end{array}\right\}\Rightarrow \vec{n_\beta}\parallel\vec{n_\alpha}\times\vec{PQ}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}:\ \vec{n_\beta}=k(\vec{n_\alpha}\times\vec{PQ})$$

$$\begin{array}{l}P(1,1,1)\\ Q(0,1,-1)\end{array}\Rightarrow \vec{PQ}=(-1,0,-2)$$

$$\vec{n_\alpha}=(1,1,1)$$

$$\vec{n_\alpha}\times\vec{PQ}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&1&1\\-1&0&-2\end{vmatrix}=\vec{i}(-2-0)-\vec{j}(-2+1)+\vec{k}(0+1)=-2\vec{i}+\vec{j}+\vec{k}$$

$$=(-2,1,1)$$

$\Rightarrow \vec{n_\beta}=k(-2,1,1)$ gdje je k neki realan broj, $k\neq 0$

$n_\beta=(-2k,k,k)$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ $\quad P(\overset{x_1}{1},\overset{y_1}{1},\overset{z_1}{1})$

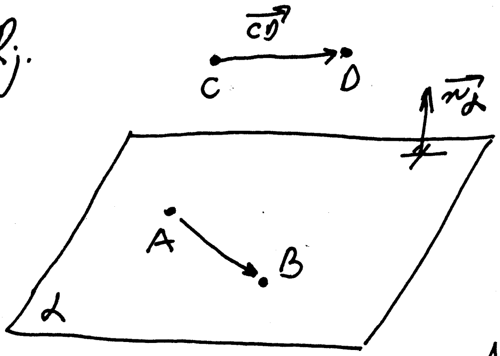
$$-2k(x-1)+k(y-1)+k(z-1)=0 \quad /:k$$

$$-2x+2+y-1+z-1=0$$

$$-2x+y+z=0$$ jednačina tražene ravni

Date su tačke $A(0,3,4)$, $B(-1,2,3)$, $C(1,-2,-1)$ i $D(4,-1,1)$. Napisati jednačinu ravni koja sadrži tačke A i B, i paralelna je sa vektorom $\overrightarrow{CD}$.

Rj.



$\alpha: ?$ $\quad A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina ravni

$$\left.\begin{array}{l}\vec{n_\alpha} \perp \overrightarrow{AB} \\ \vec{n_\alpha} \perp \overrightarrow{CD}\end{array}\right\} \Rightarrow \vec{n_\alpha} \parallel \overrightarrow{AB} \times \overrightarrow{CD}$$

$$\Downarrow$$

$$\exists k \in R \quad \vec{n_\alpha} = k(\overrightarrow{AB} \times \overrightarrow{CD})$$

$$\left.\begin{array}{l}A(0,3,4) \\ B(-1,2,3)\end{array}\right. \Rightarrow \overrightarrow{AB} = (-1,-1,-1)$$

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & -1 \\ 3 & 1 & 2 \end{vmatrix} = -\vec{i} - \vec{j} + 2\vec{k} = (-1,-1,2)$$

$$\left.\begin{array}{l}C(1,-2,-1) \\ D(4,-1,1)\end{array}\right. \Rightarrow \overrightarrow{CD} = (3,1,2)$$

$$\Downarrow \quad \vec{n_\alpha} = k(-1,-1,2), \text{ gdje je } k \text{ neki realan broj.}$$

$$= -k(1,1,-2)$$

$$B(\overset{x_1}{-1}, \overset{y_1}{2}, \overset{z_1}{3})$$

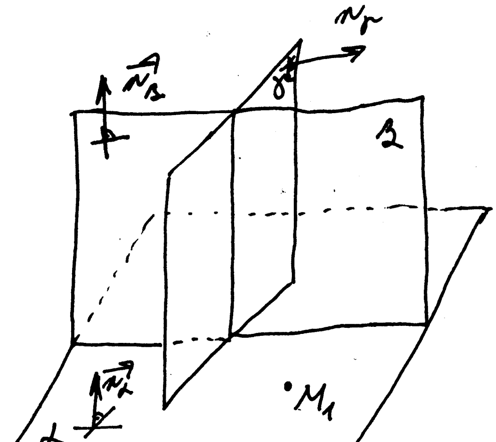
$$-k \cdot 1(x+1) - k \cdot 1 \cdot (y-2) - k \cdot (-2)(z-3) = 0 \quad /:(-k), \; k \neq 0$$

