POLINOMI

16. Odrediti koeficijente a, b i c tako da polinom $P(x)=x^3+ax^2+bx+c$ bude djeljiv sa $x-1$, $x+2$, a pri dijeljenju sa $x-4$ daje ostatak 18.

$P(x)=a_n x^n + a_{n-1}x^{n-1}+\dots+a_0$

$a_i \in \mathbb{R}$, $a_n \neq 0$ stepen $\deg = n$

Bezuov stav: Ostatak pri dijeljenju polinoma $P(x)$ sa $x-a$ je $P(a)$

Posljedica: $P(x)$ je djeljiv sa $x-a$ onda je $P(a)=0$

$$\left.\begin{aligned} P(1)&=0\\ P(-2)&=0\\ P(4)&=18\end{aligned}\right\} \quad \left.\begin{aligned} a+b+c&=-1\\ 4a-2b+c&=8\\ 16a+4b+c&=-48\end{aligned}\right\} \quad a=-2,\ b=-5,\ c=6$$

17. Odrediti ostatak pri dijeljenju polinoma:

$P(x)=x^{100}+3x^{99}+x^2-3x+9$ sa x^2+2x-3.

$P(x)=(x^2+2x-3)\cdot Q(x)+R(x)$

$\quad\;\;=(x-1)(x+3)\cdot Q(x)+ax+b$

$x=1 \quad 1+\cancel{3}+1-\cancel{3}+9=a+b \Rightarrow \boxed{a+b=11}$

$x=-3 \quad \cancel{(-3)^{100}+3\cdot(-3)^{99}}+9+9+9=-3a+b \Rightarrow \boxed{-3a+b=27}$

$\Rightarrow a=-4 \quad b=15 \quad \Rightarrow R(x)=-4x+15$

18. Naći bar jedan polinom 5. stepena sa realnim koeficijentima čije su nule: $2,\ 1+i,\ -\sqrt{3}\cdot i$

$P(x) = a_n x^n + \ldots + a_1 x + a_0$

$= a_n (x-x_1)(x-x_2)\cdot\ldots(x-x_n)$

$x_i \in \mathbb{C}$; ako su $a_0, a_1, \ldots, a_n \in \mathbb{R}$ i $a+bi$ je nula polinoma, sigurno je i $a-bi$ nula polinoma

$\Rightarrow 1-i,\ \sqrt{3}i$ su nule

$P(x) = (x-2)(x-(1+i))(x-(1-i))\cdot(x+\sqrt{3}i)(x-\sqrt{3}i)$

$= (x-2)((x-1)^2+1)(x^2+3) = \ldots$

19. Naći količnik i ostatak pri dijeljenju polinoma:

a) $2x^5 - x^4 + 8x^3 - 4x^2 + x + 3$ sa x^2+x+1 i sa $(x-2)$

b) $4x^3 + x^2$ sa $x+1+i$

	2	-1	8	-4	1	3
2	2	3	14	24	49	101

$P(x) = (x-2)(2x^4+3x^3+14x^2+24x+49)+101$

a) $(2x^5 - x^4 + 8x^3 - 4x^2 + x + 3) : (x^2+x+1) = \underbrace{2x^3 - 3x^2 + 9x - 10}_{Q(x)}$

$2x + 13 = R(x)$

b) D.Z. $(4x^3 + x^2) : (x+1+i) = \ldots$

Hornerovom shemom (kada imamo linearni polinom)

	4	1	0	0
$-1-i$	4	$-3-4i$	$7i-1$	$8-6i$

$(-1-i)(-3-4i) + 0 = -1+7i$

$(-1-i)(-1+7i) = 8-6i$

$4x^3 + x^2 = (x+1+i)\cdot(4x^2 - (3+4i)x - 1 + 7i) + 8 - 6i$

$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = (x-a)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1 x + b_0) + r$

$= b_{n-1}x^n + (b_{n-2} - ab_{n-1})x^{n-1} + \ldots + (b_0 - ab_1)x + r - ab_0$

$$\Rightarrow \left.\begin{aligned} b_{n-1} &= a_n \\ a_{n-1} &= b_{n-2} - a b_{n-1} \\ &\vdots \\ a_1 &= b_0 - a b_1 \\ a_0 &= r - a b_0 \end{aligned}\right\} \text{sis.} \qquad \begin{aligned} b_{n-1} &= a_n \\ b_{n-2} &= a_{n-1} + a\cdot b_{n-1} \\ &\vdots \\ b_0 &= a_1 + a b_1 \\ r &= a_0 + a b_0 \end{aligned}$$

$$\Rightarrow 4x^3 + x^2 = (x+1+i)\left(4x^2 + (-3-4i)x + (-1+7i)\right) + 8 - 6i$$

20. Dokazati da je polinom:

$(\cos\varphi + x\sin\varphi)^n - \cos(n\varphi) - x\cdot\sin(n\varphi)$

djeljiv sa x^2+1. (nule ovog pol. trebaju biti i nule početnog pol.)