$$x + \underline{\underline{1}} + y - \underline{\underline{2}} - 2z + \underline{\underline{6}} = 0$$

$$x + y - 2z + 5 = 0 \quad \text{jednačina tražene ravni}$$

Napisati jednačinu ravni koja prolazi kroz tačku $M_1(2,0,-1)$ i normalna je na ravnima $2x-y-3=0$ i $x+y-z+1=0$.

Rj.



$\alpha: ?$

$\beta: 2x-y-3=0, \quad \vec{n_\beta} = (2,-1,0)$

$\gamma: x+y-z+1=0, \quad \vec{n_\gamma} = (1,1,-1)$

$$\left.\begin{array}{c}\text{Ako } M_1 \text{ uvrstim u } \beta \text{ imam} \\ 2 \cdot 2 - 0 - 3 \neq 0 \\ \text{Ako } M_1 \text{ uvrstim u } \gamma \text{ imam} \\ 2 + 0 + 1 + 1 \neq 0\end{array}\right\} \Rightarrow M_1 \notin \beta \text{ i } M_1 \notin \gamma$$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

jednačina tražene ravni

$$\left.\begin{array}{l}\vec{n_\alpha}\perp\vec{n_\beta}\\ \vec{n_\alpha}\perp\vec{n_\gamma}\end{array}\right\}\Rightarrow \vec{n_\alpha}\parallel \vec{n_\beta}\times\vec{n_\gamma}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}:\ \vec{n_\alpha}=k(\vec{n_\beta}\times\vec{n_\gamma})$$

$$\vec{n_\beta}\times\vec{n_\gamma}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-1&0\\1&1&-1\end{vmatrix}=\vec{i}(1-0)-\vec{j}(-2-0)+\vec{k}(2+1)=\vec{i}+2\vec{j}+3\vec{k}$$
$$=(1,2,3)$$

$$\vec{n_\alpha}=k(1,2,3)=(\overset{A}{k},\overset{B}{2k},\overset{C}{3k})$$ gdje je k neki realan broj, $k\neq 0$

$$k(x-2)+2k(y-0)+3k(z+1)=0 \quad /:k$$

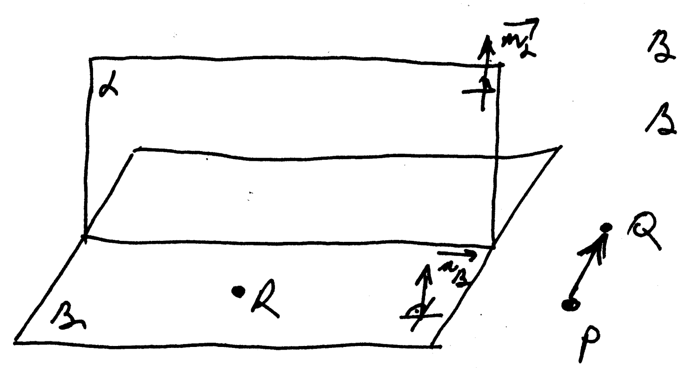
$$x+2y+3z-2+3=0$$

$$x+2y+3z+1=0$$

jednačina tražene ravni

(#) Date su tačke $P(1,1,-1)$, $Q(1,2,0)$ i $R(-1,0,0)$. Napisati jednačinu ravni koja je normalna na ravan α: $2x-y+5z-3=0$, koja je paralelna sa vektorom $\vec{PQ}$ i sadrži tačku R.

Rj.



β: ?

β: $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$$\begin{array}{l}P(1,1,-1)\\Q(1,2,0)\end{array}\Rightarrow \vec{PQ}=(0,1,1)$$

$$\vec{n_\alpha}=(2,-1,5)$$

$$\left.\begin{array}{l}\vec{n_\beta}\perp\vec{n_\alpha}\\ \vec{n_\beta}\perp\vec{PQ}\end{array}\right\}\Rightarrow \vec{n_\beta}\parallel \vec{n_\alpha}\times\vec{PQ}$$

$$\Downarrow$$

$$\exists k\in\mathbb{R}:\ \vec{n_\beta}=k(\vec{n_\alpha}\times\vec{PQ})$$

$$\vec{n_\alpha}\times\vec{PQ}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-1&5\\0&1&1\end{vmatrix}=-6\vec{i}-2\vec{j}+2\vec{k}=(-6,-2,2)$$

$$\Downarrow$$

$$\vec{n_\beta}=k(-6,-2,2)=-2k(3,1,-1)$$

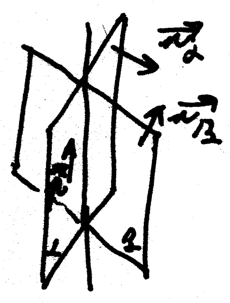
$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

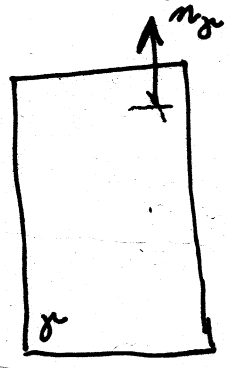
$$-2k\cdot 3(x+1)-2k\cdot 1(y-0)-2k(-1)(z-0)=0 \quad /:(-2k)$$

$$3x+y+z+3=0$$ jednačina tražene ravni

\# Kroz presjek ravni $4x-y+3z-1=0$ i $x+5y-z+2=0$ postaviti ravan koja je normalna na ravan $2x-y+5z-3=0$.

Rj. I način:





$\alpha: 4x-y+3z-1=0$

$\beta: x+5y-z+2=0$

$\gamma: 2x-y+5z-3=0$

$\vec{n}_\alpha=(4,-1,3)$

$\vec{n}_\beta=(1,5,-1)$

$\vec{n}_\gamma=(2,-1,5)$

$$\left.\begin{array}{l}\vec{p}\perp\vec{n}_\alpha\\ \vec{p}\perp\vec{n}_\beta\end{array}\right\}\Rightarrow \vec{p}\parallel \vec{n}_\alpha\times\vec{n}_\beta$$

$$\Downarrow$$

$$\vec{p}=k\cdot(\vec{n}_\alpha\times\vec{n}_\beta),\quad k\in\mathbb{R}$$

$$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\4&-1&3\\1&5&-1\end{vmatrix}=(1-15)\vec{i}-(-4-3)\vec{j}+(20+1)\vec{k}=(-14,7,21)$$

pa za $\vec{p}$ mogu uzeti $\vec{p}=(-2,1,3)$

$$\left.\begin{array}{l}\vec{n}\perp\vec{p}\\ \vec{n}\perp\vec{n}_\gamma\end{array}\right\}\Rightarrow \vec{n}\parallel\vec{p}\times\vec{n}_\gamma,\qquad \vec{p}\times\vec{n}_\gamma=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-2&1&3\\2&-1&5\end{vmatrix}=(8,16,0)\Rightarrow$$

$$\Rightarrow \vec{n}=(1,2,0)$$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ jednačina ravni kroz tačku (x_1,y_1,z_1) i vektor normale $\vec{n}=(A,B,C)$.