$x^2+1 = (x-i)(x+i)$

$P(i) = (\cos\varphi + i\sin\varphi)^n - \cos(n\varphi) - i\sin(n\varphi) = 0$ Moavrova formula

$$\begin{aligned} P(-i) &= (\cos\varphi - i\sin\varphi)^n - \cos(n\varphi) + i\sin(n\varphi) \\ &= (\cos(-\varphi) + i\sin(-\varphi))^n - \cos(n\varphi) + i\sin(n\varphi) \\ &= \cancel{\cos(-n\varphi)} + i\sin(-n\varphi) - \cancel{\cos(n\varphi)} + i\sin(n\varphi) \\ &= \cancel{-i\sin(n\varphi)} + \cancel{i\sin(n\varphi)} = 0 \end{aligned}$$

$\Rightarrow P(x)$ je djeljiv sa $(x-i)$ i sa $(x+i)$ pa je djeljiv i sa $(x-i)(x+i)$ tj. x^2+1.

21. Dokazati da je za svako $m,n,p \in \mathbb{N}_0$ polinom:

$P(x) = x^{3m} + x^{3n+1} + x^{3p+2}$ djeljiv sa x^2+x+1.

$x^2+x+1 = (x-x_1)(x-x_2)$ $\qquad x_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

ovo se pojavljuje u:

$x^3-1 = (x-1)(x^2+x+1) \Rightarrow x_1^3 - 1 = 0 \quad x_1^3 = 1$

$x_2^3 - 1 = 0 \quad x_2^3 = 1$ nula pol. x^2+x+1

$P(x_1) = \underset{=1}{(x_1^3)}^m + \underset{=1}{(x_1^3)}^n \cdot x_1 + \underset{=1}{(x_1^3)}^p \cdot x_1^2 = 1 + x_1 + x_1^2 = 0 \Rightarrow (x-x_1) \mid P(x)$

$P(x_2) = (x_2^3) \quad = 0 \Rightarrow (x-x_2) \mid P(x)$

$\Rightarrow P(x)$ je djeljiv sa $(x-x_1)(x-x_2) = x^2+x+1$

22. Odrediti brojeve a, b tako da je polinom:

$P(x) = a \cdot x^{n+1} + b \cdot x^n + 1$, $n \in \mathbb{N}$ bude djeljiv sa x^2-2x+1.

$x^2-2x+1 = (x-1)^2$ jedinica je dvostruka nula

ne dijelimo jer nemamo konkretan polinom

$P(1) = 0$ – mora biti da bi $x-1$ bio faktor makar jednom

$P(1) = a+b+1 = 0 \Rightarrow b = -a-1$

$P(x) = a \cdot x^{n+1} - ax^n - x^n + 1$

$= ax^n(x-1) - (x^n-1)$

$= ax^n(x-1) - (x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1)$

$= (x-1)\underbrace{(ax^n - x^{n-1} - \ldots - x - 1)}_{Q(x)}$

nama treba dvostruka nula pa je $x-1$ nula i pol. $Q(x)$

$Q(1) = 0$: $a - n = 0 \Rightarrow a = n \quad b = -n-1$

23. Odrediti parametar λ tako da zbir 2 rješenja jed. $2x^3 - x^2 - 7x + \lambda = 0$ bude jednak 1. Pa za takvo λ riješiti jednačinu.

Vietove formule: $P(\lambda) = a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1 x + a_0$

$-\frac{b}{a}, \frac{c}{a}$

$= a(x-x_1)(x-x_2)\cdot\ldots(x-x_m)$

$= a\left(x^m - (x_1+x_2+\ldots+x_m)x^{m-1} + \right.$

$+ (x_1x_2 + x_1x_3 + \ldots + x_1x_m + x_2x_3 + \ldots + x_{m-1}x_m)x^{m-2}$

$- \ldots \left. \underbrace{(-1)^m x_1 x_2 \cdot\ldots\cdot x_m}_{\text{slobodni član}}\right)$

$x_1 + x_2 + x_3 = \frac{1}{2}$

$x_1x_2 + x_2x_3 + x_3x_1 = -\frac{7}{2}$

$x_1 \cdot x_2 x_3 = \frac{-\lambda}{2}$

$x_1 + x_2 = 1$

$\boxed{x_3 = -\frac{1}{2}}$

$x_1x_2 - \frac{1}{2}\underbrace{(x_1+x_2)}_{1} = -\frac{7}{2} \Rightarrow 2x_1x_2 = -6 \qquad x_1x_2 = -3$

$x_1 \cdot x_2 \cdot \frac{-1}{2} = \frac{-\lambda}{2} \Rightarrow x_1 \cdot x_2 = \lambda \qquad x_1x_2 = \lambda$

$\lambda = -3$

$x_1 + x_2 = 1 \qquad x_2 = 1 - x_1$

$x_1 x_2 = -3$

$x_1 \cdot (1 - x_1) = -3$

$x_1 - x_1^2 = -3$

$x_1^2 - x_1 - 3 = 0$

$x_{1,2} = \frac{1 \pm \sqrt{13}}{2}$

$x_1 = \frac{1-\sqrt{13}}{2}, \quad x_2 = \frac{1+\sqrt{13}}{2}, \quad x_3 = \frac{-1}{2}$