nađimo tačku koja pripada presjeku ravni $\alpha\cap\beta$

$$\begin{array}{l}4x-y+3z-1=0\\ x+5y-z+2=0 \quad /\cdot 3\end{array}\qquad \begin{array}{l}4x-y+3z-1=0\\ +\;3x+15y-3z+6=0\end{array}\qquad \begin{array}{l}7x+14y+5=0\\ x=\frac{2}{7}\Rightarrow 14y=-2-5\\ y=-\frac{1}{2}\end{array}$$

$4\cdot\frac{2}{7}+\frac{1}{2}+3z-1=0\Rightarrow 3z=-\frac{8}{7}-\frac{1}{2}+1=\frac{1}{2}-\frac{8}{7}=\frac{7-16}{14}=\frac{-9}{14}\quad /\cdot\frac{1}{3}$

$\Rightarrow z=-\frac{3}{14}$

$M\left(\frac{2}{7},-\frac{1}{2},-\frac{3}{14}\right)$

$$1\cdot\left(x-\frac{2}{7}\right)+2\left(y+\frac{1}{2}\right)+0\left(z+\frac{3}{14}\right)=0$$

$$x-\frac{2}{7}+2y+1=0\Rightarrow 7x+14y+5=0$$ jednačina tražene ravni

II način: koristimo formulu pramena pravih

$$4x-y+3z-1+\lambda(x+5y-z+2)=0$$

$$(4+\lambda)x+(-1+5\lambda)y+(3-\lambda)z-1+2\lambda=0$$

$\vec{n}=(4+\lambda,-1+5\lambda,3-\lambda)$

$\vec{n}\perp\vec{n}_\gamma = \vec{n}\cdot\vec{n}_\gamma=0\Rightarrow\lambda=3$

$\Rightarrow 7x+14y+5=0$ jednačina tražene ravni

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti ravan α koja će biti okomita na ravan $\beta: 6x-4y+z=16$ i $\gamma: y+2z+1=0$. (Obavezno nacrtati sliku).

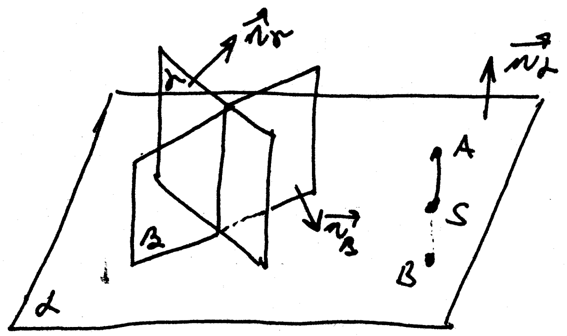
Rj:

S

$A(1,3,0)$ $B(-3,7,2)$

$$S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$S(-1, 5, 1)$$



$\beta: 6x-4y+z=16$

$\vec{n}_\beta=(6,-4,1)$

$\gamma: y+2z+1=0$

$\vec{n}_\gamma=(0,1,2)$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

jednačina ravni kroz jednu tačku

$\vec{n}_\alpha=(A,B,C)$

$$\left.\begin{array}{l}\vec{n}_\alpha \perp \vec{n}_\beta \\ \vec{n}_\alpha \perp \vec{n}_\gamma\end{array}\right\} \Rightarrow \vec{n}_\alpha \parallel \vec{n}_\beta \times \vec{n}_\gamma$$

$$\Downarrow$$

$$\exists k\in\mathbb{R} \quad \vec{n}_\alpha = k(\vec{n}_\beta \times \vec{n}_\gamma)$$

$$\vec{n}_\beta \times \vec{n}_\gamma = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -9\vec{i}-12\vec{j}+6\vec{k}=(-9,-12,6)$$

pa za $\vec{n}_\alpha$ možemo uzeti

$\vec{n}_\alpha=(3,4,-2)$

$$3(x-(-1))+4(y-5)+(-2)(z-1)=0$$

$$3x+4y-2z+3-20+2=0$$

$3x+4y-2z-15=0$ jednačina tražene ravni